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K-NACCI SEQUENCES IN MILLER'S GENERALIZATION OF POLYHEDRAL GROUPS*

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Abstract – A *k*-nacci sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \cdots, x_n, \cdots$ for which, given an initial (seed) set $x_0, x_1, x_2, \cdots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases}$$

In this paper, we examine the periods of the *k*-nacci sequences in Miller's generalization of the polyhedral groups $\langle 2,2|2;q\rangle, \langle n,2|2;q\rangle, \langle 2,n|2;q\rangle, \langle 2,2|n;q\rangle$, for any $n > 2$.

Keywords – K-nacci sequence, period, dihedral group, polyhedral group

1. INTRODUCTION

The study of Fibonacci sequences in groups began with the earlier work of Wall [1] where he considered Fibonacci sequences of the cyclic groups C_n. Wilcox extended the problem to abelian groups [2]. In [3] the Fibonacci length of a 2-generator group is defined. The concept of Fibonacci length for more than two generators has been considered, [4] and [5]. Prolific co-operation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups [3]. The theory has been generalized in [6], [7] to the ordinary 3-step Fibonacci sequences in finite nilpotent groups. Then, it is shown in [8] that the period of 2-step general Fibonacci sequence is equal to the length of the fundamental period of the 2-step general recurrence constructed by two generating elements of the group of exponent p and nilpotency class 2. Karaduman and Yavuz showed that the periods of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class *5* and a prime exponent are $p.k(p)$, for $2 < p \le 2927$, where p is prime and $k(p)$ is the periods of ordinary 2-step Fibonacci sequences [9]. The 2-step general Fibonacci sequences in finite nilpotent groups of nilpotency class 4 and exponent p and the 2-step Fibonacci sequences in finite nilpotent groups of nilpotency class n and exponent p are discussed in [10] and [11], respectively. In [12] the relationship between a number of recurrence sums involved in the j th term of the last component of the Fibonacci sequences finite nilpotent groups of nilpotency class n and exponent p and the coefficients of the binomial formula has been investigated. Knox proved that periods of the *k*-nacci (*k*-step Fibonacci) sequences in the dihedral group were equal to $2k+2$ [13]. Other work on Fibonacci length is discussed in [14] and [15]. Recently, the works have been done on the *k*-nacci sequences [16-18].

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This paper is related to the periods of the *k*-nacci sequences in Miller's generalization of the polyhedral groups $\langle 2,2|2;q\rangle, \langle n,2|2;q\rangle, \langle 2,n|2;q\rangle, \langle 2,2|n;q\rangle$, for any n.

Definition 1.1. A *k*-nacci sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \cdots, x_n, \cdots$ for which, given an initial (seed) set $x_0, x_1, x_2, \cdots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases}$$

We also require that the initial elements of the sequence, $x_0, x_1, x_2, \cdots, x_{j-1}$, generate the group, thus forcing the *k*-nacci sequence to reflect the structure of the group. It is important to note that the Fibonacci length of a group depends on the chosen generating *n*-tuple. The *k*-nacci sequence of a group generated by $x_0, x_1, x_2, \cdots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, \cdots, x_{j-1})$ and its period is denoted by $P_k(G; x_0, x_1, \cdots, x_{j-1})$.

Definition 1.2. For a finitely generated group $G = < A >$ where $A = \{a_1, a_2, ..., a_n\}$, the sequence $x_i = a_{i+1}$, $0 \le i \le n-1$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \ge 0$, is called the Fibonacci orbit of G with respect to the generating set A, denoted $F_A(G)$.

Notice that the orbit of a *k*-generated group is a *k*- nacci sequence.

2-step Fibonacci sequence in the integers modulo m can be written as $F_2(Z_m; 0,1)$. A *2*-step Fibonacci sequence of a group of elements is called a Fibonacci sequence of a finite group.

A finite group G is *k*-nacci sequenceable if there exists a *k*-nacci sequence of G such that every element of the group appears in the sequence.

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period *k* if the first *k* elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \cdots$ is simply periodic with period 6.

