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# $\Lambda BV$ AS A NON SEPARABLE DUAL SPACE<sup>\*</sup>

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Abstract – Let C be a field of subsets of a set I. Also, let  $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$  be a non-decreasing positive sequence of real numbers such that  $\lambda_1 = 1$ ,  $1/\lambda_i \to 0$  and  $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$ . In this paper we prove that  $\Lambda BV$  of all the games of  $\Lambda$ -bounded variation on C is a non-separable and norm dual Banach space of the space of simple games on C. We use this fact to establish the existence of a linear mapping T from  $\Lambda BV$  onto FA (finitely additive set functions) which is positive, efficient and satisfies a weak form of symmetry, namely invariance under a semigroup of automorphisms of (I, C).

Keywords - Set functions, duality, compactness, non separable

#### **1. INTRODUCTION**

Let *C* be a field of subsets of a nonempty set *I*. It is well-known that the space *F A* of all the finitely additive games of bounded variation on *C*, equipped with the total variation norm, is isometrically isomorphic to the norm dual of the space of all simple functions on *C*, endowed with the sup norm ([1]) (also see [2]). Maccheroni and Ruckle in [3] established a parallel result for the space *BV* of all the games of bounded variation on *C*. Indeed, they showed that *BV*, equipped with the total variation norm, is isometrically isometric to the norm dual of the space of all simple games endowed with a suitable norm where a simple game is a game which is non zero only on a finite number of elements of *C*. Let  $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$  be a non-decreasing positive sequence of real numbers such that  $\lambda_1 = 1$ ,  $1/\lambda_i \rightarrow 0$  and  $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$ . We introduce space  $\Lambda BV$  which shares many properties of space *BV*. Here, we prove that space  $\Lambda BV$  of all the games of  $\Lambda$  bounded variation on *C* equipped with the total variation norm, is isometrically isometric to the norm dual of the space of all simple games, endowed with a suitable norm. We use this fact to establish the existence of a linear mapping *T* from  $\Lambda BV$  onto *F A* (finitely additive set functions) which is positive, efficient and satisfies a weak form of symmetry, namely invariance under a semigroup of automorphisms of (I, C).

## 2. PRELIMINARIES

A set function  $v: C \to R$  is a game if  $v(\phi) = 0$ . A game on C is monotone if  $v(A) \le v(B)$  whenever  $A \subseteq B$ . A chain  $\{S_i\}_{i=0}^n$  in C is a finite strictly increasing sequence

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$$\phi = S_0 \subset S_1 \subset \ldots \subset S_n = I$$

of the elements of C. ABV is the set of all games such that

$$|| u || = \sup \left\{ \sum_{i=1}^{n} \frac{|u(S_i) - u(S_{i-1})|}{\lambda_i} : \{S_i\}_{i=0}^{n} \text{ is a chain in } C \right\} < \infty.$$

A game in  $\Lambda BV$  is said to be of  $\Lambda$  bounded variation. A game is called a simple game if it is nonzero only on a finite number of elements of C. A function u in  $\Lambda BV$  is called finitely additive if

$$u(A \cup B) = u(A) + u(B)$$

whenever A and B are in C and  $A \cap B = \phi$ .

The set FA of finitely additive functions in ABV forms a closed subspace of ABV. A function u in ABV is called increasing if  $u(A) \le u(B)$  whenever  $A \subset B$ . Each u in ABV has the form  $u = u^+ + u^-$  when  $u^+$  and  $u^-$  are increasing and  $||u|| = u^+(I) + u^-(I)$ . A linear mapping T in L(BV) is positive if Tu increases whenever u increases.

Let C denote the group of automorphisms of (I, C). A subspace X is called symmetric if  $u \circ \pi$  is in X for each x in X and each  $\pi$  in C. A value is a linear mapping T from a symmetric subspace X of  $\Lambda BV$  onto the space F A of finitely additive set functions which satisfies three conditions:

(a) T is positive: i.e., Tu increases whenever u increases.

(b) T is symmetric: i.e.,  $T(u \ 0 \ \pi) = (Tu) \ o \ \pi$  for each  $\pi$  in C and u in X.

