

## Tangent Bishop spherical images of a biharmonic B-slant helix in the Heisenberg group $\text{Heis}^3$

T. Korpınar, E. Turhan\* and V. Asil

*Department of Mathematics, Firat University, 23119, Elazığ, Turkey*  
*E-mails: talatkorpınar@gmail.com, essin.turhan@gmail.com, vasil@firat.edu.tr*

### Abstract

In this paper, biharmonic slant helices are studied according to Bishop frame in the Heisenberg group  $\text{Heis}^3$ . We give necessary and sufficient conditions for slant helices to be biharmonic. The biharmonic slant helices are characterized in terms of Bishop frame in the Heisenberg group  $\text{Heis}^3$ . We give some characterizations for tangent Bishop spherical images of B-slant helix. Additionally, we illustrate four figures of our main theorem.

**Keywords:** Biharmonic curve; Bishop frame; Heisenberg group; tangent Bishop spherical images

### 1. Introduction

Let  $(M, g)$  and  $(N, h)$  be manifolds and  $\phi: M \rightarrow N$  a smooth map. Denote by  $\nabla^\phi$  the connection of the vector bundle  $\phi^*TN$  induced from the Levi-Civita connection  $\nabla^h$  of  $(N, h)$ . The second fundamental form  $\nabla d\phi$  is defined by  $(\nabla d\phi)(X, Y) = \nabla_X^\phi d\phi(Y) - d\phi(\nabla_X Y)$ ,  $X, Y \in \Gamma(TM)$ .

Here  $\nabla$  is the Levi-Civita connection of  $(M, g)$ . The tension field  $\tau(\phi)$  is a section of  $\phi^*TN$  defined by

$$\tau(\phi) = \text{tr} \nabla d\phi. \quad (1)$$

A smooth map  $\phi$  is said to be harmonic if its tension field vanishes. It is well known that  $\phi$  is harmonic if and only if  $\phi$  is a critical point of the energy:

$$E(\phi) = \frac{1}{2} \int h(d\phi, d\phi) dv_g$$

over every compact region of  $M$ . Now let  $\phi: M \rightarrow N$  be a harmonic map. Then the Hessian  $H$  of  $E$  is given by

$$H_\phi(V, W) = \int h(J_\phi(V), W) dv_g, \quad V, W \in \Gamma(\phi^*TN)$$

Here the Jacobi operator  $J_\phi$  is defined by

$$J_\phi(V) := \bar{\Delta}_\phi V - R_\phi(V), \quad V \in \Gamma(\phi^*TN), \quad (2)$$

$$\bar{\Delta}_\phi := \sum_{i=1}^m (\nabla_{e_i}^\phi \nabla_{e_i}^\phi - \nabla_{\nabla_{e_i}^\phi e_i}^\phi), \quad R_\phi(V) = \sum_{i=1}^m R^N(V, d\phi(e_i))d\phi(e_i), \quad (3)$$

where  $R^N$  and  $\{e_i\}$  are the Riemannian curvature of  $N$ , and a local orthonormal frame field of  $M$ , respectively [1].

Let  $\phi: (M, g) \rightarrow (N, h)$  be a smooth map between two Lorentzian manifolds. The bienergy  $E_2(\phi)$  of  $\phi$  over compact domain  $\Omega \subset M$  is defined by

$$E_2(\phi) = \int_\Omega h(\tau(\phi), \tau(\phi)) dv_g.$$

A smooth map  $\phi: (M, g) \rightarrow (N, h)$  is said to be biharmonic if it is a critical point of the  $E_2(\phi)$ , [2-11].

The section  $\tau_2(\phi)$  is called the bitension field of  $\phi$  and the Euler-Lagrange equation of  $E_2$  is

$$\tau_2(\phi) := -J_\phi(\tau(\phi)) = 0. \quad (4)$$

In [12] the authors completely classified the biharmonic submanifolds of codimension greater than one in the  $n$ -dimensional sphere. The

\*Corresponding author  
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biharmonic submanifolds into a space of nonconstant sectional curvature were also investigated. The proper biharmonic curves on Riemannian surfaces were studied in [13]. Inoguchi classified the biharmonic Legendre curves and the Hopf cylinders in three-dimensional Sasakian space forms [14]. Then, Sasahara gave, in [15], the explicit representation of the proper biharmonic Legendre surfaces in five-dimensional Sasakian space forms.

