

λ –STRONGLY SUMMABLE AND λ –STATISTICALLY CONVERGENT FUNCTIONS*

F. NURAY

Afyon Kocatepe University, Mathematics Department Afyonkarahisar, Turkey
Email: fnuray@aku.edu.tr

Abstract – In this study, by using the notion of (V, λ) -summability, we introduce and study the concepts of λ -strongly summable and λ -statistically convergent functions.

Keywords – Statistical convergence, strongly summable function

1. INTRODUCTION

The idea of statistical convergence which is closely related to the concept of natural density or asymptotic density of a subset of the set of natural numbers \mathbb{N} , was first introduced by Fast [1]. The concept of statistical convergence plays an important role in the summability theory and functional analysis. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg [2]. In [3], Borwein introduced and studied strongly summable functions.

Strongly summable number sequences and statistically convergent number sequences were studied in [4] and [5], respectively. In [6], λ –statistically convergent number sequences was defined. In this paper, by taking real valued functions $x(t)$ measurable (in the Lebesgue sense) in the interval $(1, \infty)$ instead of sequences we will introduce λ –strongly summable and λ -statistically convergent functions and give some inclusion relations.

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. Λ denote the set of all such sequences. For a sequence $x = (x_k)$ the generalized de la Vallee Poussin mean is defined by

$$t_n(x) := \frac{1}{n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ) - summable to a number l if $t_n(x) \rightarrow l$ as $n \rightarrow \infty$. If $\lambda_n = n$, then (V, λ) –summability reduces to $(C, 1)$ summability.

2 λ -STRONGLY SUMMABLE FUNCTIONS

A real valued function $x(t)$, measurable(in the Lebesgue sense) in the interval $(1, \infty)$, is said to be strongly summable to l if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_1^n |x(t) - l| dt = 0.$$

*Received by the editor July 19, 2009 and in final revised form August 30, 2010

$[W]$ will denote the space of all strongly summable functions.

Also, W will denote the space of $x(t)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_1^n x(t) dt = l.$$

In this section we will introduce λ -strongly summable function.

Definition 2.1 Let $\lambda \in \Lambda$ and $x(t)$ be a real valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$, if

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \int_{n-\lambda_n+1}^n |x(t) - l| dt = 0.$$

Then we say that the function $x(t)$ is λ -strongly summable to l . In this case we write $[W, \lambda] - \lim x(t) = l$ and

$$[W, \lambda] := \{x(t): \exists l = l_x, [W, \lambda] - \lim x(t) = l\}.$$

If $\lambda_n = n$, then $[W, \lambda]$ is the same as $[W]$.

3 λ -STATISTICALLY CONVERGENT FUNCTIONS

$x(t)$ is a real valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$, if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{t \leq n: |x(t) - l| \geq \varepsilon\}| = 0$$

then we say that the function $x(t)$ is statistically convergent to l . Where the vertical bars indicate the Lebesgue measure of the enclosed set. In this case we write $S - \lim x(t) = l$ and

$$S := \{x(t): \exists l = l_x, S - \lim x(t) = l\}.$$

Definition 3.1. Let $\lambda \in \Lambda$ and $x(t)$ be a real valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$, if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{t \in I_n: |x(t) - l| \geq \varepsilon\}| = 0$$

then we say that the function $x(t)$ is λ -statistically convergent to l . In this case we write $S_\lambda - \lim x(t) = l$ and

$$(S, \lambda) := \{x(t): \exists l = l_x, S_\lambda - \lim x(t) = l\}.$$

If $\lambda_n = n$, then (S, λ) is the same as S , the set of statistically convergent functions.

Theorem 3.1. Let $\lambda \in \Lambda$ and $x(t)$ be a real valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$, then

- (i) $[W, \lambda] \subset (S, \lambda)$ and the inclusion is proper.
- (ii) If $x(t)$ is bounded and $S_\lambda - \lim x(t) = l$ then $[W, \lambda] - \lim x(t) = l$ and hence $W - \lim x(t) = l$ provided $x(t)$ is not eventually constant.
- (iii) If $x(t)$ is bounded then $(S, \lambda) = [W, \lambda]$.

Proof: (i) Let $\varepsilon > 0$ and $[W, \lambda] - \lim x(t) = l$. We write

$$\int_{t \in I_n} |x(t) - l| dt \geq \int_{\substack{t \in I_n \\ |x(t) - l| \geq \varepsilon}} |x(t) - l| dt \geq \varepsilon |\{t \in I_n : |x(t) - l| \geq \varepsilon\}|.$$

Therefore $[W, \lambda] - \lim x(t) = l$ implies $S_\lambda - \lim x(t) = l$.

Define a function $x(t)$ by

$$x(t) = \begin{cases} t, & n - (\lambda_n)^{\frac{1}{2}} + 1 \leq t \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Then $x(t)$ is not a bounded function and for every $\varepsilon (0 < \varepsilon \leq 1)$,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{t \in I_n : |x(t) - 0| \geq \varepsilon\}| = \lim_{n \rightarrow \infty} \frac{(\lambda_n)^{\frac{1}{2}}}{\lambda_n} = 0,$$

i.e., $S_\lambda - \lim x(t) = 0$. But

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \int_{n - \lambda_n + 1}^n |x(t) - 0| dt = \infty,$$

i.e., $x(t) \notin [W, \lambda]$. Therefore, the inclusion is proper.

