

Chebyshev cardinal functions for solving volterra-fredholm integro-differential equations using operational matrices

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Abstract

In this paper, an effective direct method to determine the numerical solution of linear and nonlinear Fredholm and Volterra integral and integro-differential equations is proposed. The method is based on expanding the required approximate solution as the elements of Chebyshev cardinal functions. The operational matrices for the integration and product of the Chebyshev cardinal functions are described in detail. These matrices play the important role of reducing an integral equation to a system of algebraic equations. Illustrative examples are shown, which confirms the validity and applicability of the presented technique.

Keywords: Integro-differential equation; mixed Volterra-Fredholm integral equations; chebyshev cardinal functions; operational matrix of integration

1. Introduction

Integral equations provide an important tool for modeling of the numerous problems in engineering and science [1, 2]. These equations appear in the modeling of electromagnetic and electrodynamic, elasticity and dynamic contact, heat and mass transfer, fluid mechanic, acoustic, chemical and electrochemical process, molecular physics, population, medicine and in many other phenomena [3-9]. So, it is clear that solving integral equations can be used to describe many events in real world.

In [10], some traditional methods for solving integral equations are classified and described. Recently, many researchers have focused on finding efficient numerical or analytical methods to estimate the solution of integral equations such as collocation method [1, 11], Adomian decomposition method (ADM) [12], homotopy perturbation method (HPM) [13], He's variational iteration method [14], optimal control [15], wavelets [16-19], neural networks [20], simulation methods [21], block-pulse method [22], and some other new methods [23-27].

In this study, Chebyshev cardinal functions are introduced as the efficient basis to approximate the solution of integral equations [28]. Also, the technique of solving is involved in operational matrices as a powerful tool to reduce an integral equation to a system of algebraic equations. However, operational matrices are applied with other basis [29, 30]. The coupling Chebyshev cardinal functions with operational matrices provide high accurate solutions using simple computations [31, 32].

Moreover, the new defined operational matrix utilizes the computations such that the proposed method reduces the integral equation to an algebraic system of equations without using the collocation scheme.

A nonlinear Fredholm-Volterra integro-differential equation can be considered in the following general form

$$\sum_{j=0}^m \alpha_j(t) u^{(j)}(t) = f(t) + \lambda_1 \int_a^t k_1(t, s) [u(s)]^p ds + \lambda_2 \int_a^b k_2(t, s) [u(s)]^q ds, \quad (1)$$

under the initial conditions

$$u^{(j)}(a) = \mu_j, \quad j = 0, 1, \dots, m-1, \quad (2)$$

where $u(t)$ is an unknown function, the functions $f(t)$, $k_1(t, s)$ and $k_2(t, s)$ are defined on an interval $a \leq t, s \leq b$ and also λ_1, λ_2 and $\mu_j, j = 0, 1, \dots, m-1$, are constants. Though different choices of the parameters lead to various problems, the method can afford to approximate the solution.

The presented paper is organized in 6 sections. In Section 2, we introduce the Chebyshev cardinal functions and the matrix form of an approximated function. Section 3 includes some useful property of the Chebyshev cardinal functions. In Section 4,

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we show how to approximate the solutions of the integral equation by the mentioned basis through the operational matrices. Numerical results are shown in Section 5. Finally, Section 6 concludes this paper with a brief summary and more discussion of the numerical results.

2. Chebyshev cardinal functions

Chebyshev cardinal functions of order N in $[-1, 1]$ are defined as [28]

$$C_j(x) = \frac{T_{N+1}(x)}{T_{N+1}(x_j)(x-x_j)}, \quad j=1,2,\dots,N+1, \quad (3)$$

where $T_{N+1}(x)$ is the first kind Chebyshev function of order $N+1$ in $[-1, 1]$ defined by

$$T_{N+1}(x) = \cos((N+1)\arccos(x)), \quad (4)$$

subscript x denotes x -differentiation and x_j , $j=1,2,\dots,N+1$, are the zeros of $T_{N+1}(x)$ defined by $\cos(\frac{(2j-1)\pi}{2N+2})$, $j=1,2,\dots,N+1$, with the Kronecker property

$$C_j(x_i) = \delta_{ji} = \begin{cases} 1, & \text{if } j=i, \\ 0, & \text{if } j \neq i, \end{cases} \quad (5)$$

where δ_{ji} is the Kronecker delta function. We change the variable $t = \frac{b-a}{2}x + \frac{b+a}{2}$ to use these functions on $[a, b]$. Now any function $g(t)$ on $[a, b]$ can be approximated as