The second variation formula for biharmonic maps in spheres was deduced [8] and the stability of certain classes of biharmonic maps in spheres was discussed in [4]. Also, in [16] there were given some sufficient conditions for the instability of Legendre proper biharmonic submanifolds in Sasakian space forms and the author proved the instability of Legendre curves and surfaces in Sasakian space forms.

Biharmonic functions are utilized in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them. In linear elasticity, if the equations are formulated in terms of displacements for two-dimensional problems, then the introduction of a stress function leads to a fourth-order equation of biharmonic type. For instance, the stress function is proved to be biharmonic for an elastically isotropic crystal undergoing phase transition, which follows spontaneous dilatation. Biharmonic functions arise when dealing with transverse displacements of plates and shells. They can describe the deflection of a thin plate subjected to uniform loading over its surface with fixed edges. Biharmonic functions also arise in fluid dynamics, particularly in Stokes flow problems (i.e., low-Reynolds-number flows). There are many applications for Stokes flow such as in engineering and biological transport phenomena (for details, see [17-19]). Fluid flow through a narrow pipe or channel, such as that used in micro-fluidics, involves low Reynolds number. Seepage flow through cracks and pulmonary alveolar blood flow can also be approximated by Stokes flow. Stokes flow also arises in flow through porous media, which have been long applied by civil engineers to groundwater movement. The industrial applications include the fabrication of microelectronic components, the effect of surface roughness on lubrication, the design of polymer dies and the development of peristaltic pumps for sensitive viscous materials. In natural systems, creeping

flows are important in biomedical applications and studies of animal locomotion.

In this paper, biharmonic slant helices are studied according to Bishop frame in the Heisenberg group  $\text{Heis}^3$ . Necessary and sufficient conditions are given for slant helices to be biharmonic. We characterize the biharmonic slant helices in terms of Bishop frame in the Heisenberg group  $\text{Heis}^3$ , and give some characterizations for tangent Bishop spherical images of B-slant helix. Additionally, four figures of our main theorem are illustrated (Fig. 1 and Fig. 2).

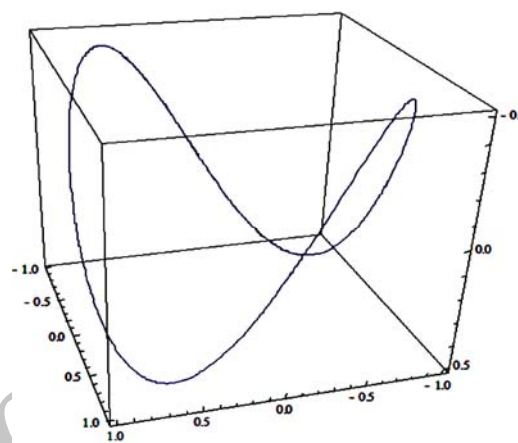


Fig. 1. Biharmonic slant helices according to Bishop frame for different constants with the help of the programme of Mathematica

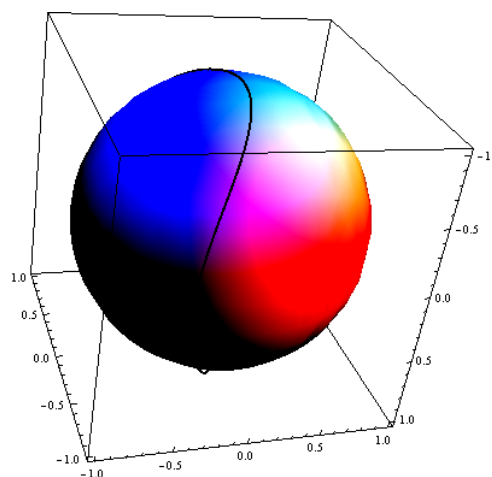


Fig. 2. Tangent spherical indicatrix of  $\gamma$  according to Bishop frame with the help of the programme of Mathematica

