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A NOVEL NAVIGATION METHOD FOR PURSUING A MOVING TARGET*

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Abstract – The most current pursuit algorithms for moving targets which are presented so far in the literature are Pure Pursuit and Pure Rendezvous navigations. Recently, one of the present authors has introduced a geometric model for the Pure Pursuit navigation algorithm. Here, in this paper, we study a new algorithm for the pursuit navigation problem which is a combination of both of the above algorithms. We study its geometric properties, as well as the trajectories as time optimal paths. Finally, we compare this algorithm with well-known algorithms in some real examples.

Keywords – Pure Pursuit, Pure Rendezvous, Composed Pursuit, Finsler metric

1. INTRODUCTION

One of the topics in navigation is the "act of guiding" a pursuer for pursuing a moving point. This act has applications in structured footballer robots, rescue robots and also in military equipment. The most current pursuit algorithms for moving targets presented thus far in the literature are Pure Pursuit [1] and Pure Rendezvous [2, 3] navigation. In this paper, an ephemeral review of these two algorithms is given. A new pursuit algorithm is also presented as a composite of Pure Pursuit and Pure Rendezvous navigation. This composite algorithm will be referred to as *Composed Pursuit* in the sequel. The Composed Pursuit is planned to benefit the advantages of these two algorithms. In section 6, we will see that the cost function of the Composed Pursuit algorithm is not only a function of time but also the accuracy of reaching the target. For example, in Pure Pursuit algorithm the cost function is just a function of accuracy. In Composed Pursuit navigation, the pursuer has a smooth trajectory whenever the target has a smooth path. After a brief review on Finsler structure and navigation problems, the kinematics of the Composed Pursuit algorithm is studied and it's proved that the related metric to the Composed Pursuit is a Finsler metric. The Finsler structure of this algorithm is calculated as well. Finally, we will compare, in three real examples, these three algorithms using the *maple* software. These examples can lead to a much deeper understanding of the Composed Pursuit model and verify our predictions for having better performance in pursuing a target in Composed Pursuit navigation. In this paper, all paths of the target are supposed to be smooth.

2. PRELIMINARIES AND TERMINOLOGY

2.1. *Kinematic analysis*

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Kinematic is a branch of Mechanics which describes the motion of objects without considering the masses or the forces that bring about this motion. In a kinematic analysis, position, velocity and acceleration are calculated without considering the causes of the motion. Here, we study the kinematic of trajectories produced when an object pursues another one. We shall refer to the target as **G** and to the pursuer as **R** and their velocity vectors as v_G and v_R, respectively. In this work, we consider the composed algorithm in two dimensional space $\mathbb{R}^2$. To begin, we set up a coordinate system called reference frame of coordinates. When considering the planar motion, we shall use Cartesian coordinates (x, y) with the origin **O**, and the angles will be positive if measured counterclockwise.

In Figure 1, the ray that starts at the pursuer **R** and is directed at the target **G** is called the *line of sight* and is denoted by RG. We can also consider the line of sight as a vector in the two dimensional vector space $\mathbb{R}^2$, in each case it is presented by $\overrightarrow{\mathrm{RG}}$. Suppose that θ_G, θ_R and η denote the angles between v_G and the horizontal axis, v_R and the horizontal axis and $\overrightarrow{\mathrm{RG}}$ and the horizontal axis, respectively. We denote the vector $\overrightarrow{\mathrm{RG}}$ by the vector valued function $\vec{r} = \vec{r}(t)$ and its length by the real valued function $r = r(t)$.