$$g(t) \approx \sum_{j=1}^{N+1} g(t_j) C_j(t) = G^T \Theta_N(t), \quad (6)$$

where t_j , $j=1,2,\dots,N+1$, are the shifted points of x_j , $j=1,2,\dots,N+1$, by transforming

$$t = \frac{b-a}{2}x + \frac{b+a}{2},$$

$$G = [g(t_1), g(t_2), \dots, g(t_{N+1})]^T, \quad (7)$$

and

$$\Theta_N(t) = [C_1(t), C_2(t), \dots, C_{N+1}(t)]^T. \quad (8)$$

Also, we choose t_j so that, $t_1 < t_2 < \dots < t_{N+1}$.

3. Some new properties of chebyshev cardinal functions

In this section, some operational matrices of integration and product will be derived.

Lemma 1. The integration of the vector $\Theta_N(t)$ defined in (8) can be approximated as

$$\int_a^t \Theta_N(s) ds \approx P \Theta_N(t), \quad (9)$$

where P is the $(N+1) \times (N+1)$ operational matrix of integration for Chebyshev cardinal functions.

Proof: Let

$$\int_a^t \Theta_N(s) ds = \left[\int_a^t C_1(s) ds, \int_a^t C_2(s) ds, \dots, \int_a^t C_{N+1}(s) ds \right]^T. \quad (10)$$

Using (6), any function $\int_a^t C_j(s) ds$ can be approximated as

$$\int_a^t C_j(s) ds \approx \sum_{k=1}^{N+1} \alpha_{jk} C_k(t), \quad (11)$$

where

$$\alpha_{jk} = \int_a^{t_k} C_j(s) ds = \frac{\beta}{T_{N+1}(t_j)} \int_a^{t_k} \prod_{i=1, i \neq j}^{N+1} (s-t_i) ds, \quad j, k=1, 2, \dots, N+1. \quad (12)$$

and $\beta = \frac{2^{2N+1}}{(b-a)^{N+1}}$. Comparing (9) and (11), we obtain

$$P = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1,N+1} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2,N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N+1,1} & \alpha_{N+1,2} & \dots & \alpha_{N+1,N+1} \end{bmatrix} \quad (13)$$

Remark: The elements of the matrix P can be also found without integration equivalent to (12). Let $L_{M+1}(x)$ be the Legendre polynomial of order $M+1$ on $[-1, 1]$. Then the Legendre-Gauss nodes are

$$-1 < \tau_1 < \tau_2 < \dots < \tau_{M+1} < 1, \quad (14)$$

where $\{\tau_i\}_{i=1}^{M+1}$ are the zeros of $L_{M+1}(x)$. No explicit formulas are known for the points τ_i , and so they are computed numerically using subroutines [33]. Also, we approximate the integral of f on $[-1,1]$ as

$$\int_{-1}^1 f(x)dx \approx \sum_{i=1}^{M+1} w_i f(\tau_i), \tag{15}$$

where τ_i are Legendre-Gauss nodes in (14) and the weights w_i given in [33]

$$w_i = \frac{2}{(1-\tau_i^2)[L'_{M+1}(\tau_i)]^2}, \quad i = 1, 2, \dots, M+1. \tag{16}$$

It is well known that the integration in (15) is exact whenever $f(x)$ is a polynomial of degree smaller than $2M+1$.

By change of variable $s = \frac{t_k - a}{2}\tau + \frac{t_k + a}{2}$, (12) can be written as

$$\alpha_{jk} = \frac{\beta}{T_{N+1,s}(t_j)} \frac{t_k - a}{2} \int_{-1}^1 \prod_{i=1, i \neq j}^{N+1} \left(\frac{t_k - a}{2}\tau + \frac{t_k + a}{2} - t_i \right) d\tau. \tag{17}$$

Now, using the Gaussian integration formula in (15) with $M = \left\lfloor \frac{N}{2} \right\rfloor$ leads to

$$\alpha_{jk} = \frac{\beta}{T_{N+1,s}(t_j)} \frac{t_k - a}{2} \sum_{l=1}^{M+1} w_l \prod_{i=1, i \neq j}^{N+1} \left(\frac{t_k - a}{2}\tau_l + \frac{t_k + a}{2} - t_i \right), \tag{18}$$

for $j, k = 1, 2, \dots, N+1$.