## 2. The Heisenberg group $\text{Heis}^3$

Heisenberg group  $\text{Heis}^3$  can be seen as the space  $\mathbb{R}^3$  endowed with the following multiplication:

$$(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \frac{1}{2}x\bar{y} + \frac{1}{2}x\bar{y}) \quad (5)$$

$\text{Heis}^3$  is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric  $g$  is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$

The Lie algebra of  $\text{Heis}^3$  has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, e_3 = \frac{\partial}{\partial z}, \quad (6)$$

for which we have the Lie products [20, 21]

$$[e_1, e_2] = e_3, [e_2, e_3] = [e_3, e_1] = 0$$

with

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

We obtain

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \frac{1}{2} e_3,$$

$$\nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -\frac{1}{2} e_2,$$

$$\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2} e_1.$$

We adopt the following notation and sign convention for Riemannian curvature operator on  $\text{Heis}^3$  defined by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z,$$

while the Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W),$$

where  $X, Y, Z, W$  are smooth vector fields on  $\text{Heis}^3$ .

The components  $\{R_{ijkl}\}$  of  $R$  relative to  $\{e_1, e_2, e_3\}$  are defined by

$$g(R(e_i, e_j)e_k, e_l) = R_{ijkl}.$$

The non vanishing components of the above tensor fields are

$$R_{121} = -\frac{3}{4}e_2, \quad R_{131} = \frac{1}{4}e_3, \quad R_{122} = \frac{3}{4}e_1, \\ R_{232} = \frac{1}{4}e_3, \quad R_{133} = -\frac{1}{4}e_1, \quad R_{233} = -\frac{1}{4}e_2,$$

and

$$R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}. \quad (7)$$

### 3. Biharmonic B-Slant Helices with Bishop Frame In The Heisenberg Group $\text{Heis}^3$

Let  $\gamma : I \rightarrow \text{Heis}^3$  be a non geodesic curve on the Heisenberg group  $\text{Heis}^3$  parametrized by arc length. Let  $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$  be the Frenet frame fields tangent to the Heisenberg group  $\text{Heis}^3$  along  $\gamma$  defined as follows:

$\mathbf{T}$  is the unit vector field  $\gamma'$  tangent to  $\gamma$ ,  $\mathbf{N}$  is the unit vector field in the direction of  $\nabla_{\mathbf{T}} \mathbf{T}$  (normal to  $\gamma$ ), and  $\mathbf{B}$  is chosen so that  $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$  is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{T} &= \kappa \mathbf{N}, \\ \nabla_{\mathbf{T}} \mathbf{N} &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\ \nabla_{\mathbf{T}} \mathbf{B} &= -\tau \mathbf{N}, \end{aligned} \quad (8)$$

where  $\kappa$  is the curvature of  $\gamma$  and  $\tau$  is its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, g(\mathbf{N}, \mathbf{N}) = 1, g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \quad (9)$$

In the rest of the paper, we suppose everywhere  $\kappa \neq 0$  and  $\tau \neq 0$ .

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined, even when the curve has vanishing second derivative [22]. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{T} &= k_1 \mathbf{M}_1 + k_2 \mathbf{M}_2, \\ \nabla_{\mathbf{T}} \mathbf{M}_1 &= -k_1 \mathbf{T}, \\ \nabla_{\mathbf{T}} \mathbf{M}_2 &= -k_2 \mathbf{T}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, g(\mathbf{M}_1, \mathbf{M}_1) = 1, g(\mathbf{M}_2, \mathbf{M}_2) = 1, \\ g(\mathbf{T}, \mathbf{M}_1) &= g(\mathbf{T}, \mathbf{M}_2) = g(\mathbf{M}_1, \mathbf{M}_2) = 0. \end{aligned} \quad (11)$$

Here, we shall call the set  $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$  as Bishop trihedra,  $k_1$  and  $k_2$  as Bishop curvatures [23, 24], where  $\zeta(s) = \arctan \frac{k_2}{k_1}$ ,  $\tau(s) = \zeta'(s)$  and  $\kappa(s) = \sqrt{k_1^2 + k_2^2}$ . Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s) \cos \zeta(s), \\ k_2 &= \kappa(s) \sin \zeta(s). \end{aligned} \quad (12)$$