Remark 1.1. The polyhedral group (l, m, n), for $l, m, n > 1$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz = 1 \rangle$$

or

$$\langle x, y, z : x^l = y^m = (xy)^n = 1 \rangle.$$

The polyhedral group (l, m, n) is finite if, and only if, the number $k = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn$ is positive and the order of (l, m, n) being $2lmn/k$.

These groups are also called *triangle* groups and are denoted by $T(l, m, n)$.

Remark 1.2. Miller's generalization of the polyhedral group $\langle l, m|n\rangle$, for $l, m, n > 1$ is defined by the presentation

$$\langle x, y : x^l = y^m, (xy)^n = 1 \rangle.$$

Its order is that of (l,m,n) multiplied by the period of central element

$$S = x^l = y^m.$$

If this period is finite, any divisor q yields a factor group

$$\langle l,m|n;q\rangle \cong \langle m,l|n;q\rangle$$

defined by

$$\langle x,y : x^l = y^m = S, (xy)^n = S^q = 1\rangle.$$

For more information on these groups, see [19].

2. MAIN RESULTS AND PROOFS

Theorem 2.1. Let G be the group defined by the presentation $G = \langle x,y : x^2 = y^2 = S, (xy)^2 = S^q = 1\rangle$. We get

$$P_k(G,x,y) = \begin{cases} 4k+4, & q=4, \\ 2k+2, & q=2, \\ k+1, & q=1. \end{cases}$$

Proof: We first note that $|x| = 2q, |y| = 2q, |xy| = 2, xy = y^{2q-1}x^{2q-1}, yx = x^{2q-1}y^{2q-1}$.

If $k = 2$, the sequence will be as follows:

$$x, y, xy, yxy, y^4y, y^7x, y^{12}x, y^{20}y, y^{32}xy, y^{52}yxy, y^{88}y, y^{143}x, y^{232}x, y^{376}y, y^{608}xy, \cdots.$$

If $q=4$, $P_k(G;x,y)=12$ because of $y^8 = x^8 = 1$.
If $q=2$, $P_k(G;x,y)=6$ because of $y^4 = x^4 = 1$.
If $q=1$, $P_k(G;x,y)=3$ because of $y^2 = x^2 = 1$.
If $k = 3$, the sequence will be as follows:

$$x, y, xy, 1, yxy, y^4y, y^7x, y^{12}, y^{24}x, y^{44}y, y^{80}xy, y^{148}, y^{272}yxy, y^{5040}y,$$
$$y^{927}x, y^{1704}, y^{3136}x, y^{5768}y, y^{10608}xy, \cdots.$$

If $q=4$, $P_k(G;x,y)=16$ because of $y^8 = x^8 = 1$.
If $q=2$, $P_k(G;x,y)=8$ because of $y^4 = x^4 = 1$.
If $q=1$, $P_k(G;x,y)=4$ because of $y^2 = x^2 = 1$.
Let $k \geq 4$.
If $q=4$, the first k elements of the sequence are $x_0 = x, x_1 = y, x_2 = xy, x_3 = (xy)^2 = 1, 1, \cdots, 1$ where $x_j = 1$ for $4 \leq j \leq k-1$. Thus, we have the sequence

$$x_k = \prod_{i=0}^{k-1} x_i = 1, x_{k+1} = \prod_{i=1}^{k} x_i = yxy, x_{k+2} = \prod_{i=2}^{k+1} x_i = y^5,$$

$$x_{k+3} = \prod_{i=3}^{k+2} x_i = x^3y, x_{k+4} = \prod_{i=4}^{k+3} x_i = x^4 = y^4, \cdots, x_{2k+2} = \prod_{i=k+2}^{2k+1} x_i = x,$$

$$x_{2k+3}=\prod_{i=k+3}^{2k+2}x_i=y^5,\ x_{2k+4}=\prod_{i=k+4}^{2k+3}x_i=xy,\ x_{2k+5}=\prod_{i=k+5}^{2k+4}x_i=x^4=y^4,\cdots,$$