Lemma 2. Assume $\Theta_N(t)$ in (8) and $F = [f_1, f_2, \dots, f_{N+1}]^T$ as the column vectors, then

$$\Theta_N(t)\Theta_N^T(t)F \approx \tilde{F}\Theta_N(t), \tag{19}$$

where \tilde{F} is a $(N+1) \times (N+1)$ product operational matrix as follows

$$\tilde{F} = \text{diag}[f_1, f_2, \dots, f_{N+1}]. \tag{20}$$

Proof: First, by using the definition of $\Theta_N(t)$ in (8) we obtain

$$\Theta_N(t)\Theta_N^T(t) = \begin{bmatrix} C_1(t)C_1(t) & C_1(t)C_2(t) & \dots & C_1(t)C_{N+1}(t) \\ C_2(t)C_1(t) & C_2(t)C_2(t) & \dots & C_2(t)C_{N+1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ C_{N+1}(t)C_1(t) & C_{N+1}(t)C_2(t) & \dots & C_{N+1}(t)C_{N+1}(t) \end{bmatrix}. \tag{21}$$

Using (6), any function $C_j(t)C_k(t)$, $j, k = 1, 2, \dots, N+1$ can be approximated as

$$C_j(t)C_k(t) \approx \sum_{i=1}^{N+1} \theta_{jk}^{(i)} C_i(t), \tag{22}$$

where

$$\theta_{jk}^{(i)} = C_j(t_i)C_k(t_i) = \delta_{ji}\delta_{ki}. \tag{23}$$

So, from (21) and (22), we have

$$\Theta_N(t)\Theta_N^T(t) \approx \begin{bmatrix} C_1(t) & 0 & \dots & 0 \\ 0 & C_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{N+1}(t) \end{bmatrix}. \tag{24}$$

Clearly, by using the vector F , we find the $(N+1) \times (N+1)$ matrix \tilde{F} as follows

$$\tilde{F} = \text{diag}[f_1, f_2, \dots, f_{N+1}]. \tag{25}$$

Lemma 3. Assume $\Lambda_{(N+1) \times (N+1)}$ is an arbitrary matrix, then

$$\Theta_N^T(t)\Lambda\Theta_N(t) \approx \hat{\Lambda}\Theta_N(t), \tag{26}$$

where $\Theta_N(t)$ is defined in (8) and $\hat{\Lambda}$ is a $(N+1)$ -row vector including elements equal to the diagonal entries of Λ matrix.

Proof: To prove the identity, we expand the formula as follows

$$\begin{aligned} \Theta_N^T(t)\Lambda\Theta_N(t) &= \Theta_N^T(t) \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \dots & \Lambda_{1,N+1} \\ \Lambda_{21} & \Lambda_{22} & \dots & \Lambda_{2,N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{N+1,1} & \Lambda_{N+1,2} & \dots & \Lambda_{N+1,N+1} \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_{N+1}(t) \end{bmatrix} \\ &= \Theta_N^T(t) \begin{bmatrix} \sum_{j=1}^{N+1} \Lambda_{1j} C_j(t) \\ \sum_{j=1}^{N+1} \Lambda_{2j} C_j(t) \\ \vdots \\ \sum_{j=1}^{N+1} \Lambda_{N+1,j} C_j(t) \end{bmatrix} = \left[\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \Lambda_{ij} C_i(t) C_j(t) \right] \stackrel{(22)}{\approx} \left[\sum_{i=1}^{N+1} \Lambda_{ii} C_i(t) \right] = \hat{\Lambda}\Theta_N(t). \end{aligned}$$

Lemma 4. Suppose $U^T = [u_1, u_2, \dots, u_{N+1}]$ and $e_j^T = [\mu_j, \mu_j, \dots, \mu_j]$. If we consider

$$u^{(m)}(t) \approx U^T \Theta_N(t), \quad j = 0, 1, \dots, m-1, \quad (27)$$

then

$$\begin{aligned} u^{(m-1)}(t) &\approx \left[U^T P^l + \sum_{k=1}^l e_{m-k}^T P^{l-k} \right] \Theta_N(t) \\ &= L_l \Theta_N, \quad l = 0, 1, \dots, m. \end{aligned} \quad (28)$$

In particular, for $l = m$, we have

$$u(t) \approx [U^T P^m + e_{m-1}^T P^{m-1} + \dots + e_1^T P + e_0^T] \Theta_N(t) = L_m \Theta_N, \quad (29)$$

where L_l , $l = 0, 1, \dots, m$, are the $N+1$ -row vectors.