With respect to the orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  we can write

$$\begin{aligned} \mathbf{T} &= T^1 \mathbf{e}_1 + T^2 \mathbf{e}_2 + T^3 \mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1 \mathbf{e}_1 + M_1^2 \mathbf{e}_2 + M_1^3 \mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1 \mathbf{e}_1 + M_2^2 \mathbf{e}_2 + M_2^3 \mathbf{e}_3. \end{aligned} \quad (13)$$

**Theorem 3.1.**  $\gamma: I \rightarrow Heis^3$  is a biharmonic curve with Bishop frame if and only if

$$\begin{aligned} k_1^2 + k_2^2 &= \text{constant} = C \neq 0, \\ k_1'' - Ck_1 &= k_1 \left[ \frac{1}{4} - (M_2^3)^2 \right] - k_2 M_1^3 M_2^3, \\ k_2'' - Ck_2 &= k_1 M_1^3 M_2^3 + k_2 \left[ \frac{1}{4} - (M_1^3)^2 \right]. \end{aligned} \quad (14)$$

**Proof:** Using (10), we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla_{\mathbf{T}}^3 \mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}} \mathbf{T}) \mathbf{T} \\ &= (-3k_1' k_1 - 3k_2' k_2) \mathbf{T} + (k_1'' - k_1^3 - k_1 k_2^2) \mathbf{M}_1 \\ &\quad + (k_2'' - k_2^3 - k_2 k_1^2) \mathbf{M}_2 \\ &\quad - k_1 R(\mathbf{T}, \mathbf{M}_1) \mathbf{T} - k_2 R(\mathbf{T}, \mathbf{M}_2) \mathbf{T}. \end{aligned}$$

By (4), we see that  $\gamma$  is a biharmonic curve if and only if

$$\begin{aligned} k_1' k_1 + k_2' k_2 &= 0, \\ k_1'' - k_1^3 - k_1 k_2^2 &= k_1 R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_1) \\ &\quad + k_2 R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_1), \\ k_2'' - k_2^3 - k_2 k_1^2 &= k_1 R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_2) \\ &\quad + k_2 R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_2). \end{aligned} \quad (15)$$

Making necessary calculations from (15), we have

$$\begin{aligned} k_1^2 + k_2^2 &= \text{constant} = C \neq 0, \\ k_1'' - Ck_1 &= k_1 R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_1) \\ &\quad + k_2 R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_1), \\ k_2'' - Ck_2 &= k_1 R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_2) + k_2 R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_2). \end{aligned} \quad (16)$$

A direct computation using (7) yields

$$\begin{aligned} R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_1) &= \frac{1}{4} - (M_2^3)^2, \\ R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_1) &= -M_1^3 M_2^3, \\ R(\mathbf{T}, \mathbf{M}_1, \mathbf{T}, \mathbf{M}_2) &= M_1^3 M_2^3, \\ R(\mathbf{T}, \mathbf{M}_2, \mathbf{T}, \mathbf{M}_2) &= \frac{1}{4} - (M_1^3)^2. \end{aligned} \quad (17)$$

These, together with (11), complete the proof of the theorem.

**Definition 3.2.** A regular curve  $\gamma: I \rightarrow Heis^3$  is called a slant helix, provided the unit vector  $\mathbf{M}_1$  of the curve  $\gamma$  has constant angle  $\theta$  with some fixed spacelike unit vector  $u$ , that is

$$g(\mathbf{M}_1(s), u) = \cos \theta \text{ for all } s \in I. \quad (18)$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as B-slant helix.

**Theorem 3.3.** Let  $\gamma: I \rightarrow Heis^3$  be a unit speed curve with non-zero natural curvatures. Then  $\gamma$  is a B-slant helix if and only if

$$\frac{k_1}{k_2} = \text{constant}. \quad (19)$$

**Proof:** Differentiating (18) and by using the Bishop frame (3.3), we find

$$g(\nabla_{\mathbf{T}} \mathbf{M}_1, u) = g(k_1 \mathbf{T}, u) = k_1 g(\mathbf{T}, u) = 0.$$

From the above equation, we get

$$g(\mathbf{T}, u) = 0.$$

Again differentiating from the last equality, we obtain

$$\begin{aligned} g(\nabla_{\mathbf{T}} \mathbf{T}, u) &= g(k_1 \mathbf{M}_1 + k_2 \mathbf{M}_2, u) \\ &= k_1 g(\mathbf{M}_1, u) + k_2 g(\mathbf{M}_2, u) \\ &= k_1 \cos \theta + k_2 \sin \theta = 0. \end{aligned}$$

Using the above equation, we get

$$\frac{k_1}{k_2} = -\tan \theta = \text{constant}. \quad (20)$$

The converse statement is trivial. This completes the proof.