$$x_{3k+6}=\prod_{i=2k+6}^{3k+5}x_i=1,1,\cdots,1\ (\text{where } x_j=1 \text{ for } 3k+6\le x_j\le 4k+2),\ x_{4k+3}=\prod_{i=3k+3}^{4k+2}x_i=1,$$

$$x_{4k+4}=\prod_{i=3k+4}^{4k+3}x_i=x,\ x_{4k+5}=\prod_{i=3k+5}^{4k+4}x_i=y,\ x_{4k+6}=\prod_{i=3k+6}^{4k+5}x_i=xy,$$

$$x_{4k+7}=\prod_{i=3k+7}^{4k+6}x_i=1,1,\cdots,1\ (\text{where } x_j=1 \text{ for } 4k+7\le x_j\le 5k+4),\cdots.$$

Since the elements succeeding $x_{4k+4}, x_{4k+5}, x_{4k+6}$ depend on x, y and xy for their values, the cycle begins again with the $4k+4^{nd}$ element; that is, $x_0=x_{4k+4}, x_1=x_{4k+5},\cdots$. Thus, $P_k(G;x,y)=4k+4$.

If $q=2$, then the first k elements of the sequence are $x_0=x, x_1=y, x_2=xy, x_3=(xy)^2=1,1,\cdots,1$ where $x_j=1$ for $4\le j\le k-1$. Thus, we have the sequence

$$x_k=\prod_{i=0}^{k-1}x_i=1,\ x_{k+1}=\prod_{i=1}^{k}x_i=yxy,\ x_{k+2}=\prod_{i=2}^{k+1}x_i=y^5, x_{k+3}=\prod_{i=3}^{k+2}x_i=x^3y,$$

$$x_{k+4}=\prod_{i=4}^{k+3}x_i=x^4=y^4=1,1,\cdots 1\ (\text{where } x_j=1 \text{ for } k+4\le j\le 2k),\ x_{2k+1}=\prod_{i=k+1}^{2k}x_i=1,$$

$$x_{2k+2}=\prod_{i=k+2}^{2k+1}x_i=x,\ x_{2k+3}=\prod_{i=k+3}^{2k+2}x_i=y^5=y,\ x_{2k+4}=\prod_{i=k+4}^{2k+3}x_i=xy,$$

$$x_{2k+5}=\prod_{i=k+5}^{2k+4}x_i=x^4=y^4=1,1,\cdots,1\ (\text{where } x_j=1 \text{ for } 2k+5\le j\le 3k+2),\cdots.$$

Since the elements succeeding $x_{2k+2}, x_{2k+3}, x_{2k+4}$ depend on x, y and xy for their values, the cycle begins again with the $2k+2^{nd}$ element; that is $x_0=x_{2k+2}, x_1=x_{2k+3},\cdots$. Thus, $P_2(G;x,y)=2k+2$.

If $q=1$, the first k elements of the sequence are $x_0=x, x_1=y, x_2=xy, x_3=(xy)^2=1,1,\cdots,1$ where $x_j=1$ for $4\le j\le k-1$. Thus, we have the sequence

$$x_k=\prod_{i=0}^{k-1}x_i=1,\ x_{k+1}=\prod_{i=1}^{k}x_i=yxy=x,$$

$$x_{k+2}=\prod_{i=2}^{k+1}x_i=y^5=y,\ x_{k+3}=\prod_{i=3}^{k+2}x_i=x^3y=xy,$$

$$x_{k+4}=\prod_{i=4}^{k+3}x_i=x^4=y^4=1,1,\cdots,1\ (\text{where } x_j=1 \text{ for } k+4\le j\le 2k+1),\cdots.$$

Since the elements succeeding $x_{k+1}, x_{k+2}, x_{k+3}$ depend on x, y and xy for their values, the cycle begins again with the $k+1^{nd}$ element; that is, $x_0=x_{k+1}, x_1=x_{k+2},\cdots$. Thus, $P_2(G;x,y)=k+1$. Also, see [18] for a different proof when $q=1$ since $\langle 2,2|2;1\rangle\cong(2,2,2)$.