Proof: Clearly, by integration of (27) we will have

$$\int_a^t u^{(m)}(s) ds \approx u^{(m-1)}(t) - \mu_{m-1}. \quad (30)$$

Also, from (9), we get

$$\int_a^t U^T \Theta_N(s) ds \approx U^T P \Theta_N(t), \quad (31)$$

which concludes

$$u^{(m-1)}(t) \approx U^T P \Theta_N(t) + \mu_{m-1}. \quad (32)$$

From (6), μ_{m-1} as the constant function has the vector form $e_{m-1}^T \Theta_N(t)$ which gives

$$u^{(m-1)}(t) \approx U^T P \Theta_N(t) + e_{m-1}^T \Theta_N(t) = [U^T P + e_{m-1}^T] \Theta_N(t). \quad (33)$$

Similarly, we can obtain vector forms of other differentiations of $u(t)$.

Lemma 5. If we consider

$$u(t) \approx U^T \Theta_N(t), \quad (34)$$

then for every $n \in N$ we have

$$[u(t)]^n \approx U^T (\tilde{U})^{n-1} \Theta_N(t), \quad (35)$$

or

$$[u(t)]^n \approx [u_1^n, u_2^n, \dots, u_{N+1}^n] \Theta_N(t), \quad (36)$$

where $\tilde{U} = \text{diag}(u_1, u_2, \dots, u_{N+1})$.

Proof: By induction, for $k = n+1$, we will have

$$\begin{aligned} [u(t)]^{n+1} &= [u(t)]^n u(t) \approx U^T (\tilde{U})^{n-1} \frac{\tilde{U} \Theta_N(t)}{\Theta_N(t)} \Theta_N^T(t) U \\ &= U^T (\tilde{U})^n \Theta_N(t) \\ &= [u_1^{n+1}, u_2^{n+1}, \dots, u_{N+1}^{n+1}] \Theta_N(t). \end{aligned}$$

Lemma 6. If we consider

$$u^{(m)}(t) \approx U^T \Theta_N(t), \quad (37)$$

then for every $q \in N$ we have

$$[u(t)]^q \approx U_q^T \Theta_N(t), \quad (38)$$

where $(U_q)_i = [(L_m)_i]^q$ and L_m is defined in Lemma 3.

Proof: According to Lemma 4, hypostasis lead to $u(t) \approx L_m \Theta_N(t)$.

So, considering the previous Lemma 5 gives

$$[u(t)]^q \approx [(L_m)_1^q, (L_m)_2^q, \dots, (L_m)_{N+1}^q] \Theta_N(t),$$

and the proof is completed.

4. Direct method to solve integral equations

In this section, by using results obtained in the previous section concerning Chebyshev cardinal functions, an effective and accurate direct method for solving nonlinear Fredholm, Volterra and Fredholm-Volterra integral and integro-differential equations is presented.

Consider the following integral equation

$$\sum_{j=0}^m \alpha_j(t) u^{(j)}(t) = f(t) + \lambda_1 \int_a^b k_1(t,s) [u(s)]^p ds + \lambda_2 \int_a^b k_2(t,s) [u(s)]^q ds, \quad (39)$$

under the initial conditions

$$u^{(j)}(a) = \mu_j, \quad j = 0, 1, \dots, m-1, \quad (40)$$

as before.

We first reform the proposed integral equation to utilize for solving. This process is described in the following steps.

Step 1.

Since $f(t)$ is a given function, according to (6) we can approximate it as follows

$$f(t) \approx \sum_{j=1}^{N+1} f(t_j) C_j(t) = F^T \Theta_N(t), \quad (41)$$

where $F^T = [f(t_1), f(t_2), \dots, f(t_{N+1})]$ and $t_j, j = 1, \dots, N+1$, are shifted points of $x_j, j = 1, \dots, N+1$, by $t = \frac{b-a}{2}x + \frac{b+a}{2}$.

Step 2.

From (6), $\alpha_j(t)$ as the given functions can be rewritten in the following form

$$\alpha_j(t) \approx \hat{A}_j^T \Theta_N(t) = \Theta_N^T(t) \hat{A}_j, \quad j = 0, 2, \dots, m, \quad (42)$$

where $\hat{A}_j^T = [\alpha_j(t_1), \alpha_j(t_2), \dots, \alpha_j(t_{N+1})]$. Also, by Lemma 4 we have

$$u^{(j)}(t) = L_{m-j} \Theta_N(t), \quad j = 0, 1, \dots, m.$$

Therefore, the right side of the equation can be approximated as

$$\sum_{j=0}^m \alpha_j(x) u^{(j)}(t) \approx \sum_{j=0}^m L_j \overline{\hat{A}_j \Theta_N^T(x) \Theta_N^T(t) \hat{A}_j} \approx \left[\sum_{j=0}^m L_j \tilde{A}_j \right] \Theta_N(t), \quad (43)$$

where \tilde{A}_j is the $(N+1) \times (N+1)$ diagonal matrix whose entries correspond to the elements of \hat{A}_j . Note that the points t_j related to x_j , $j = 1, 2, \dots, N+1$, and the vectors L_j , $j = 0, 2, \dots, m$, are defined in (28).