**Theorem 3.4.** Let  $\gamma : I \rightarrow Heis^3$  be a unit speed biharmonic B-slant helix with non-zero natural curvatures. Then

$$k_1 = \text{constant and } k_2 = \text{constant}. \quad (21)$$

**Proof:** Suppose that  $\gamma$  be a unit speed biharmonic slant helix. From (19), we have

$$k_1 = -\tan \theta k_2. \quad (22)$$

On the other hand, using the first equation of (14), we obtain that  $k_2$  is a constant. Similarly,  $k_1$  is a constant.

Hence, the proof is completed.

**Corollary 3.5.**  $\gamma : I \rightarrow Heis^3$  is biharmonic B-slant helix if and only if

$$\begin{aligned} k_1 &= \text{constant} \neq 0, \\ k_2 &= \text{constant} \neq 0, \\ \frac{M_1^3 M_2^3}{C + \frac{1}{4} - (M_2^3)^2} &= -\tan \theta, \\ \frac{C + \frac{1}{4} - (M_1^3)^2}{M_1^3 M_2^3} &= \tan \theta, \end{aligned} \quad (23)$$

where  $-\tan \theta = \frac{k_1}{k_2}$  and  $C = k_1^2 + k_2^2$ .

**Corollary 3.6.** If  $\gamma : I \rightarrow Heis^3$  is biharmonic B-slant helix, then

$$(M_1^3 M_2^3)^2 + \left( C + \frac{1}{4} - (M_2^3)^2 \right) \left( C + \frac{1}{4} - (M_1^3)^2 \right) = 0, \quad (24)$$

where  $C = k_1^2 + k_2^2$ .

**Theorem 3.7.** Let  $\gamma : I \rightarrow Heis^3$  be a unit speed biharmonic B-slant helix with non-zero natural curvatures. Then, the parametric equation of  $\gamma$  are

$$\begin{aligned} x(s) &= \frac{1}{k} \cos \theta \sin \Omega(s) + c_1, \\ y(s) &= -\frac{1}{k} \cos \theta \cos \Omega(s) + c_2, \\ z(s) &= (\sin \theta + \cos^2 \theta) s - \frac{1}{4k} \cos 2\Omega(s) - \frac{c_1}{k} \cos \Omega(s) + c_3, \end{aligned} \quad (25)$$

where  $c_1, c_2, c_3$  are constants of integration,

$$\begin{aligned} \Omega(s) &= \left( \sqrt{k_1^2 + k_2^2 - \cos^2 \theta} - \sin \theta \right) s + \sigma \quad \text{and} \\ k &= \sqrt{k_1^2 + k_2^2 - \cos^2 \theta} - \sin \theta. \end{aligned}$$

**Proof:** Assume that  $\gamma : I \rightarrow Heis^3$  be a unit speed biharmonic B-slant helix. So, without loss of generality, we take the axis of  $\gamma$  parallel to the vector  $\mathbf{e}_3$ . Then,

$$g(\mathbf{M}_1, \mathbf{e}_3) = M_1^3 = \cos \theta, \quad (26)$$

where  $\theta$  is constant angle.

On the other hand, the vector  $\mathbf{M}_1$  is a unit vector, so the following condition is satisfied

$$(M_1^1)^2 + (M_1^2)^2 = 1 - \cos^2 \theta. \quad (27)$$

Noting that  $\cos^2 \theta + \sin^2 \theta = 1$ , we have

$$(M_1^1)^2 + (M_1^2)^2 = \sin^2 \theta. \quad (28)$$

The general solution of (28) can be written in the following form

$$\begin{aligned} M_1^1 &= \sin \theta \cos \Omega(s), \\ M_1^2 &= \sin \theta \sin \Omega(s), \end{aligned} \quad (29)$$

where  $\Omega$  is an arbitrary function of  $s$ .