Theorem 2.2. Let G be the group defined by the presentation $G=\langle x,y : x^2=y^n=S,(xy)^2=S^q=1\rangle$. Then the following are true.

i. If $q=1$, $P_k(G;x,y)=(2k+2)$.
ii. If $q=2^u$, $u\in N$, $P_k(G;x,y)=(2k+2)2^{u-1}$.
iii. If $p>2$ is a prime number and $q=2p$, then $P_k(G;x,y)$ are the same for both q and p.
iv. If $q=p_1^{u_1}p_2^{u_2}\cdots p_j^{u_j}$ and $p_i>2$ $(1\le i\le j)$ is the biggest of $p_1, p_2, \cdots, p_j$ prime numbers, then either $P_k(G;x,y)$ are the same for both q and p_i or $P_{k,p_i}(G;x,y)\,|\,P_{k,q}(G;x,y)$. Where $P_{k,q}(G;x,y)$ denote period of G for q and $P_{k,p_i}(G;x,y)\,|\,P_{k,q}(G;x,y)$ means that $P_{k,p_i}(G;x,y)$ divides $P_{k,q}(G;x,y)$

Proof: We first note that $|x|=2q,\ |y|=qn,\ yxy=y^{(q-1)n}x,\ xyx=y^{-1}$.

If $k=2$, the sequence will be as follows:

$$\begin{aligned}&x,\ y,\ xy,\ y^{(q-1)n}x,\ y^{(q-1)n}xyx,\ y^{(2q-1)n}yx,\ y^{(3q-2)n}x,\ y^{(5q-2)n}y,\ y^{(8q-4)n}xy,\\&y^{(14q-7)n}x,\ y^{(22q-11)n}xyx,\ y^{(36q-17)n}yx,\ y^{(58q-28)n}x,\ y^{(94q-44)n}y,\ y^{(52q-72)n}xy,\ y^{(147q-117)n}x,\\&y^{(199q-189)n}xyx,\ y^{(346q-305)n}yx,\ y^{(545q-494)n}x,\ y^{(891q-798)n}y,\ y^{(1436q-1292)n}xy,\ y^{(2328q-2091)n}x,\\&y^{(3764q-3383)n}xyx,\ y^{(6092q-5473)n}yx,\ y^{(9856q-8856)n}x,\ y^{(15948q-14328)n}y,\ \cdots.\end{aligned}\tag{1}$$

i. If $q=1$, $P_2(G;x,y)=6$ because of $\langle 2,n|2;1\rangle\cong(2,n,2)\cong D_n$.
ii. If $q=2^u$, $u\in N$, the sequence reduces to

$$\begin{aligned}&x_0=x,\ x_1=y,\ x_2=xy,\ x_3=y^{-n}x,\ x_4=y^{-n}xyx,\ x_5=y^{-n}yx,\ x_6=y^{-2n}x,\ x_7=y^{-2n}y,\ \cdots,\\&x_{12}=y^{-2^2 7n}x,\ x_{13}=y^{-2^2 11n}y,\ \cdots,\ x_{24}=y^{-2^3 1107n}x,\ x_{25}=y^{-2^3 1791n}y,\ \cdots,\ x_{48}=y^{-2^4 a_1 n}x,\\&x_{49}=y^{-2^4 a_2 n}y,\ \cdots,\ x_{6.2^{u-1}}=y^{-2^u a_1 n}x,\ x_{6.2^{u-1}+1}=y^{-2^u a_2 n}y,\ \cdots.\end{aligned}$$

Where $a_1, a_2\in N$.