(c) T is efficient: (Tu)(I) = u(I) for each u in X.

In this note we establish the existence of linear operations from all of  $\Lambda BV$  onto FA which satisfy (a), (b) and a weaker form of (c), namely symmetry under a semigroup of C. In addition, these linear operators are projections (i.e., Tu = u for u in FA). Our main result is that, given any locally finite subgroup  $\Phi$  of C there is a projection T from  $\Lambda BV$  onto FA which is symmetric under  $\Phi$ . Since  $\Lambda BV$  is a (proper) subspace of  $R^{C}$ , it inherits a topology from the product topology of  $R^{C}$ . This is the weak topology generated by the projection functional

$$P_A : \Lambda BV \to R$$
$$u \to u(A)$$

where  $A \in C$ . A net  $\{u_{\alpha}\}$  converges to u in this topology if  $u_{\alpha}(A) \rightarrow u(A)$  for all  $A \in C$  (we write  $u_{\alpha} \xrightarrow{C} u$ ). This topology is called  $\Lambda$ -*vague* topology for the analogy with the vague topology on the set of probability measures.

### 3. ABV AS A NON SEPARABLE DUAL SPACE

In [4], Aumann and Shapley proved that BV is a Banach space. Here, we show  $\Lambda BV$  is a Banach space too.

Let  $\Omega = \{S_i\}_{i=0}^n$  be a chain. For any set function  $\nu$  we define

$$\|\nu\|_{\Omega} = \left\{ \sum_{i=1}^{n} \frac{|\nu(S_i) - \nu(S_{i-1})|}{\lambda_i} \right\} < \infty.$$

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This shows that a necessary and sufficient condition,  $\nu \in \Lambda BV$ , is that  $\|\nu\|_{\Omega}$  be bounded over all chain  $\Omega$ . Then,  $\nu \in \Lambda BV$  if and only if  $\|\nu\| = \sup \|\nu\|_{\Omega}$ , where the sup is taken over all chains  $\Omega$ .

It is obvious that this defines a norm on  $\Lambda BV$ . Now, we show that with this norm,  $\Lambda BV$  is a complete space.

## **Theorem 3.1.** $\Lambda BV$ is complete, hence a Banach space.

**Proof:** Let  $\{v_n\}$  be a Cauchy sequence of elements of ABV. For any subset S of I, we show that sequence  $\{v_n(S)\}$  is a Cauchy sequence in R.

Let S be a subset of I. For the chain

$$\Phi \subset S \subset I;$$

We have

$$||v_{n} - v_{m}|| \ge \frac{|(v_{n}(S) - v_{m}(S)) - (v_{n}(\Phi) - v_{m}(\Phi))|}{\lambda_{1}}$$
$$= |(v_{n}(S) - v_{m}(S))|.$$

Then the sequence  $\{v_n(S)\}$  is a Cauchy sequence in R and is convergent; denote it's limit by v(S). We must first show that v is  $\Lambda$ -bounded variation. Let N be such that  $||v_n - v_m|| \le 1$  whenever  $n \ge N$ . Then for each chain  $\Omega$  and each  $n \ge N$  we have

$$\| \boldsymbol{v}_n \|_{\Omega} - \| \boldsymbol{v}_N \| \leq \| \boldsymbol{v}_n \|_{\Omega} - \| \boldsymbol{v}_N \|_{\Omega}$$

$$\leq \| \boldsymbol{v}_n - \boldsymbol{v}_N \|_{\Omega}$$

$$\leq \| \boldsymbol{v}_n - \boldsymbol{v}_N \|$$

$$\leq 1$$

letting  $n \to \infty$ , we deduce

$$\left\| v \right\|_{\Omega} \le 1 + \left\| v_N \right\|_{\cdot}$$

Hence  $\nu$  is  $\Lambda$ -bounded variation. That  $||\nu_n - \nu|| \rightarrow 0$  is now easily verified, so the theorem is proved.

Here, we show that  $\Lambda BV$  is a non separable space. So, the dual of  $\Lambda BV$  is non separable too.