So, substituting the components  $M_1^1, M_1^2$  and  $M_1^3$  in the second equation of (3.6), we have the following equation

$$\begin{aligned} \mathbf{M}_1 &= \sin \theta \cos \Omega(s) \mathbf{e}_1 \\ &+ \sin \theta \sin \Omega(s) \mathbf{e}_2 + \cos \theta \mathbf{e}_3, \end{aligned} \quad (30)$$

On the other hand, using Bishop formulas (10) and (6), we have

$$\mathbf{M}_2 = \sin \Omega(s) \mathbf{e}_1 - \cos \Omega(s) \mathbf{e}_2,$$

$$\mathbf{T} = \cos \theta \cos \Omega(s) \mathbf{e}_1 + \cos \theta \sin \Omega(s) \mathbf{e}_2 - \sin \theta \mathbf{e}_3.$$

The covariant derivative of the vector field  $\mathbf{T}$  is:

$$\nabla_{\mathbf{T}} \mathbf{T} = (T_1' + T_2 T_3) \mathbf{e}_1 + (T_2' - T_1 T_3) \mathbf{e}_2 + T_3' \mathbf{e}_3.$$

Therefore, we use Bishop formulas (10) and the above equation we get

$$\Omega(s) = \left( \sqrt{k_1^2 + k_2^2 - \cos^2 \theta} - \sin \theta \right) s + \sigma,$$

where  $\sigma$  is a constant of integration.

Thus proof is complete.

#### 4. Tangent Bishop spherical images of Biharmonic B-Slant Helix in the Heisenberg group $Heis^3$

**Definition 4.1.** Let  $\gamma : I \rightarrow Heis^3$  be a regular curve in  $Heis^3$ . If we translate the first (tangent) vector field of Bishop frame to the center O of the unit sphere  $S^2$ , we obtain a spherical image  $\xi = \xi(s_\xi)$ . This curve is called Tangent Bishop spherical image or indicatrix of the curve  $\gamma = \gamma(s)$ .

Let  $\xi = \xi(s_\xi)$  be tangent Bishop spherical image of a regular curve  $\gamma = \gamma(s)$ . One can differentiate  $\xi$  with respect to  $s$ :

$$\xi' = \frac{d\xi}{ds_\xi} \frac{ds_\xi}{ds} = k_1 \mathbf{M}_1 + k_2 \mathbf{M}_2. \quad (31)$$

Here, we shall denote differentiation according to  $s$  by a dash, and differentiation according to  $s_\xi$  by a dot. In terms of Bishop frame vector fields (10), we have the tangent vector of the spherical image as follows:

$$\mathbf{T}_\xi = \frac{k_1 \mathbf{M}_1 + k_2 \mathbf{M}_2}{\sqrt{k_1^2 + k_2^2}}, \quad (32)$$

where

$$\frac{ds_\xi}{ds} = \sqrt{k_1^2 + k_2^2} = \kappa(s).$$

**Theorem 4.2.** Let  $\gamma : I \rightarrow Heis^3$  be a unit speed biharmonic slant helix with non-zero natural

curvatures. Then the parametric equation of the tangent spherical indicatrix of  $\gamma$  are

$$\begin{aligned} x_\xi(s) &= \cos \theta \cos \Omega(s), \\ y_\xi(s) &= \cos \theta \sin \Omega(s), \\ z_\xi(s) &= \cos^2 \theta \cos \Omega(s) \sin \Omega(s) - \sin \theta, \end{aligned} \quad (33)$$

where

$$\Omega(s) = \left( \sqrt{k_1^2 + k_2^2 - \cos^2 \theta} - \sin \theta \right) s + \sigma.$$

**Proof:** Using (31) we have

$$\xi' = \nabla_{\mathbf{T}} \mathbf{T} = k_1 \mathbf{M}_1 + k_2 \mathbf{M}_2. \quad (34)$$

Substituting (20) in (34), we get

$$\xi = \cos \theta \cos \Omega(s) \mathbf{e}_1 + \cos \theta \sin \Omega(s) \mathbf{e}_2 - \sin \theta \mathbf{e}_3. \quad (35)$$

These, together with (30) give (33).