Since the elements succeeding $x_{6.2^{u-1}}$, $x_{6.2^{u-1}+1}$ depend on x, y for their values, the cycle begins again with the 6.2^{u-1} element; that is, $x_0=x_{6.2^{u-1}}$, $x_1=x_{6.2^{u-1}+1}, \cdots$. Thus, $P_2(G;x,y)=6.2^{u-1}$.
iii. If $p>2$ is a prime number and $q=2p$, then we have the sequence

$$\begin{aligned}&x_0=x,\ x_1=y,\ x_2=xy,\ x_3=y^{-n}x,\ x_4=y^{-n}xyx,\ x_5=y^{-n}yx,\ x_6=y^{-2n}x,\\&x_7=y^{-2n}y,\ x_8=y^{-4n}xy,\ x_9=y^{-7n}x,\ x_{10}=y^{-11n}xyx,\ x_{11}=y^{-17n}yx,\\&x_{12}=y^{-2.14n}x,\ x_{13}=y^{-2.22n}y, x_{14}=y^{-72n}xy,\ x_{15}=y^{-117n}x,\\&x_{16}=y^{-189n}xyx,\ x_{17}=y^{-305n}yx,\ x_{18}=y^{-2.247n}x,\ x_{19}=y^{-2.399n}y,\ \cdots,\\&x_{24}=y^{-2.4428n}x,\ x_{25}=y^{-2.7164n}y,\ \cdots, x_{6.i}=y^{-2.b_1 n}x,\ x_{6i+1}=y^{-2b_2 n}y,\ \cdots.\end{aligned}\tag{2}$$

Where $b_1, b_2\in N$.

If $p|2.247$ and $p|2.399$ or $p|2.4428$ and $p|2.7164$ or … $p|2.b_1$ and $p|2.b_2$, then $2p|2.247$ and $2p|2.399$ or $2p|2.4428$ and $2p|2.7164$ or … $2p|2.b_1$ and $2p|2.b_2$. So, it can be seen that from (2), $P_2(G;x,y)$ are the same, for both q and p.
iv. By computing b_1, b_2 in (2), it can be seen that either $P_2(G;x,y)$ are the same, for both q and p_i or $P_{2,p_i}(G;x,y)\,|\,P_{2,q}(G;x,y)$.

Let $k\ge 3$.
i. If $q=1$, then $P_k(G;x,y)=2k+2$ because of $\langle 2,n|2;1\rangle\cong(2,n,2)\cong D_n$.
ii. If $q=2^u$, $u\in N$, the first k elements of the sequence are $x_0=x,\ x_1=y,\ x_2=xy,\ x_3=(xy)^2=1, 1, \cdots, 1$ where $x_j=1$ for $4\le j\le k-1$. Thus, we have the sequence

$$x_k = \prod_{i=0}^{k-1} x_i = 1,\ x_{k+1} = \prod_{i=1}^{k} x_i = y^{(q-1)n}x,\ x_{k+2} = \prod_{i=2}^{k+1} x_i = y^{(q-1)n}xyx,$$

$$x_{k+3} = \prod_{i=3}^{k+2} x_i = y^{(2q-1)n}yx,\ x_{k+4} = \prod_{i=4}^{k+3} x_i = y^{(4q-2)n},\ \cdots,\ x_{2k+2} = \prod_{i=k+2}^{2k+1} x_i = x,$$

$$x_{2k+3} = \prod_{i=k+3}^{2k+2} x_i = y^{2n}y,\ x_{2k+4} = \prod_{i=k+4}^{2k+3} x_i = y^{4n}xy,$$

$$x_{2k+5} = \prod_{i=k+5}^{2k+4} x_i = y^{2n},\ x_{2k+6} = 1, 1, \cdots, 1 \text{ (where } x_j = 1 \text{ for } 2k+6 \le j \le 3k+2), \cdots,$$

$$x_{2(2k+2)} = \prod_{i=3k+4}^{4k+3} x_i = x,\ x_{2(2k+2)+1} = \prod_{i=3k+5}^{4k+4} x_i = y^{4n}y,\ x_{2(2k+2)+2} = \prod_{i=3k+6}^{4k+5} x_i = y^{8n}xy,$$

