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Multiplication lattice modules

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Abstract

Let M be a lattice module over the multiplicative lattice L. An $L-$module M is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$. Our objective is to investigate properties of prime elements of multiplication lattice modules.

Keywords: Multiplicative lattice; lattice modules; maximal element; prime element

1. Introduction

A multiplicative lattice L is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has greatest element 1_L (least element 0_L) as a multiplicative identity (zero). For L a multiplicative lattice and $a \in L, L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \vee a$. Multiplicative lattices have been studied extensively by E. W. Johnson, C.Jayaram, the current authors, and others, see, for example, $[1-8]$.

An element $a \in L$ is said to be proper if $a < 1$. An element $p < 1$ in L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1$ in L is said to be maximal if $m < x \leq 1$ implies $x = 1$. It is easily seen that maximal elements are prime.

If a,b belong to L, $(a : b)$ is the join of all $c \in L$ such that $cb \leq a$. An element e of L is called meet principal if $a \wedge be = ((a:e) \wedge b)e$ for all $a, b \in L$. An element e of L is called join principal if $((ae \vee b):e) = a \vee (b:e)$ for all $a, b \in L$. $e \in L$ is said to be principal if e is both meet principal and join principal.

$e \in L$ is said to be week meet (join) principal if $a \wedge e = e(a:e)$ ($a \vee (0_L:e) = (ea:e)$) for all $a \in L$. An element a of a multiplicative lattice L is called compact if $a \leq \vee b_\alpha$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \ldots \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If each element of L is a join of principal (compact) elements of L, then L is called a PG–lattice (CG –lattice).

A multiplicative lattice L is called an $r-$lattice if it is modular, principally generated, compactly generated and has 1_L compact.

Let M be a complete lattice. Recall that M is a lattice module over the multiplicative lattice L, or simply an $L-$module in case there is a multiplication between elements of L and M, denoted by lB for $l \in L$ and $B \in M$, which satisfies the following properties:

(i) $(lb)B = l(bB)$;
(ii) $(\vee_\alpha l_\alpha)(\vee_\beta B_\beta) = \vee_{\alpha,\beta} l_\alpha B_\beta$;
(iii) $1_L B = B$;
(iv) $0_L B = 0_M$;
for all l, l_α, b in L and for all B, B_β in M.

Let M be an $L-$module. If $N \in M$ and $b \in L$, $(N : b)$ is the join of all $X \in M$ such that $bX \leq N$. An element $e \in L$ is said to be $M-principal$ if $A \wedge eB = e((A:e) \wedge B)$ and $((eA \vee B):e) = A \vee (B:e)$ for all $A, B \in M$. If each element of L is a join of $M-$principal elements of L, then L is called $M-$principally generated $[see,\ 9]$.

Let M be an $L-$module. If N, K belong to M, $(N:K)$ is the join of all $a \in L$ such that $aK \leq N$. An element N of M is called meet principal if $(b \wedge (B:N))N = bN \wedge B$ for all $b \in L$ and for all $B \in M$. An element N of M is called join principal if $b \vee (B:N) = ((bN \vee B):N)$ for all $b \in L$ and for all $N \in M$. N is said to be principal if it is both meet principal and join principal. In a special case an element N of M is called weak meet principal (weak join principal) if $(B:N)N = B \wedge N$ ($(bN:N) = b \vee (0_M:N)$) for all $B \in M$ and for all $b \in L$. N is said to be weak principal if N is both weak meet principal and weak join principal.

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Let M be an $L-$module. An element N in M is called compact if $N \leq \vee_\alpha B_\alpha$ implies $N \leq B_{\alpha_1} \vee B_{\alpha_2} \vee \ldots \vee B_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. The greatest element of M will be denoted by 1_M. If each element of M is a join of principal (compact) elements of M, then M is called a $PG-$lattice ($CG-$lattice). M is called an R-lattice if it is modular, principally generated, compactly generated and has 1_M compact.

Let M be an $L-$module. An element $N \in M$ is said to be proper if $N < 1_M$. If $(0_M: 1_M) = 0_L$, M is called a faithful L-module. If $cm = 0_M$ implies $m = 0_M$ or $c = 0_L$ for any $c \in L$ and $m \in M$, M is called a torsion-free L-module.