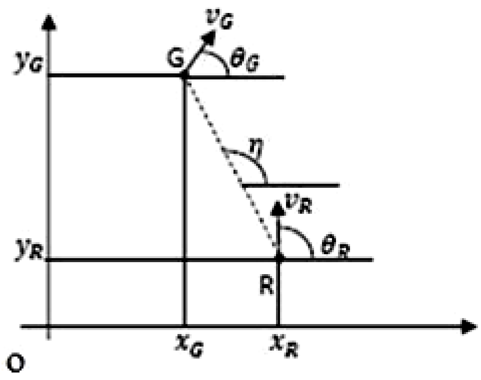

Fig. 1. The line of sight $\overrightarrow{\mathrm{RG}}$ between pursuer R and target G

2.2. *Finslerian structure*

Recently, Finsler metrics appear very often, both on the theory and the applications of differential geometry, as well as navigation problems. For more references see the bibliographies of [4-6]. A Finsler fundamental function on M is a function $F\colon TM \longrightarrow [0,\infty)$ with properties: F is C^∞ on TM_0, F is positively 1-homogeneous on the fiber bundle TM and Hessian F^2 with elements $g_{ij}(x,y) := 1/2[F^2(x,y)]_{y^i y^j}$ is positively defined on TM_0.

3. A SHORT REVIEW OF KNOWN NAVIGATIONS

In this section, we recall the definition of Pure Pursuit and Pure Rendezvous navigation. These two algorithms are described in [1] and [2, 3] in more detail. In Pure Pursuit navigation, v_R is always directed towards the target, while in Pure Rendezvous navigation, v_R is directed towards some points in front of the target which are achieved by prediction of the path of the target. This prediction forces the angle η to be constant during the process of the pursuit.

If Δt is sufficiently small, then v_R and v_G can be considered (or at least approximately) as constants. According to the first specific navigation law, in order for **R** to reach **G**, along with the line of sight RG, $\Delta r(t)$ should be equal to the difference of displacement vectors of $\overrightarrow{\mathrm{OR}}$ and $\overrightarrow{\mathrm{OG}}$, along with the line of sight. That is, $\Delta r = (|v_G|\cos(\theta_G - \eta) - |v_R|\cos(\theta_R - \eta))\Delta t$. Dividing both sides by Δt, as Δt approaches zero

$$\dot{r} = |v_G|\cos(\theta_G - \eta) - |v_R|\cos(\theta_R - \eta). \tag{1}$$

In addition, the variation $\dot{\eta}$ of the angle between the line of sight and the horizontal axis, is computed by displacement of **G** and **R** in the vertical direction with respect to the line of sight direction. That is,

$$\dot{\eta} = \left(|v_G| \sin(\theta_G - \eta) - |v_R| \sin(\theta_R - \eta)\right) r^{-1}. \tag{2}$$

Moreover, for sufficiently small Δt, $\vec{\boldsymbol{r}}(t_0 + \Delta t) = \vec{\boldsymbol{r}}(t_0) + v_G(t_0)\Delta t - v_R(t_0)\Delta t$ is true for all $t_0 \geq 0$. From which we obtain,

$$\dot{\vec{\boldsymbol{r}}}(t_0) = v_G(t_0) - v_R(t_0). \tag{3}$$

Where we have put $\dot{\vec{\boldsymbol{r}}}(t_0) = \frac{d\vec{r}}{dt}|_{t=t_0}$.

3.1. *Pure Pursuit navigation*

There is a fundamental law in Pure Pursuit navigation which claims that the line of sight RG and the velocity of pursuer v_R are always in the same direction. This means that $\theta_R = \eta$. Next, by inserting this equation in Eq. (1), we obtain

$$\dot{r} = |v_G|\cos(\theta_G - \eta) - |v_R|. \tag{4}$$

Therefore, by inequality $|v_R| > |v_G|$, which is always supposed to be true in pursuit problems, we can see in Eq. (4) that $\dot{r}$ is negative. Indeed in the Pure Pursuit navigation, the fact $\dot{r} < 0$ asserts that the pursuer will reach the target along the line of sight.