$$x_{2(2k+2)+3} = \prod_{i=3k+7}^{4k+6} x_i = y^{4n},\ x_{2(2k+2)+4} = 1, 1, \cdots, 1 \text{ (where } x_j = 1 \text{ for } 2(2\mathrm{k}+2)+4 \le j \le 5k+4), \cdots,$$

$$x_{4(2k+2)} = \prod_{i=7k+8}^{8k+7} x_i = x,\ x_{4(2k+2)+1} = \prod_{i=7k+9}^{8k+8} x_i = y^{8n}y,\ x_{4(2k+2)+2} = \prod_{i=7k+10}^{8k+9} x_i = y^{16n}xy,$$

$$x_{4(2k+2)+3} = \prod_{i=7k+11}^{8k+10} x_i = y^{8n},\ x_{4(2k+2)+4} = 1, 1, \cdots, 1 \text{ (where } x_j = 1 \text{ for } 4(2k+2)+4 \le j \le 9k+8), \cdots,$$

$$x_{2^u(k+1)+2-k} = \prod_{\mathrm{i}=2^u(k+1)+2-k-k}^{2^u(k+1)+1-k} x_i = 1,\ x_{2^{u-1}(2k+2)-1} = 1, 1, \cdots 1$$

$$\text{(where } x_j = 1 \text{ for } 2^{u-1}(k+1)+3-k \le j \le 2^{u-1}(2k+2)\text{-1 and } u \in N \text{)},$$

$$x_{2^{u-1}(2k+2)} = \prod_{i=2^u k-k+2^u}^{2^u k+2^u-1} x_i = x,\ x_{2^{u-1}(2k+2)+1} = \prod_{i=2^u k-k+2^u+1}^{2^u k+2^u} x_i = y^{2^u n}y = y,$$

$$x_{2^{u-1}(2k+2)+2} = \prod_{i=2^u k-k+2^u+2}^{2^u k+2^u+1} x_i = y^{2^{u+1} n}xy = xy,\ x_{2^{u-1}(2k+2)+3} = \prod_{i=2^u k-k+2^u+3}^{2^u k+2^u+2} x_i = y^{2^u n} = 1,$$

$$x_{2^{u-1}(2k+2)+4} = 1, 1, \cdots, 1 \text{ (where } x_j = 1 \text{ for } 2^{u-1}(2k+2)+4 \le j \le k+2^u(k+1) \text{ and } u \in N \text{)}, \cdots.$$

Since the elements succeeding $x_{2^{u-1}(2k+2)}, x_{2^{u-1}(2k+2)+1}, x_{2^{u-1}(2k+2)+2}$ depend on x, y, xy for their values, the cycle begins again with the $2^{u-1}(2k+2)$ element; that is $x_0 = x_{2^{u-1}(2k+2)}$, $x_1 = x_{2^{u-1}(2k+2)+1}$, $x_2 = x_{2^{u-1}(2k+2)+2}, \cdots$. Thus, $P_k(G; x, y) = (2k+2).2^{u-1}$.

iii. If $p > 2$ is prime number and $q = 2p$, the first k elements of the sequence are $x_0 = x, x_1 = y, x_2 = xy, x_3 = (xy)^2 = 1, 1, \cdots, 1$ where $x_j = 1$ for $4 \le j \le k-1$. Thus, we have the sequence

$$x_k=1,\ x_{k+1}=y^{(q-1)n}x,\ x_{k+2}=y^{(q-1)n}xyx,\ x_{k+3}=y^{(2q-1)n}yx,$$
$$x_{k+4}=y^{(4q-2)n},\cdots,\ x_{2k+2}=y^{-2c_1n}x,\ x_{2k+2+1}=y^{-2c_2n}y,\ x_{2k+2+2}=y^{-2c_3n}xy,$$
$$x_{2k+2+3}=y^{-2c_4n},x_{2k+2+4}=y^{-2c_5n},\cdots,x_{3k+2}=y^{-c_{k+1}n},\cdots, \tag{3}$$
$$x_{\beta(2k+2)}=y^{-2c_1n}x,\ x_{\beta(2k+2)+1}=y^{-2c_2n}y,\ x_{\beta(2k+2)+2}=y^{-2c_3n}xy,\ x_{\beta(2k+2)+3}=y^{-2c_4n},$$
$$x_{\beta(2k+2)+4}=y^{-2c_5n},\cdots,\ x_{\beta(2k+2)+k}=y^{-2c_{k+1}n}\ (\text{where } c_1,c_2,c_3,c_4,\cdots,c_{k+1},\beta\in N),\cdots.$$