**Theorem 3.2.**  $\Lambda BV[a,b]$  is non separable.

**Proof:** For each a satisfying a < s < b and subset A of [a,b], let  $\chi_s(A)$  be the set function defined by

$\chi_s(A) = \langle$	1	if	[a,s]	$\subseteq A$
	0	oth	erwise.	

We see that  $\chi_s$  is a monotone set function and belongs to the  $\Lambda BV[a,b]$ . For any *s* and *r* with a < s < r < b, let  $\Omega$  be the chain  $\emptyset \subseteq [a,s] \subseteq I$ . Then

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$$\|\chi_{r} - \chi_{s}\| \geq \|\chi_{r} - \chi_{s}\|_{\Omega}$$

$$\geq \frac{|(\chi_{r} - \chi_{s})([a, s]) - (\chi_{r} - \chi_{s})(\emptyset)|}{\lambda_{1}}$$

$$\geq 1$$

This completes the proof.

# 4. $\Lambda BV$ AS A DUAL SPACE

In [3], Maccheroni and Ruckle showed that BV is a dual Banach space. Indeed, they showed that BV is isometrically isomorphic to the norm dual of space of all simple games. Here, we establish this result for  $\Lambda BV$ .

We define the game  $e_A : C \to R$  by

$$e_{A}(B) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{otherwise} \end{cases}$$

Let X be the space of all simple games. For all  $A \in C - \{\phi\}$  and  $e_{\phi} = 0$  being  $x = \sum_{A \in C} x(A)e_A$  for all  $x \in X$ , we have  $X = \langle e_A : A \in C \rangle$ . For each chain  $\Omega = \{S_i\}_{i=0}^n$  in C, define a semi norm on X by

$$||x||_{\Omega} = \max_{0 \le k \le n} \left| \sum_{i=k}^{n} x(s_i) \right|.$$
(1)

For all  $x \in X$ . Let  $X_{\Omega} = \langle e_A : A \in \Omega \rangle$ . If  $x \in X_{\Omega}$ , we say that X depends on the chain  $\Omega$ . For all  $x \in X$ , set

$$\|x\| = \inf \sum_{e=1}^{L} \|x_e\| \Omega_e$$

where the inf is taken over all finite decompositions  $x = \sum_{e=1}^{L} x_e$  in which  $x_e$  depends on the chain  $\Omega_e$  and  $\|.\|_{\Omega_e}$  is defined as in (1) for all e = 1, 2, ..., L.

Lemma 4 of [3] showed that this equation defines a norm on X.

**Lemma 4.1.** The function  $\| . \| : X \to R$  is a norm on X.

Given a linear continuous functional  $f: X \to R$ , define the game  $G_f$  as follows

$$G_f(A) = f(e_A)$$

For all  $A \in C$ .

**Theorem 4.2.** Let  $X^*$  be the norm dual of  $(X, \|.\|)$ . The operator

$$G: X^* \to \Lambda BV$$

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 $f \mapsto G_f$ 

is an isometric isomorphism from  $X^*$  onto ABV.

**Proof:** We first show that if  $\Omega = \{S_i\}_{i=0}^n$  is a chain in C, then

$$\sum_{k=1}^{n} \frac{\left|G_{f}(S_{k}) - G_{f}(S_{k-1})\right|}{\lambda_{k}} \leq ||f||,$$

which implies that  $G_f \in \Lambda BV$  and  $\parallel G_f \parallel \,\leq \, \parallel f \parallel$ .

Define  $x \in X_{\Omega}$  by

$$x(S_{n}) = Sgn(f(eS_{n}) - f(eS_{n-1})),$$
  

$$x(S_{n}) + x(S_{n-1}) = Sgn(f(eS_{n-1}) - f(eS_{n-2})),$$
  

$$\vdots$$
  

$$x(S_{n}) + x(S_{n-1}) + \dots + x(S_{1}) = Sgn(f(eS_{1}) - f(eS_{0})),$$
  

$$x(S_{0}) = 0.$$

Obviously  $||x||_{\Omega} \le 1$ , so that ||x|| < 1. Similar to proof of theorem 5 of [3], we have,