3.2. *Pure Rendezvous navigation*

The fundamental law in the Pure Rendezvous navigation claims that, η, the angle between the line of sight RG and the horizontal axis, is constant or equivalently $\dot{\eta} = 0$. Therefore, if we assume that $\dot{\eta}$ in Eq. (2) is zero, then we have

$$\theta_R = \eta + \sin^{-1}\left(\frac{|v_G|}{|v_R|}\sin(\theta_G - \eta)\right). \tag{5}$$

With respect to the assumption $|v_R| > |v_G|$ mentioned in the subsection 3.1 and in Eq. (5), we have

$$\sin(\theta_R - \eta) = \frac{|v_G|}{|v_R|}\sin(\theta_G - \eta) < |\sin(\theta_G - \eta)|,$$
$$\frac{-\pi}{2} < \theta_R - \eta < \frac{\pi}{2}.$$

With a simple calculation, these inequalities imply

$$\cos(\theta_G - \eta) < \cos(\theta_R - \eta). \tag{6}$$

By substituting Eq. (6) and $|v_R| > |v_G|$ in Eq. (1), we can see that $\dot{r}$ is negative for Pure Rendezvous navigation. This fact ensures that in the Pure Rendezvous navigation, the pursuer will reach the target.

4. KINEMATIC OF COMPOSED PURSUIT NAVIGATION

In this section a new algorithm for the pursuit navigation problem is introduced which is a combination of both Pure Pursuit and Pure Rendezvous algorithms. In this algorithm, both methods presented in the subsections 3.1 and 3.2 are considered to introduce an algorithm called here *Composed Pursuit* navigation. In the Composed Pursuit navigation, the velocity vector of the pursuer would not be directed to the target, but to some points in front of the target by the prediction of the target path.

In Composed Pursuit Navigation, the angle θ_R must be calculated in order to find the velocity vector v_R. For this purpose, some new variables will be needed as follows. Let us put

$$\alpha = \cos^{-1}(|\cos(\theta_G - \eta)|). \tag{7}$$

We consider our frame work space $\mathbb{R}^2$ as the (x,y,0) subspace of $\mathbb{R}^3$ and we let $\vec{k} = (0,0,1)$, then we define

$$a = \vec{k}.\left(\overrightarrow{RG} \times v_G\right). \tag{8}$$

Now, we define θ_R as follows,

$$\theta_R = \eta + sign(a)\left(\frac{\sin\left(\alpha/_2\right)}{\sin\left(\alpha/_2\right) + \cos\left(\alpha/_2\right)}\right)\alpha\,. \tag{9}$$

The equation (9) shows that the sign of the variable a is related to the sign of $\theta_R - \eta$ and the value of a has no influence on the value of $\theta_R - \eta$. It can be easily checked that θ_R is well-defined. In fact, $sin(\alpha/2) + cos(\alpha/2)$ are different by zero for $0 \le \alpha \le \pi/2$ and the discontinuity point of the function $sign(a)$ is $a = 0$. In this case the equations (7) and (8) imply $\alpha = 0$. Clearly $\left(\frac{\sin\left(\alpha/_2\right)}{\sin\left(\alpha/_2\right)+\cos\left(\alpha/_2\right)}\right)\alpha$ is a continuous function of α and is equal to zero if $\alpha = 0$. Therefore, $sign(a)\left(\frac{\sin\left(\alpha/_2\right)}{\sin\left(\alpha/_2\right)+\cos\left(\alpha/_2\right)}\right)\alpha$ will be continuous everywhere, even at the point $a = 0$.

Proposition 4.1. In the kinematic of the Composed Pursuit navigation, the pursuer converges to the target.

Proof: In all pursuit algorithms, we have the $|v_R| > |v_G|$ assumption. The equation (9) implies

$$\cos(\theta_R - \eta) = \cos\left(\left(\frac{\sin\left(\alpha/_2\right)}{\sin\left(\alpha/_2\right) + \cos\left(\alpha/_2\right)}\right)\alpha\right).$$

On the other hand, from the inequality $0 \le \alpha \le \frac{\pi}{2}$ we obtain

$$0 \le \left(\frac{\sin\left(\alpha/_2\right)}{\sin\left(\alpha/_2\right) + \cos\left(\alpha/_2\right)}\right)\alpha \le \alpha \le \frac{\pi}{2}.$$

Therefore

$$\cos(\theta_G - \eta) \le \cos(\alpha) \le \cos\left(\left(\frac{\sin\left(\alpha/_2\right)}{\sin\left(\alpha/_2\right) + \cos\left(\alpha/_2\right)}\right)\alpha\right),$$
$$\cos(\theta_G - \eta) \le \cos(\theta_R - \eta). \tag{10}$$

Thus, by replacing Eq. (10) and $|v_R| > |v_G|$ in Eq. (1) we will find that $\dot{r}$is negative and hence *r*(*t*) is decreasing. This shows that the pursuer gets closer to the target in the Composed Pursuit navigation.