For various characterizations of lattice modules, the reader is referred to $[10-14]$.

2. The prime elements in lattice modules

Definition 1. Let M be an $L-$module. An element $N < 1_M$ in M is said to be prime if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e $a \leq (N: 1_M)$ for every $a \in L, X \in M$.

Let M be an $L-$module. If N is a prime element of $L-$module M, then $(N: 1_M)$ is a prime element of L [11, *Proposition* 3.6].

Example 1. Let L be an $L-$module. If $p \in L$ is a prime element, then p is also a prime element as an $L-$module .

Example 2. Let M be an $L-$module. If $L = \{0_L, 1_L\}$ is a field, then every element of M is a prime element.

Definition 2. Let M be an $L-$module. An element $N < 1_M$ in M is said to be primary, if $aX \leq N$ and $X \nleq N$ implies $a^k 1_M \leq N$, for some $k \geq 0$ i.e $a^k \leq (N: 1_M)$ for every $a \in L, X \in M$.

Proposition 1. Let M be an $L-$module and $N < 1_M$ be an element of M. If $(N: 1_M)$ is a prime element of L and N is primary, then N is prime.

Proof: Let $aX \leq N$ and $X \nleq N$ for $a \in L$ *and* $X \in M$. Since N is primary, $aX \leq N$ and $X \nleq N$ implies $a^k 1_M \leq N$, for some $k \geq 0$ i.e $a^k \leq (N: 1_M)$. Since $(N: 1_M)$ is a prime element of L, $a \leq (N: 1_M)$. Consequently, N is prime element of M.

Let M be an $L-$module and $N \in M$. Then $M/N = \{B \in M: N \leq B\}$ is an $L-$module with multiplication $c \circ D = cD \vee N$ for every $c \in L$ and for every $N \leq D \in M$. Similarly, M/N is an $L/(N: 1_M)$-module with $a \circ N^* = aN^* \vee N$ for all $N \leq N^* \in M$ and $(N: 1_M) \leq a$.

Theorem 1. Let M be an $L-$module and $N \in M$. Then N is a prime element if and only if M/N is a torsion-free $L/(N: 1_M)$-module.

Proof: Suppose that $N \in M$ is a prime element. For $(N: 1_M) \leq a$ in L and $N < N^*$ in M, if $a \circ N^* = aN^* \vee N = N$, we have $aN^* \leq N$. Since N is prime, $a = (N: 1_M)$. Conversely, suppose that M/N is a torsion-free $L/(N: 1_M)$-module. If $aX \leq N$ and $X \nleq N$ for $a \in L$ and $X \in M$, then $(a \vee (N: 1_M)) \circ (X \vee N) = N$. Since M/N is a torsion-free $L/(N: 1_M)$ -module, $a \leq (N: 1_M)$.

Lemma 1. Let M be an $L-$module and let B be an element of M. If 1_M is weak principal, then there exists a lattice isomorphism $M/B \cong L/(B: 1_M)$.

Proof: [see 11, Lemma 2.1].

Let M be an $L-$module. Recall that an element $N < 1_M$ of M is called a maximal element if for every element B of M such that $N \leq B$, then either $N = B$ or $B = 1_M$.

Proposition 2. Let M be an $L-$module and $N \in M$. Then,

(i) If $(N: 1_M)$ is maximal in L, then N is prime in M.

(ii) If a is maximal in L and $a1_M < 1_M$, then $a1_M$ is prime in M.

(iii) If N is maximal in M, then N is prime in M.

Proof: (i) If $(N: 1_M)$ is maximal in L, then $L/(N: 1_M)$ is a field. Then M/N is a torsion-free $L/(N: 1_M)$ – module and hence N is prime in M by Theorem 1.

(ii) Since $a \leq (a1_M: 1_M) < 1_L$ and a is maximal in L, $a = (a1_M: 1_M)$. This implies that $a1_M$ is prime in M by (i).

(iii) Let $aX \leq N$ and $X \nleq N$ for $a \in L$ *and* $X \in M$. Since N is maximal, $N \vee X = 1_M$ and so $aN \vee aX = a1_M \leq N$. This implies that $a \leq (N: 1_M)$.

Theorem 2. Let L be an $r-$ lattice and $M-$principally generated, and M be an R- lattice $L-$module. *If* $p1_M$ is compact for every prime element $p \in L$, then every element in M *is* compact.