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#### References

- [1] Eells, J. & Sampson, J. H. (1964). Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, 86, 109-160.
- [2] Jiang, G. Y. (1986). 2-Harmonic maps and their first and second variation formulas, *Chinese Ann. Math. Ser. A7*, 389-402.
- [3] Körpınar, T. & Turhan, E. (2010). On Horizontal Biharmonic Curves In The Heisenberg Group  $Heis^3$ , *Arab. J. Sci. Eng. Sect. A Sci.*, 35(1) 79-85.
- [4] Loubeau, E. & Oniciuc, C. (2007). On the biharmonic and harmonic indices of the Hopf map, *Trans. Amer. Math. Soc.* 359, 5239-5256.
- [5] Milnor, J. (1976). Curvatures of Left-Invariant Metrics on Lie Groups. *Advances in Mathematics*, 21, 293-329.
- [6] Nutbourne, A. W. & Martin, R. R. (1988). *Differential Geometry Applied to the Design of Curves and Surfaces*. UK: Ellis Horwood, Chichester.
- [7] O'Neill, B. (1983). *Semi-Riemannian Geometry*. New York, Academic Press.
- [8] Oniciuc, C. (2002). On the second variation formula for biharmonic maps to a sphere. *Publ. Math. Debrecen* 61, 613-622.
- [9] Ou, Y.L. (2006). p-Harmonic morphisms, biharmonic morphisms, and nonharmonic biharmonic maps, *J. Geom. Phys.* 56, 358-374.
- [10] Turhan, E. & Körpınar, T. (2009). Characterize on the Heisenberg Group with left invariant Lorentzian metric. *Demonstratio Mathematica*, 42(2), 423-428
- [11] Turhan, E. & Körpınar, T. (2010). On Characterization Of Timelike Horizontal Biharmonic

- Curves In The Lorentzian Heisenberg Group Heis<sup>3</sup>, *Zeitschrift für Naturforschung A-A Journal of Physical Sciences*, 65a, 641-648.
- [12] Caddeo, R., Montaldo, S. & Oniciuc, C. (2002). Biharmonic submanifolds in spheres, *Israel J. Math.* 130, 109-123.
- [13] Caddeo, R., Montaldo, S. & Piu, P. (2001). Biharmonic curves on a surface., *Rend. Mat. Appl.*, 21, 143-157.
- [14] Inoguchi, J. (2004). Submanifolds with harmonic mean curvature in contact 3-manifolds. *Colloq. Math.* 100, 163-179.
- [15] Sasahara, T. (2005). Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors, *Publ. Math Debrecen*, 67, 285-303.
- [16] Sasahara, T. Stability of biharmonic Legendre submanifolds in Sasakian space forms, *preprint*.
- [17] Estabrook, F. B. & Wahlquist, H. D. (1976). Prolongation structures of nonlinear evolution equations II. *J. Math Phys.*, 17, 1293-1297.
- [18] Happel, J. & Brenner, H. (1965). *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media*. New Jersey, Prentice-Hall.
- [19] Langlois, W. E. (1964). *Slow Viscous Flow*. Macmillan, New York; London, Collier-Macmillan.
- [20] Rahmani, S. (1992). Métriques de Lorentz sur les groupes de Lie unimodulaires, de dimension trois. *J. Geom. Phys.* 9, 295-302.
- [21] Turhan, E. (2008). Completeness of Lorentz Metric on 3-Dimensional Heisenberg Group, *International Mathematical Forum*, 3(13), 639-644.
- [22] Bükcü, B. & Karacan, M. K. (2008). Special Bishop motion and Bishop Darboux rotation axis of the space curve. *J. Dyn. Syst. Geom. Theor.* 6(1), 27-34.
- [23] Yılmaz, S. & Turgut, M. (2010). A new version of Bishop frame and an application to spherical images, *J. Math Anal. Appl.* 371, 764-776.
- [24] Yılmaz, S., Özyılmaz, E. & Turgut, M. (2010). New spherical indicatrices and their characterizations, *An. St. Univ. Ovidius Constanta*, 18(2), 337-354.

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