To find the corresponding metric to the Composed Pursuit navigation, without loss of generality, we assume $|v_R| = 1 > |v_G|$. Now, using Okubo's technic cf. [6] and [8], we can find the related metric to the Composed Pursuit navigation. By means of Eq. (3) and the assumption $|v_R| = 1$, we obtain

$$1 = < v_G - \dot{\vec{r}}, v_G - \dot{\vec{r}} >,$$

$$\| \dot{\vec{r}} \|^2 - 2|v_G| \, \| \dot{\vec{r}} \| \cos(\beta) + |v_G|^2 - 1 = 0, \tag{11}$$

where, β is the angle between $\dot{\vec{r}}$ and v_G. Solving this equation for $\| \dot{\vec{r}} \|$ we get

$$\| \dot{\vec{r}} \| = |v_G| \cos(\beta) + \sqrt{|v_G|^2 cos^2(\beta) + 1 - |v_G|^2}. \tag{12}$$

If $\| \dot{\vec{r}} \| = 0$, then from Eq. (11) we have $|v_G| = 1$ and it is a contradiction to the assumption $1 > |v_G|$. So $\| \dot{\vec{r}} \| \neq 0$ and we can divide both sides of Eq. (12) by $|v_G| \cos(\beta) + \sqrt{|v_G|^2 cos^2(\beta) + 1 - |v_G|^2}$, and obtain

$$1 = \frac{\| \dot{\vec{r}} \|}{|v_G| \cos(\beta) + \sqrt{|v_G|^2 cos^2(\beta) + 1 - |v_G|^2}},$$

$$= \frac{\| \dot{\vec{r}} \|^2}{< \dot{\vec{r}}, v_G > + \sqrt{< \dot{\vec{r}}, v_G >^2 + \| \dot{\vec{r}} \|^2 - \| \dot{\vec{r}} \|^2 \; |v_G|^2}}. \tag{13}$$

Now, we are in a position to determine the Finsler fundamental function. Assuming $\dot{\vec{r}} = \vec{v}$ as the direction, we obtain

$$\mathrm{F}(\boldsymbol{x}, \vec{\boldsymbol{v}}) = \frac{<\vec{v},\vec{v}>}{<\vec{v},v_G> + \|\vec{v}\| \sqrt{|v_G|^2 \cos(\beta) + 1 - |v_G|^2}}. \tag{14}$$

This function can be written in the following form

$$\mathrm{F}(\boldsymbol{x}, \vec{\boldsymbol{v}}) = \frac{< \vec{\boldsymbol{v}}, \vec{\boldsymbol{v}} >}{\mathrm{W}(\vec{\boldsymbol{v}}) + f \, \| \vec{\boldsymbol{v}} \|},$$

where f is a smooth function and W is a 1-form.

One can check that the function F obtained in this way is C^∞ and homogeneous of degree one, verifying the convexity property. Hence it is a Finsler fundamental function, known in literature as Matsumoto metric, cf. [7]. This Matsumoto metric can be written in the following form

$$F = \frac{\alpha^2}{(r.\alpha + \beta)}.$$

where α is a Riemannian metric, β is a 1-form and r is a constant.
This completes the proof of the following theorem.

Theorem 4.1. The corresponding metric to the Composed Pursuit navigation is a Matsumoto metric.