Proof: Let $\Omega = \{K \in M: K \textit{ is not compact}\}$. Suppose that $\Omega \neq \emptyset$. Since 1_M is compact, Ω has a maximal element by the Zorn Lemma. Suppose that N is a maximal in Ω.

Let $p = (N: 1_M)$. We first show that p is prime. If p is not prime, there exists M-principal elements $a, b \in L$ such that $a \nleq p$, $b \nleq p$ and $ab \leq p$. Hence $N < N \vee a1_M$. Therefore $N \vee a1_M$ is a compact element of M. Since $(ab)1_M \leq N$, $b1_M \leq (N: a)$. Then $N < N \vee b1_M \leq (N: a)$ Hence $(N: a)$ is also compact. Since $N = \vee C_\alpha$ is compactly generated, then $N \vee a1_M = (\vee_{finite} C_\alpha) \vee a1_M$ and we have $N = (\vee_{finite} C_\alpha) \vee (a1_M \wedge N)$. Since a is an M-principal element of L, $a1_M \wedge N = a(N:a)$. Since $(N:a)$ is the finite join of principal elements of M and a is M-principal element in L, $a(N:a)$ is compact [9, *Proposition* 1 *and Proposition* 3]. The finite join of compact elements is compact, so N is compact. This contradiction shows that p is prime.

Since 1_M is compact, 1_M is a join of finite principal elements K_i. Then $p = (N:1_M) = (N:\vee_{finite} K_i) = \wedge_{finite}(N:K_i)$ and $p = (N:K_j)$ for some $K_j \nleq N$, since p is prime. Hence $N < N \vee K_j$ is compact and as is shown in the preceding paragraph, $N = (\vee_{finite} C_\alpha) \vee (K_j \wedge N)$ and $K_j \wedge N = (N:K_j)K_j = pK_j$. Since $N = (\vee_{finite} C_\alpha) \vee pK_j \leq (\vee_{finite} C_\alpha) \vee p1_M \leq N$, $N = (\vee_{finite} C_\alpha) \vee p1_M$ is compact by hypothesis. This is a contradiction. Therefore, Ω is empty.

3. Multiplication lattice modules

In this section we study the concept of multiplication lattice module over a multiplicative lattice and generalize the important results for multiplication modules over commutative rings, obtained by Z. A. El-Bast and P. F. Smith [15], to the lattice modules over multiplicative lattices.

Definition 3. Let M be an $L-$module. If 1_M is a principal element in M, M is called a cyclic lattice module.

Definition 4. An $L-$module M is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$.

Proposition 3. Let M be an $L-$module. Then M is a multiplication lattice module if and only if $N = (N:1_M)1_M$ for all $N \in M$.

Proof: $\Rightarrow$**:** Let M be a multiplication lattice $L-$module and $N \in M$. Then, $N = a1_M$ for some $a \in L$. Hence $a \leq (N:1_M)$ and so $N = a1_M \leq (N:1_M)1_M \leq N$. Therefore $N = (N:1_M)1_M$.

$\Leftarrow$: Clear.

It is clear that an $L-$module M is a multiplication lattice module if and only if 1_M is weak meet principal. If M is a cyclic lattice $L-$module, then M is a multiplication lattice $L-$module.

Proposition 4. Let M be a multiplication lattice $L-$module. If $p \in L$ is maximal and $p1_M < 1_M$, then $p1_M$ is maximal element in M.

Proof: Since p is maximal such that $p \leq (p1_M:1_M) \neq 1_L$, $p = (p1_M:1_M)$. Let $p1_M \leq B$. Then $p = (p1_M:1_M) \leq (B:1_M)$. Since p is maximal, $p = (B:1_M)$ or $(B:1_M) = 1_L$. Therefore, $p1_M = (B:1_M)1_M = B$ or $(B:1_M)1_M = B = 1_M$. Consequently, $p1_M$ is maximal element in M.

Theorem 3. Let L be a multiplicative lattice with 1_L compact, and M be a non-zero multiplication $PG-$lattice $L-$module. Then M contains a maximal element.