(ii) Suppose that $S_\lambda - \lim x(t) = l$ and $x(t)$ be a bounded function, say $|x(t) - l| \leq M$ for all t . Given $\varepsilon > 0$, we have that

$$\begin{aligned} \frac{1}{\lambda_n} \int_{t \in I_n} |x(t) - l| dt &= \frac{1}{\lambda_n} \int_{\substack{t \in I_n \\ |x(t) - l| \geq \varepsilon}} |x(t) - l| dt + \frac{1}{\lambda_n} \int_{\substack{t \in I_n \\ |x(t) - l| < \varepsilon}} |x(t) - l| dt \\ &\leq \frac{M}{\lambda_n} |\{t \in I_n : |x(t) - l| \geq \varepsilon\}| + \varepsilon \end{aligned}$$

which implies that $[W, \lambda] - \lim x(t) = l$.

Also, we have, since $\frac{\lambda_n}{n} \leq 1$ for all n ,

$$\begin{aligned} \frac{1}{n} \int_1^n (x(t) - l) dt &= \frac{1}{n} \int_1^{n - \lambda_n} (x(t) - l) dt + \frac{1}{n} \int_{t \in I_n} (x(t) - l) dt \\ &\leq \frac{1}{n} \int_1^{n - \lambda_n} |x(t) - l| dt + \frac{1}{n} \int_{t \in I_n} |x(t) - l| dt \\ &\leq \frac{2}{\lambda_n} \int_{t \in I_n} |x(t) - l| dt. \end{aligned}$$

Hence $[W] - \lim x(t) = l$, since $[W, \lambda] - \lim x(t) = l$.

(iii) This follows from (i) and (ii).

It is easy to see that $(S, \lambda) \subset S$ for all λ , since $\frac{\lambda_n}{n} \leq 1$. Now we prove the following inclusion.

Theorem 3.2. $S \subset (S, \lambda)$ if and only if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} > 0. \quad (*)$$

Proof: For given $\varepsilon > 0$, we have

$$\{t \leq n: |x(t) - l| \geq \varepsilon\} \supset \{t \in I_n: |x(t) - l| \geq \varepsilon\}.$$

Therefore,

$$\begin{aligned} \frac{1}{n} |\{t \leq n: |x(t) - l| \geq \varepsilon\}| &\geq \frac{1}{n} |\{t \in I_n: |x(t) - l| \geq \varepsilon\}| \\ &\geq \frac{\lambda_n}{n} \frac{1}{\lambda_n} |\{t \in I_n: |x(t) - l| \geq \varepsilon\}|. \end{aligned}$$

Hence by using (*) and taking the limit as $n \rightarrow \infty$, we get $x(t) \rightarrow l(S)$ implies $x(t) \rightarrow l(S, \lambda)$. Conversely, suppose that $\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} = 0$. We can choose a subsequence (n_j) such that $\frac{\lambda_{n_j}}{n_j} < \frac{1}{j}$. Define a function $x(t)$ by $x(t) = 1$ if $t \in I_{n_j}$, $j = 1, 2, \dots$ and $x(t) = 0$ otherwise. Then $x(t) \in [W]$ and hence $x(t) \in S$. But $x(t) \notin [W, \lambda]$ and Theorem 3.1(ii) implies that $x(t) \notin (S, \lambda)$. Hence (*) is necessary.

Finally, we conclude this paper with a definition which generalizes Definition 2.1 of Section 2 and two theorems related to this definition.

Definition 3.2. Let $\lambda \in \Lambda$, p is a real number and $x(t)$ be a real valued function which is measurable (in the Lebesgue sense) in the interval $(1, \infty)$, if

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \int_{n-\lambda_n+1}^n |x(t) - l|^p dt = 0.$$

Then we say that the function $x(t)$ is λp -strongly summable to l . In this case we write $[W_p, \lambda] - \lim x(t) = l$ and

$$[W_p, \lambda] := \{x(t): \exists l = l_x, [W_p, \lambda] - \lim x(t) = l\}.$$

If $\lambda_n = n$, then $[W_p, \lambda]$ is the same as $[W_p]$, the set of strongly p -Cesaro summable functions.

Theorem 3.3. Let $1 \leq p < \infty$. If a function $x(t)$ is λp -strongly summable to l , then it is λ -statistically convergent to l .

The proof of the theorem is similar to that of Theorem 3.1.(i). So it was omitted.

Theorem 3.4. Let $1 \leq p < \infty$. If a bounded function $x(t)$ is λ -statistically convergent to l , then it is λp -strongly summable to l .

The proof of the theorem is similar to that of Theorem 3.1.(ii). So it was omitted.

REFERENCES

1. Fast, H. (1951). Sur la convergence statistique. *Colloq. Math.*, 2, 241-244.
2. Schoenberg, I. J. (1959). The integrability of certain functions and related summability methods. *Amer. Math. Monthly*, 66, 361-375.
3. Borwein, D. (1965). Linear functionals with strong Cesaro summability. *Journal London Math. Soc.*, 40, 628-634.
4. Maddox, I. J. (1967). *Space of strongly summable sequences*. Oxford(2), Quart. J. Math., 345-355.
5. Fridy, J. A. (1985). On statistical convergence. *Analysis*, 5, 301-313.
6. Mursaleen, M. (2000). λ -Statistical convergence. *Math. Slovaca*, 50(11), 111-115.