Step 3.

Now we focus on the Integral part with the constant limits of integration of the proposed integral equation. It is clear that $k_2(x, t)$, as a kernel of the Fredholm part, can be written as follows

$$\begin{aligned} k_2(t, s) &\approx \sum_{j=1}^{N+1} \sum_{i=1}^{N+1} k_{ij}^f C_i(t) C_j(s) \\ &= [C_1(t), C_2(t), \dots, C_{N+1}(t)] K^f [C_1(s), C_2(s), \dots, C_{N+1}(s)]^T \\ &= \Theta_N^T(t) K^f \Theta_N(s), \end{aligned} \quad (44)$$

where $k_2(t_i, s_j) = k_{ij}^f = (K^f)_{ij}$.

Now we describe the most important part of the process

$$\begin{aligned} \int_a^b k_2(t, s) [u(s)]^q ds &\approx \int_a^b \Theta_N^T(t) K^f \Theta_N(s) \Theta_N^T(s) U_q ds \\ &= \Theta_N^T(t) K^f \left(\int_a^b \Theta_N(s) \Theta_N^T(s) ds \right) U_q \\ &= \Theta_N^T(t) K^f S U_q, \end{aligned} \quad (45)$$

where S is defined as follows

$$S = \int_a^b \Theta_N(s) \Theta_N^T(s) ds \approx \text{diagonal} \left[\int_a^b C_1(s) ds, \int_a^b C_2(s) ds, \dots, \int_a^b C_{N+1}(s) ds \right],$$

which is obtained from (24).

Note that the integral has the scalar value. So it is equal to its transpose. Finally, we have

$$\int_a^b k_2(t, s) [u(s)]^q ds \approx \left[K^f S U_q \right]^T \Theta_N(t). \quad (46)$$

Step 4.

This step explains how to reduce the Volterra Integral part of the proposed mixed Volterra-Fredholm integral equation. Similarly, $k_1(t, s)$, as a kernel of the Volterra part, can be approximated as

$$k_1(t, s) \approx \sum_{j=1}^{N+1} \sum_{i=1}^{N+1} k_{ij}^v C_i(t) C_j(s) = \Theta_N^T(t) K^v \Theta_N(s), \quad (47)$$

where $k_1(x_i, t_j) = k_{ij}^v = (K^v)_{ij}$. Therefore, we have

$$\begin{aligned} \int_a^b k_1(t, s) [u(s)]^p ds &\approx \int_a^b \Theta_N^T(t) K^v \overline{\Theta_N(s) \Theta_N^T(s) U_p} ds \approx \Theta_N^T(t) K^v \tilde{U}_p \int_a^b \Theta_N^T(s) ds \\ &\approx \Theta_N^T(t) K^v \tilde{U}_p P \Theta_N(t) \approx \hat{H} \Theta_N(t), \end{aligned} \quad (48)$$

Where, according to Lemma 2 \tilde{U}_p is the diagonal matrix constructed by entries of U_p . Also, \hat{H} is a column vector including the entries of the main diagonal of $K^v \tilde{U}_p P$. It is a direct result of Lemma 3.

Step 5.

Now we substitute the obtained reformed parts into (39). So, we have

$$\left[\left(\sum_{j=0}^m L_j \tilde{A}_j \right) - F^T - \lambda_1 \hat{H} - \lambda_2 (K^f S U_q)^T \right] \Theta_N(t) = 0. \quad (49)$$

Since the above equation is satisfied for every $t \in [a, b]$, we have

$$\left(\sum_{j=0}^m L_j \tilde{A}_j \right) - F^T - \lambda_1 \hat{H} - \lambda_2 (K^f S U_q)^T = 0. \quad (50)$$

Solving the obtained system included $N + 1$ unknowns and $N + 1$ equations, leading to the solution of the integral equation. Newton's method can fulfill the accurate solutions of nonlinear systems.