Proof: There exists a non-zero principal element X in M. Let $p \in L$ be a maximal element such that $(0_M:X) \leq p$. We show that $p1_M < 1_M$. Suppose that $p1_M = 1_M$. Since M is a multiplication lattice $L-$module, $X = a1_M$ for some $a \in L$. Then $pX = ap1_M = a1_M = X$ and so $1_L = (pX:X) = p \vee (0_M:X) = p$. This is a contradiction. Since p is maximal and $p1_M < 1_M$, $p1_M$ is maximal in M by proposition 4.

Theorem 4. Let L be a $PG-$lattice with 1_L compact, and M be a $PG-$lattice $L-$module. Then M is a multiplication lattice $L-$module if and only if for every maximal element $q \in L$,

(i) For every principal element $Y \in M$, there exists a principal element $q_Y \in L$ with $q_Y \nleq q$ such that $q_Y Y = 0_M$ or

(ii) There exists a principal element $X \in M$ and a principal element $b \in L$ with $b \nleq q$ such that $b1_M \leq X$.

Proof: $\Rightarrow$**:** Let M be a multiplication lattice $L-$module. We have two cases.

Case 1. Let $q1_M = 1_M$ where q is a maximal element of L. For every principal element $Y \in M$, there exists an element $a \in L$ such that $Y = a1_M$. Then $Y = a1_M = aq1_M = qY$. Therefore, $1_L = (qY:Y) = q \vee (0_M:Y)$. Hence $(0_M:Y) \nleq q$. There exists a principal element q_Y such that $q_Y \leq (0_M:Y)$ and $q_Y \nleq q$. Consequently, $q_Y Y = 0_M$ and $q_Y \nleq q$.

Case 2. Let $q1_M < 1_M$. There exists a principal element $X \in M$ such that $X = j1_M \nleq q1_M$, with $j \in L, j \nleq q$. There exists a principal element $b \in L$ with $b \leq j$ and $b \nleq q$. We obtain $b1_M \leq j1_M = X$.

$\Leftarrow$: Let $N \in M$. Put $a = (N:1_M)$. Clearly $a1_M = (N:1_M)1_M \leq N$. Take any principal element $Y \leq N$. We will show that $(a1_M:Y) = 1_L$.

Suppose there exists a maximal element $q \in L$ such that $(a1_M:Y) \leq q$. We have two cases.

Case 1. Suppose that (i) is satisfied. There exists a principal element $q_Y \in L$ with $q_Y \nleq q$ such that $q_Y Y = 0_M$ for every principal element $Y \in M$. Then $q_Y \leq (0_M:Y) \leq (a1_M:Y) \leq q$. This is a contradiction.

Case 2. Suppose that (ii) is satisfied. There exists a principal element $X \in M$ and a principal element $b \in L$ with $b \nleq q$ such that $b1_M \leq X$. Then $bN \leq b1_M \leq X$ for any $N \in M$. Since X is a principal element of $M, bN = (bN:X)X$. Then $b(bN:X)1_M \leq (bN:X)X = bN \leq N$ and so $b(bN:X) \leq a = (N:1_M)$. Therefore, $b^2Y \leq b^2N = b(bN:X)X \leq aX \leq a1_M \Rightarrow b^2 \leq (a1_M:Y) \leq q$. Since q is maximal (and so, the prime) element of $L, b \leq q$. This is a contradiction.

Recall that a multiplicative lattice L is called local if it contains precisely one maximal element.

Corollary 1. Let L be a multiplicative lattice with 1_L compact. Let M be a multiplication PG −lattice L −module. If (L,p) is a local PG −lattice, then M is a cyclic L −module.

Proof: Suppose that $M \neq \{0_M\}$. First, assume that there exists a principal element $q_Y \in L$ with $q_Y \nleq p$ such that $q_Y Y = 0_M$ for every principal element $Y \in M$. Since (L,p) is a local lattice, $q_Y = 1_L$. Then every principal element $Y = 0_M$. This is a contradiction.

Now assume that there exists a principal element $X \in M$ and a principal element $b \in L$ with $b \nleq p$ such that $b1_M \leq X$. Since $b \nleq p$, $b = 1_L$. Therefore, $1_M = X$ is principal.