5. A COMPARISON OF COMPOSED NAVIGATION WITH PURE PURSUIT AND RENDEZVOUS NAVIGATION

In this section, the trajectories of a pursuer in these three algorithms are computed in real examples by *Maple* software. We recall that there is a specific differential equation for each algorithm. The solutions of these differential equations are the trajectories of the pursuer. For Pure Pursuit algorithm, in the section 3.1, we saw that if the path of the pursuer is $(x(t), y(t))$ then the corresponding differential equation should be

$$\big(\dot{x}(t), \dot{y}(t)\big) = \frac{\overrightarrow{RG}}{\left|\overrightarrow{RG}\right|}. \tag{15}$$

The differential equations corresponding to the Pure Rendezvous and Composed Pursuit algorithms are given by the following equation

$$\big(\dot{x}(t), \dot{y}(t)\big) = M\frac{\overrightarrow{RG}}{|\overrightarrow{RG}|}, \tag{16}$$

where, M is a rotation matrix with the angle $\theta_R - \eta$, given in Eq. (5) and Eq. (9) for Pure Rendezvous and Composed Pursuit algorithms, respectively. In these two differential equations, the velocity of the pursuer is equal to one. We will compare these three algorithms in three different examples. We assume everywhere that the initial position of the pursuer at the time *t=0* is the origin, and its speed is equal to one unit, that is, $\dot{x}^2(t) + \dot{y}^2(t) = 1$ for all *t>0*.

5.1. *Example 1*

Consider a target moving along the line $x = x(t)$, $y = 10$ parallel to the positive direction of x-axis, with the velocity 2/3 unit. Assume that the initial position of the target at the time *t=0* is the point (-5, 10). By means of Eq. (15), the differential equation of Pure Pursuit algorithm is given by

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = \begin{bmatrix}2/3t - 5 - x(t)\\ 10 - y(t)\end{bmatrix}. \tag{17}$$

And the differential equation of Pure Rendezvous and Composed Pursuit algorithms are obtained from Eq. (16) as follows

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = \begin{bmatrix}\cos(\theta_R - \eta) & -\sin(\theta_R - \eta)\\ \sin(\theta_R - \eta) & \cos(\theta_R - \eta)\end{bmatrix}\begin{bmatrix}2/3t - 5 - x(t)\\ 10 - y(t)\end{bmatrix}, \tag{18}$$

where, in the case of Pure Rendezvous and Composed Pursuit algorithms the function $\theta_R - \eta$ is defined by Eq. (5) and Eq. (9), respectively.

In Fig. 2, the path of the target is denoted by a solid line. The dash-dot curve shows the path of the pursuer in the Pure Pursuit navigation, the wasted time in this algorithm is 14.7 units. The long-dash curve corresponds to the Pure Rendezvous navigation and wasted time in this algorithm is equal to 10.2 units. Finally, the dash curve shows the path of the Composed Pursuit navigation and the wasted time is 10.45 units. Using Maple program we can easily get the graph of these algorithms.

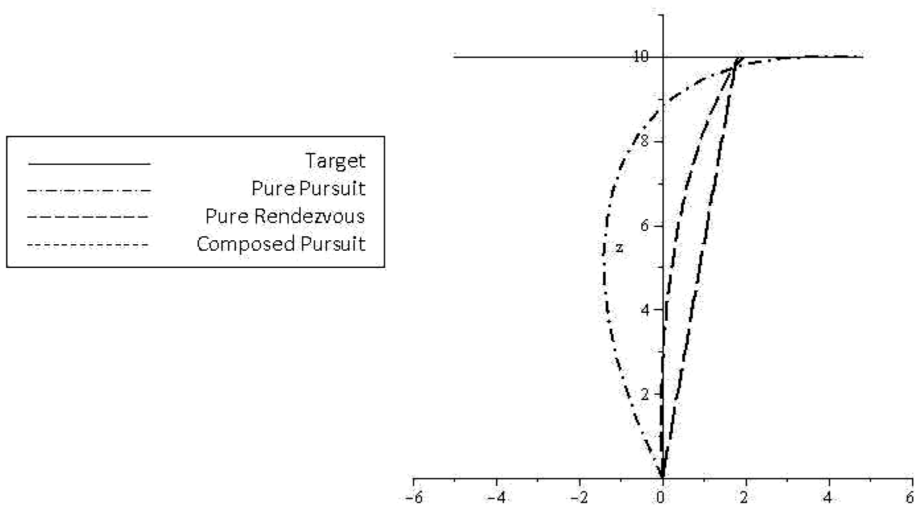

Fig. 2. Comparison of optimal paths of three algorithms when target is moving in a straight line