5. Numerical results

In this section, some different examples which have been solved with other usual methods are considered. Thus, the obtained numerical results can be compared with other methods. The associated computations with the examples were performed using MAPLE 13 with 64 digit precision on a Personal Computer. Although 64 digits were used in the computations, only 2 digits are legit in the illustrative examples.

Example 1. As the first example, consider the following integral equation of the first kind [34]

$$\frac{e^{t^2+1} - 1}{t^2 + 1} - \int_0^1 e^{t^2s} u(s) ds = 0, \quad t \in [0, 1],$$

where the exact solution is $u(t) = e^t$. We proceed with the process of solving corresponding to the described steps for $N = 4$. According to (39)

$$k_2(t, s) = e^{t^2s}, \quad q = 1, \quad \lambda_1 = 0, \quad \lambda_2 = -1,$$

$$f(t) = \frac{e^{t^2+1} - 1}{t^2 + 1},$$

and we approximate the solution as follows

$$u(t) \approx U^T \Theta_N(t).$$

Now, we obtain the needed vectors and matrices as follows

$$K^f = \begin{bmatrix} 2.53 & 2.13 & 1.61 & 1.22 & 1.02 \\ 1.85 & 1.65 & 1.37 & 1.14 & 1.02 \\ 1.28 & 1.22 & 1.13 & 1.05 & 1.01 \\ 1.04 & 1.03 & 1.02 & 1.01 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}, \quad F = \begin{bmatrix} 3.09 \\ 2.52 \\ 1.99 \\ 1.76 \\ 1.72 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.08 & 0 & 0 & 0 & 0 \\ 0 & 0.26 & 0 & 0 & 0 \\ 0 & 0 & 0.31 & 0 & 0 \\ 0 & 0 & 0 & 0.26 & 0 \\ 0 & 0 & 0 & 0 & 0.08 \end{bmatrix}, \quad U_q = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}.$$

Substituting these values in (53) gives the following system of linear equations

$$F^T - (K^f S U_q)^T = 0,$$

which is equivalent to

$$\begin{cases} 0.21u_1 + 0.56u_2 + 0.49u_3 + 0.32u_4 + 0.09u_5 - 3.09 = 0, \\ 0.16u_1 + 0.43u_2 + 0.42u_3 + 0.30u_4 + 0.09u_5 - 2.52 = 0, \\ 0.11u_1 + 0.32u_2 + 0.35u_3 + 0.28u_4 + 0.08u_5 - 1.99 = 0, \\ 0.09u_1 + 0.27u_2 + 0.31u_3 + 0.27u_4 + 0.08u_5 - 1.76 = 0, \\ 0.08u_1 + 0.26u_2 + 0.31u_3 + 0.26u_4 + 0.08u_5 - 1.72 = 0. \end{cases}$$

From this system, the coefficients $u_i, i = 1, 2, \dots, 5$ are computed as

$$U = [2.64 \quad 2.22 \quad 1.64 \quad 1.24 \quad 1.01],$$

and the approximate solution of the integral equation is obtained by

$$u(t) \approx \sum_{i=1}^5 u_i C_i(t),$$

which is fitted on the exact solution demonstrated in Fig. 1. In addition, error function is shown for $N = 10$. Further investigations will be described in the conclusion.

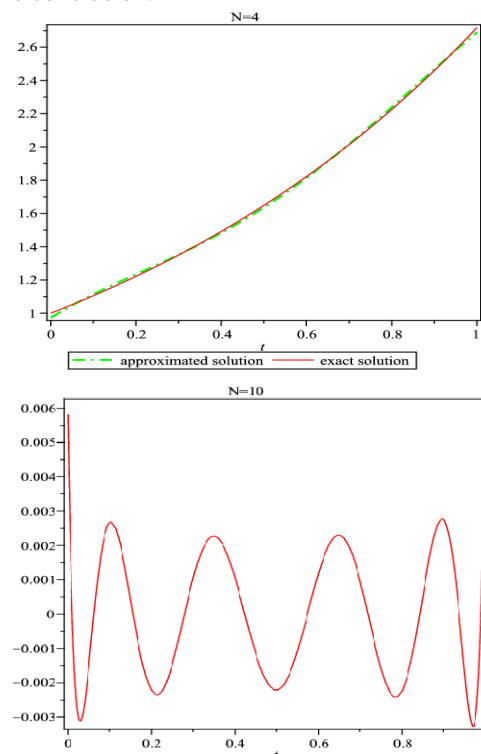


Fig. 1. The exact and approximated solution cover each other for $N = 4$. The error function for $N = 10$ is also shown. See the Example 1