Corollary 2. Let L be a PG −lattice with 1_L compact, and M be a PG −lattice and CG −lattice L −module. Suppose that $1_M = \vee_{i\in I} Y_i$ for some principal elements Y_i in M. Then M is a multiplication lattice L −module if and only if there exist $a_i \in L$ such that $Y_i = a_i 1_M$ for all $i \in I$.

Proof: $\Rightarrow$: Clear.

$\Leftarrow$: Suppose that there exist $a_i \in L$ such that $Y_i = a_i 1_M$ for all $i \in I$. Let q be a maximal element in L. We have two cases.

Case 1. Suppose that $a_i \leq q$ for all $i \in I$. Then $1_M = \vee_{i\in I} Y_i = \vee_{i\in I} (a_i 1_M) = (\vee_{i\in I} a_i)1_M \leq q1_M$. Hence $1_M = q1_M$ and $Y_i = qY_i$. Therefore, there exists a principal element $q_{Y_i} \nleq q$, with $q_{Y_i} Y_i = 0_M$ for all $i \in I$ as is shown in the theorem. Let X be any principal element in M. Since $X \leq 1_M = \vee_{i\in I} Y_i$ and X is principal, X is compact and so $X \leq \vee_{i=1}^n Y_i$ [13, $Corollary$ 2.2]. Put $t = q_{Y_1} q_{Y_2} \ldots q_{Y_n}$. Then $tX \leq t(\vee_{i=1}^n Y_i) = 0_M$ and $t \nleq q$. Since, finite product of principal elements is principal, t is principal. So M is a multiplication lattice L −module by theorem.

Case 2. Suppose that $a_j \nleq q$ for some $j \in I$. Then there exists a principal element $b_j \in L$ with $b_j \leq a_j$ and $b_j \nleq q$ such that $b_j 1_M \leq a_j 1_M = Y_j$. Therefore, M is a multiplication lattice L −module by theorem.

Theorem 5. Let L be a PG −lattice with 1_L compact, and M be a faithful multiplication PG −lattice L −module. Then the following conditions are equivalent.

(i) 1_M is a compact element of M.
(ii) If $a, c \in L$ such that $a1_M \leq c1_M$, then a $\leq c$.
(iii) For each element N of M there exists a unique element a of L such that $N = a1_M$.
(iv) $1_M \neq a1_M$ for any proper element a of L.
(v) $1_M \neq p1_M$ for any maximal element p of L.

Proof: (i) $\Rightarrow$ (ii): Suppose 1_M is compact. Let a and c be elements of L such that $a1_M \leq c1_M$. We will show that $(c:a) = 1_L$. Suppose that $(c:a) \neq 1_L$. Then there exist a maximal element p of L such that $(c:a) \leq p$. We have two cases.

Case 1. Suppose that $1_M = p1_M$. Then $Y = a'1_M = a'p1_M = pa'1_M = pY$ for any principal element $Y \in M$. Then $1_L = (pY:Y) = p \vee (0_M:Y)$ for all principal elements $Y \in M$. Since 1_M is a compact element of M, $1_M = \vee_{i=1}^k Y_i$ for some principal elements Y_i of M. For any principal elements $Y_i (1 \leq i \leq k)$, $1_L = (pY_i:Y_i) = p \vee (0_M:Y_i)$ and so $(0_M:Y_i) \nleq p$. Therefore, there exist $q_{Y_i} \leq (0_M:Y_i)$ such that $q_{Y_i} \nleq p$ for all $i \in \{1,2,\ldots,k\}$. Hence $q_{Y_i} Y_i = 0_M$ and so $\left(\prod_{i=1}^k q_{Y_i}\right)1_M = 0_M$. Since M is a faithful L −module, $\prod_{i=1}^k q_{Y_i} = 0_L \leq p$, and p is a prime element of L, so $q_{Y_i} \leq p$ for some $i \in \{1,2,\ldots,k\}$. This is a contradiction.

Case 2. Suppose that $p1_M < 1_M$. There exists a principal element $X \in M$ and a principal element $s \in L$ with $s \nleq p$ such that $s1_M \leq X$.