$$|| f || \ge f(x) = \sum_{j=1}^{n} |G_{f}(S_{j}) - G_{f}(S_{j-1})|$$
$$\ge \sum_{j=1}^{n} \frac{|G_{f}(S_{j}) - G_{f}(S_{j-1})|}{\lambda_{j}}$$

which implies that  $|| f || \ge || G_f ||$ . Then G is well defined and obviously linear and injective. Given  $u \in \Lambda BV$ , we can define  $f_u$  on X by

$$f_u(x) = \sum_{A_j \in C} \frac{u(A_j)}{\lambda_j} x(A_j),$$

for all  $x \in X$ . It is trivial that  $f_u$  is linear. If x depends on  $\Omega = \{S_j\}_{j=0}^n$ , then

$$f_{u}(x) = \sum_{j=0}^{n} \frac{u(S_{j})}{\lambda_{j}} x(S_{j})$$
$$= \frac{u(S_{0})}{\lambda_{0}} \sum_{k=0}^{n} x(S_{k}) + \sum_{j=1}^{n} \left[ \left( \frac{u(S_{j}) - u(S_{j-1})}{\lambda_{j}} \right) \sum_{k=j}^{n} x(S_{k}) \right]$$
$$= \sum_{j=1}^{n} \left[ \frac{u(S_{j}) - u(S_{j-1})}{\lambda_{j}} \sum_{j=1}^{n} x(S_{k}) \right]$$

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$$\leq \sum_{j=1}^{n} \left[ \frac{|u(S_j) - u(S_{j-1})|}{\lambda_j} \left| \sum_{k=j}^{n} x(S_k) \right| \right]$$

$$\leq ||x||_{\Omega}||u||_{.}$$

If  $x = \sum_{e=1}^{L} x_e$  with  $x_e \in X_{\Omega_e}$  for all e = 1, 2, ..., L, then

$$f_u(x) = \sum_{e=1}^{L} f_u(x_e)$$
$$\leq \sum_{e=1}^{L} ||u|| ||x_e||_{\Omega_e}$$
$$\leq ||u|| \sum_{e=1}^{L} ||x_e||_{\Omega_e}$$

and so

$$f_u(x) \le \inf \left\{ \| u \| \sum_{e=1}^{L} \| x_e \|_{\Omega_e} : x = \sum_{e=1}^{L} x_e, x_e \in X_{\Omega_e} \right\}$$
$$= \| u \| \| x \|.$$

We conclude that  $f_u \in X^*$ ,  $G(f_u) = u$  and G is onto. For all  $u \in \Lambda BV$ ,  $f_u = G_u^{-1}$  and  $||G_u^{-1}|| = ||f_u|| \le ||u||$ .

Therefore, for all  $f \in X^*$ ,  $||f|| = ||G_{(G_f)}^{-1}|| \le ||G_f||$  and G is an isometry. Let G be similar to the previous theorem. We show that,

**Theorem 4.3.** G is weak  $^* \Lambda$  – vague homeomorphism.

**Proof:** Let  $\{f^a\}$  be a net in  $X^*$ . By using the notations of the previous theorem, we have that  $f^a \xrightarrow{w^*} f$  iff  $f^a(x) \to f(x)$  for all  $x \in X$  iff  $f^a(e_A) \to f(e_A)$  for all  $A \in C$  iff  $G_{f^a}(A) \to G_f(A)$  for all  $A \in C$  iff  $G_{f^a} \xrightarrow{C} G_f$ .

In Theorem 4.2, together with the Alaoghlu theorem, we have the compactness of the unit ball U(BV) in the  $\Lambda$ -vague topology.