Example 2. Consider the following integral equation of the first kind [17]

$$\frac{-1}{15}(-8e^{2t} + 6\sin t + 3\cos t + 5e^{-t}) - \int_0^t (e^{s-t} + \sin(t-s))u(s)ds = 0,$$

where the exact solution is $u(t) = e^{2t}$. We describe the steps of solving for $N = 4$. According to (39)

$$k_1(t, s) = e^{s-t} + \sin(t-s), \quad p = 1, \quad \lambda_1 = -1, \quad \lambda_2 = 0,$$

$$f(t) = \frac{-1}{15}(-8e^{2t} + 6\sin t + 3\cos t + 5e^{-t}),$$

and we approximate the solution as follows

$$u(t) \approx U^T \Theta_N(t).$$

Now we obtain the needed vectors and matrices as follows

$$K^v = \begin{bmatrix} 1.0 & 1.01 & 1.08 & 1.16 & 1.20 \\ 1.02 & 1.0 & 1.04 & 1.11 & 1.16 \\ 1.15 & 1.05 & 1.0 & 1.04 & 1.08 \\ 1.46 & 1.25 & 1.05 & 1.0 & 1.01 \\ 1.77 & 1.46 & 1.15 & 1.02 & 1.0 \end{bmatrix}, \quad F = \begin{bmatrix} 3.18 \\ 2.03 \\ 0.88 \\ 0.26 \\ 0.03 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.06 & -0.01 & 0.01 & -0.0 & 0.0 \\ 0.27 & 0.13 & -0.02 & 0.01 & -0.0 \\ 0.30 & 0.33 & 0.15 & -0.03 & 0.0 \\ 0.26 & 0.25 & 0.29 & 0.13 & -0.0 \\ 0.08 & 0.09 & 0.08 & 0.10 & 0.03 \end{bmatrix}, \quad \tilde{U}_p = \begin{bmatrix} u_1 & 0 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 \\ 0 & 0 & 0 & u_4 & 0 \\ 0 & 0 & 0 & 0 & u_5 \end{bmatrix},$$

and \hat{H} constructed by $K^v \tilde{U}_p P$

$$\hat{H}^T = \begin{bmatrix} 0.06u_1 + 0.27u_2 + 0.33u_3 + 0.31u_4 + 0.10u_5 \\ -0.01u_1 + 0.13u_2 + 0.35u_3 + 0.28u_4 + 0.10u_5 \\ 0.01u_1 - 0.02u_2 + 0.15u_3 + 0.30u_4 + 0.08u_5 \\ -0.01u_1 + 0.02u_2 - 0.03u_3 + 0.13u_4 + 0.10u_5 \\ 0.03u_5 \end{bmatrix}.$$

Substituting these values in (53) gives the following system of linear equations

$$F^T - \hat{H}^T = 0,$$

which is equivalent to

$$\begin{cases} 0.06u_1 + 0.27u_2 + 0.33u_3 + 0.31u_4 + 0.10u_5 - 3.18 = 0, \\ -0.01u_1 + 0.13u_2 + 0.35u_3 + 0.28u_4 + 0.10u_5 - 2.03 = 0, \\ 0.01u_1 - 0.02u_2 + 0.15u_3 + 0.30u_4 + 0.08u_5 - 0.88 = 0, \\ -0.01u_1 + 0.02u_2 - 0.03u_3 + 0.13u_4 + 0.10u_5 - 0.26 = 0, \\ 0.03u_5 - 0.03 = 0. \end{cases}$$

From this system, the coefficients u_i , $i = 1, 2, \dots, 5$ are computed as

$$U = [6.99 \quad 4.91 \quad 2.71 \quad 1.52 \quad 1.05],$$

and the approximate solution of the integral equation is obtained by

$$u(t) \approx \sum_{i=1}^5 u_i C_i(t),$$

which is fitted on the exact solution demonstrated in Fig. 2. In addition, error function is shown for $N = 15$.