Suppose that α is any principal element of L such that α $\leq a$. Then, $\alpha 1_M \leq a1_M \leq c1_M$. Therefore, $s\alpha X \leq s\alpha 1_M \leq sa1_M \leq sc1_M \leq cX$. Since X is a principal element of M, $s\alpha \vee (0_M:X) = (s\alpha X:X) \leq (cX:X) = c \vee (0_M:X)$. Hence $s^2\alpha \vee s(0_M:X) \leq sc \vee s(0_M:X)$. But $s(0_M:X) = 0_L$. Indeed, let $r \leq (0_M:X)$. Since $s1_M \leq X, rs1_M \leq rX = 0_M$ and so $rs \leq (0_M:1_M)$. Since M is faithful, $(0_M:1_M) = 0_L$. This implies that $s(0_M:X) = 0_L$. Then $s^2\alpha \leq sc \leq c$ for any principal element α$\leq a$ and so $s^2 a \leq c$. Then $s^2 \leq (c:a) \leq p$. Since p is a prime element of $L, s \leq p$. This is a contradiction.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v): Clear.

(v) $\Rightarrow$ (i): Suppose $1_M \neq p1_M$ for every maximal element p of L. Let q be a maximal element of L. Since $q1_M < 1_M$, there is a principal element $Y_q \nleq q1_M$. Since M is a multiplication lattice L −module, $(Y_q:1_M) \nleq q$. There is not a maximal element such that $\vee_{q\, max} (Y_q:1_M) \leq q$. This implies that $\vee_{q\, max} (Y_q:1_M) = 1_L$. Since 1_L is compact, we have finitely maximal elements q_i

such that $1_L = \bigvee_{i=1}^{k}(Y_{q_i}:1_M)$. Since $Y_{q_i} = (Y_{q_i}:1_M)1_M, 1_M = \bigvee_{i=1}^{k} Y_{q_i}$.

Theorem 6. Let L be a PG-lattice with 1_L compact and M be a PG −lattice L −module. Let M be a multiplication lattice L −module. Suppose that p is a prime element in L with $(0_M:1_M) \leq p$. If $aX \leq p1_M$ where $a \in L, X \in M$, then $X \leq p1_M$ or $a \leq p$.

Proof: We may suppose that X is principal in M. Suppose that $aX \leq p1_M$ with $a \nleq p$. We will show that $(p1_M:X) = 1_L$. Suppose that there exists a maximal element $q \in L$ such that $(p1_M:X) \leq q$. We have two cases.

Case 1. If there exists a principal element $q_X \in L$ with $q_X \nleq q$ such that $0_M = q_X X$, then $q_X \leq (0_M:X) \leq (p1_M:X) \leq q$. This is a contradiction.

Case 2. If there exists a principal element $Y \in M$ and a principal element $b \in L$ with $b \nleq q$ such that $b1_M \leq Y$, then $bX \leq b1_M \leq Y$. Since Y is principal, $bX = (bX:Y)Y$. Put $(bX:Y) = s$. Then $abX = asY$. Since Y is join principal, $(asY:Y) = as \vee (0_M:Y)$. Since Y is meet principal, $abX = (abX:Y)Y$. Put $c = (abX:Y)$. Since $cY = abX \leq bp1_M \leq pY, c \vee (0_M:Y) = (cY:Y) \leq (pY:Y) = p \vee (0_M:Y)$.Since $b(0_M:Y)1_M = (0_M:Y)b1_M \leq (0_M:Y)Y = 0_M$, $b(0_M:Y) \leq (0_M:1_M) \leq p$. Hence $bc \vee b(0_M:Y) \leq bp \vee b(0_M:Y) \leq p$. Therefore, $bc \leq p$. On the other hand, $c = (abX:Y) = (asY:Y) = as \vee (0_M:Y)$ and so $abs \leq abs \vee b(0_M:Y) = bc \leq p$. If $b \leq p$, then $b \leq p \leq (p1_M:X) \leq q$. This is a contradiction. Therefore $b \nleq p$. Since p is prime, $s \leq p$. Therefore, $bX = sY \leq pY \leq p1_M$ and so $b \leq (p1_M:X) \leq q$. This is a contradiction.

Corollary 3. Let L be a PG-lattice with 1_L compact. Let M be a multiplication PG −lattice L −module and $N < 1_M$. Then the following conditions are equivalent.
(i) N is a prime element in M,
(ii) $(N:1_M)$ is a prime element in L,
(iii) There exists a prime element p in L with $(0_M:1_M) \leq p$ such that $N = p1_M$.

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