5.2. *Example 2*

In this example, the path of target is on a circle which is centered at origin and has the radius equal to 2 units. Also, we suppose that the target is in the point (2, 0) at t=0 with a velocity equal to 2/3 unit and moves toward positive trigonometric direction. Figure 3 shows the path of the pursuer in three algorithms. The formats of curves are similar to those of example 1.

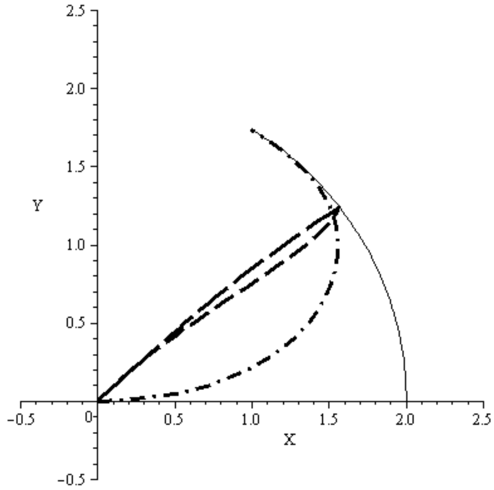

Fig. 3. Comparison of optimal paths of three algorithms when target is moving in a circle graph

5.3. *Example 3*

Finally, we assume the path of the target has a sinusoid graph with the parametric equation $(t/2 + 5, \sin(t)/2)$ in the Cartesian coordinate. We keep the line notations of the previous examples in Fig. 4, which shows the paths of those navigations of Example 3.

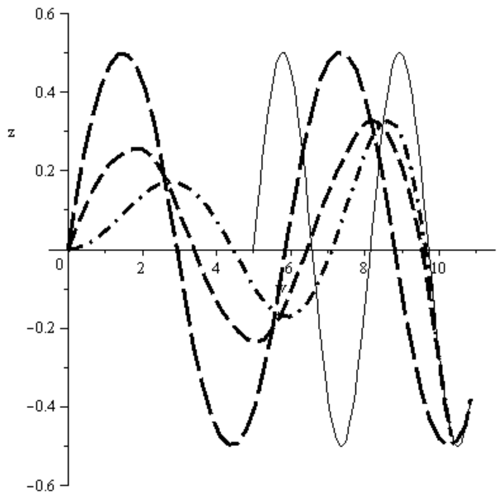

Fig. 4. Comparison of optimal paths of three algorithms when target is moving in a sinusoid graph

Table 1. Comparison of wasted traveling time between three navigations

	Pure Pursuit	**Pure Rendezvous**	**Composed Pursuit**
Wasted Time in example 1	14.7	10.2	10.45
Wasted Time in example 2	3.1	2	2
Wasted Time in example 3	10.7	11.7	10.7

Table 2. Overall rating of three navigations in different paths of target

Performance	Pure Pursuit	Pure Rendezvous	Composed Pursuit
predictable paths	Poor	Excellent	Excellent
unpredictable paths	Very good	good	Excellent

6. CONCLUSION

Comparing these three examples in Table 1, we find that Composed Pursuit navigation has some advantages when the target has a predictable path as in example 1. In this case, the Pure Rendezvous navigation also has a trajectory very close to that of Composed Pursuit. In other words, there is not much difference between Composed Pursuit and Pure Rendezvous navigation whenever the target has predictable paths. But for targets with unpredictable paths, in example 3, the Composed Pursuit algorithm has better performance. By comparing these results in Tables 1 and 2, obviously the Composed Pursuit navigation has the best performance among the other ones.

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