**Theorem 4.4.** The unit ball U(BV) is compact with respect to the  $\Lambda$  – *vague* topology.

## **5. PROJECTIONS FROM** ABV **ONTO** FA

Given I and C as in §1, let  $\Theta$  denote the set of one-to-one functions  $\Theta$  from I into I such that  $\pi(S) \in C$  if and only if  $S \in C$ . Then  $\Theta$  forms a group under composition. For each  $\pi$  in  $\Theta$  the function  $T_{\pi}$  defined by  $T_{\pi}u = u \circ \pi$  is a linear operator from  $\Lambda BV$  into  $\Lambda BV$  with  $||T_{\pi}|| = 1$ . A function u in  $\Lambda BV$  is called finitely additive if

$$u(A \cup B) = u(A) + u(B)$$

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whenever A and B are in C an  $A \cap B = \phi$ . The set FA of finitely additive functions in  $\Lambda BV$  forms a closed subspace of  $\Lambda BV$ . A function u in  $\Lambda BV$  is called increasing if  $u(A) \le u(B)$  whenever  $A \subset B$ . Each u in  $\Lambda BV$  has the form  $u = u^+ - u^-$  when  $u^+$  and  $u^-$  are increasing and  $||u|| = u^+(I) - u^-(I)$ . A linear mapping T in  $L(\Lambda BV)$  is positive if Tu is increasing whenever u is increasing.

**Definition 5.1.** Let  $\Phi$  be a subgroup of  $\Theta$ . A  $\Phi$ -value is a projection P from  $\Lambda BV$  onto FA which fulfills the following conditions:

$$\|Pu\| \le \|u\|, \qquad u \in \Lambda BV.$$
<sup>(2)</sup>

$$Pu(I) = u(I), \quad u \text{ in } \Lambda BV.$$
 (3)

$$PT_{\pi} = T_{\pi}P \quad for \quad all \quad \pi \quad in \quad \Phi.$$
(4)

**Definition 5.2.** For each finite partition D of I into members of C,  $\Gamma_{\Lambda} - set[\Gamma_{\Lambda}(D)]$  is the set of all T in  $L(\Lambda BV)$  for which

$$Tu(I) = u(I)$$
 for  $u$  in  $ABV$ ; (5)

$$||Tu|| \le ||u||, \qquad u \in \Lambda BV; \tag{6}$$

*Tu is additive on the algebra of sets determined by D;* (7)

$$Tu(B) = u(B) \quad for \quad u \quad in \quad FA, \quad B \quad in \quad D.$$
(8)

**Lemma 5.3.** No set  $\Gamma_{\Lambda}(D)$  is empty.

**Proof:** Suppose  $D = \{D_1, D_2, ..., D_k\}$  (any order). Let  $E_0 = \phi$ ,  $E_1 = D_1, ..., E_n = D_1 \cup D_2 \cup ... \cup D_n$ , ...,  $E_k = I$ . For each  $D_i$  in C let  $d_{D_i}$  be the function

$$d_{D_j}(A) = \begin{cases} \lambda_i & \text{if } D_j \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

Define  $Q_D$  from  $\Lambda BV$  into  $\Lambda BV$  by

$$Q_{D^{u}} = \sum_{j=1}^{k} \frac{(u(E_{j}) - u(E_{j-1}))d_{D_{j}}}{\lambda_{j}}$$

It is clear that  $Q_D$  is linear and satisfied (5) since the sum for  $Q_{D^u}(I)$  collapses to u(I). Since each  $d_{D_j}$  is increasing, and each coefficient is positive when V is increasing it follows that  $Q_D$  is positive. If  $u = u^+ - u^-$  when  $u^+$  and  $u^-$  are increasing and  $||u|| = u^+(I) + u^-(I)$  we have

$$||Q_{D^{u}}|| \leq ||Q_{D}u^{+}|| + ||Q_{D}u^{-}||$$
  
=  $Q_{D}u^{+}(I) + Q_{D}u^{-}(I)$   
=  $u^{+}(I) + u^{-}(I)$   
=  $||u||.$ 

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Thus (6) is valid. We omit the straightforward arguments which show  $Q_D$  satisfies (6) and (7). Now with a similar proposition 2.2 and theorem 2.3 of [5], one can prove that

**Theorem 5.4.** There exists a projection Q from ABV onto FA satisfying (2) and (3).

**Theorem 5.5.** If  $\Phi$  is a locally finite subgroup there is a  $\Phi$ -value P from  $\Lambda BV$  onto FA.

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