If $p|2c_1,\ p|2c_2,\ p|2c_3,\ p|2c_4,\ p|2c_5,\cdots,\ p|2c_{k+1}$, then $2p|2c_1,\ 2p|2c_2,\ 2p|2c_3,\ 2p|2c_4,\ 2p|2c_5,\cdots,\ 2p|2c_{k+1}$. So, it can be seen that from (3) $P_k(G;x,y)$ are the same, for both q and p.

iv. By computating $c_1,c_2,c_3,c_4,\cdots,c_{k+1}$ in (3), it can be seen that either $P_k(G;x,y)$ are the same, for both q and p_i or $P_{k,p_i}(G;x,y)\,|\,P_{k,q}(G;x,y)$.

The *i,ii,iii* and *iv* axioms in the Theorem 2.2 are valid for both $\langle 2,n|2;q\rangle$ and $\langle n,2|2;q\rangle$ because of $\langle n,2|2;q\rangle\cong\langle 2,n|2;q\rangle$.

Theorem 2.3. Let G be the group defined by the presentation $G=\langle x,y:x^2=y^2=S,(xy)^n=S^q=1\rangle$. Then the following are true.

i. If $q=2$, then

i'. $P_2(G;x,y)=6$.

ii'. $P_{3,4}(G;x,y)=\begin{cases} n\left(\dfrac{k+1}{2}\right), & n\equiv 0 \bmod 4,\\ n(k+1), & n\equiv 2 \bmod 4,\\ 2n(k+1), & \text{otherwise.}\end{cases}$

iii'. Let $k\geq 5$.

1. If there is no $t\in[3,k-2]$ such that t is a odd factor of n, then

$$P_k(G;x,y)=\begin{cases} n\left(\dfrac{k+1}{2}\right), & n\equiv 0 \bmod 4,\\ n(k+1), & n\equiv 2 \bmod 4,\\ 2n(k+1), & \text{otherwise.}\end{cases}$$

2. Let α be the biggest odd factor of n in $[3,k-2]$, then two cases occur:

i''. If $\alpha.3^j\notin[3,k-2]$ for $j\in N$, then

$$P_k(G;x,y)=\begin{cases} \alpha\left(n\left(\dfrac{k+1}{2}\right)\right), & n\equiv 0 \bmod 4,\\ \alpha(n(k+1)), & n\equiv 2 \bmod 4,\\ \alpha(2n(k+1)), & \text{otherwise.}\end{cases}$$

ii''. If β is the biggest odd number which is in $[3,k-2]$ and $\beta=\alpha 3^j$ for $j\in N$, then

$$P_k(G;x,y)=\begin{cases}\beta\left(n\left(\frac{k+1}{2}\right)\right), & n\equiv 0 \bmod 4,\\ \beta\left(n\left(k+1\right)\right), & n\equiv 2 \bmod 4,\\ \beta\left(2n\left(k+1\right)\right), & \text{otherwise.}\end{cases}$$

ii. If $p>2$ is a prime number and $q=2p$, then $P_k(G;x,y)$ are the same, for both q and p.
iii. If $q=p_1^{u_1}p_2^{u_2}\cdots p_j^{u_j}$ and $p_i>2$ $(1\le i\le j)$ is the biggest of $p_1, p_2, \cdots, p_j$ prime numbers, then either $P_k(G;x,y)$ are the same, for both q and p_i or $P_{k,p_i}(G;x,y) \mid P_{k,q}(G;x,y)$.

Proof: The proof is similar to the proofs of Theorem 2.1. and Theorem 2.2.

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