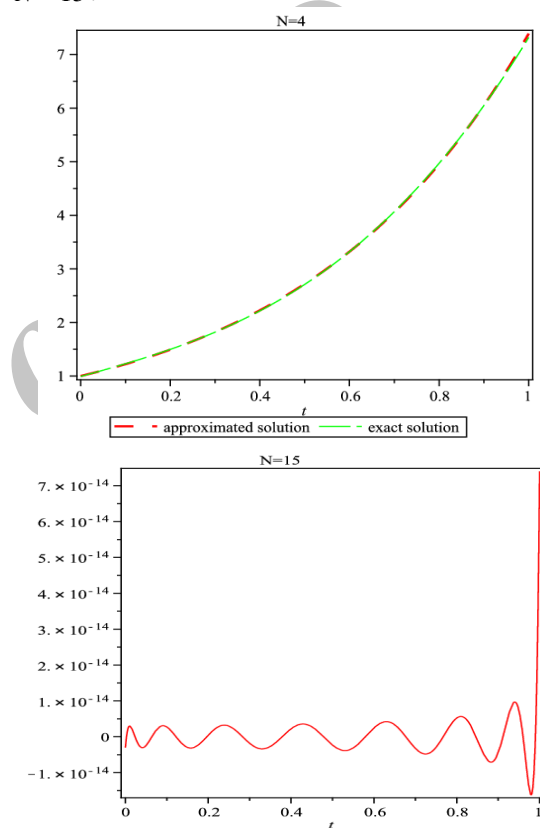


Fig. 2. The exact and approximated solution coincided with each other for $N = 4$. The error function for $N = 15$ is also shown. See the Example 2.

Example 3. Consider the Volterra integral equation of the second kind as follows [35, 36]

$$u(t) = \cos(t) - \int_0^t (t-s)\cos(t-s)u(s)ds,$$

where the exact solution is $u(t) = \frac{1}{3}(2\cos\sqrt{3}t + 1)$.

Corresponding to (39)

$$k_1(t, s) = (t-s)\cos(t-s), \quad p = 1, \\ \lambda_1 = -1, \quad \lambda_2 = 0 \quad m = 0,$$

$$f(t) = \cos t, \quad \alpha_0(t) = 1,$$

and we approximate the solution as follows

$$u(t) \approx U^T \Theta_N(t).$$

Similarly, we obtain the needed matrices and vectors for $N = 4$ as follows

$$\hat{A}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \tilde{A}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_0 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}, \quad F = \begin{bmatrix} 0.56 \\ 0.70 \\ 0.88 \\ 0.98 \\ 1.0 \end{bmatrix},$$

$$K^v = \begin{bmatrix} 0.0 & 0.18 & 0.42 & 0.55 & 0.55 \\ -0.18 & 0.0 & 0.28 & 0.49 & 0.55 \\ -0.42 & -0.28 & 0.0 & 0.28 & 0.42 \\ -0.55 & -0.49 & -0.28 & 0.0 & 0.18 \\ -0.55 & -0.55 & -0.42 & -0.18 & 0.0 \end{bmatrix}, \quad P = \begin{bmatrix} 0.06 & -0.01 & 0.01 & -0.0 & 0.0 \\ 0.27 & 0.13 & -0.02 & 0.01 & -0.0 \\ 0.30 & 0.33 & 0.15 & -0.03 & 0.0 \\ 0.26 & 0.25 & 0.29 & 0.13 & -0.0 \\ 0.08 & 0.09 & 0.08 & 0.10 & 0.03 \end{bmatrix},$$

$$\tilde{U}_p = \begin{bmatrix} u_1 & 0 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 \\ 0 & 0 & 0 & u_4 & 0 \\ 0 & 0 & 0 & 0 & u_5 \end{bmatrix}, \quad \hat{H}^T = \begin{bmatrix} 0.05u_2 + 0.13u_3 + 0.15u_4 + 0.05u_5 \\ 0.09u_3 + 0.12u_4 + 0.05u_5 \\ 0.01u_2 + 0.08u_4 + 0.03u_5 \\ -0.01u_2 + 0.01u_3 + 0.02u_5 \\ 0.0 \end{bmatrix}.$$

Substituting these values in (53) gives the nonlinear system of equations

$$\begin{cases} u_1 + 0.05u_2 + 0.13u_3 + 0.15u_4 + 0.05u_5 - 0.56 = 0, \\ u_2 + 0.09u_3 + 0.12u_4 + 0.05u_5 - 0.70 = 0, \\ u_3 + 0.01u_2 + 0.08u_4 + 0.03u_5 - 0.88 = 0, \\ u_4 - 0.01u_2 + 0.01u_3 + 0.02u_5 - 0.98, \\ u_5 - 1.0 = 0. \end{cases}$$

From this system, the coefficients u_i , $i = 1, 2, \dots, 5$ are computed as

$$U = [0.25 \quad 0.46 \quad 0.77 \quad 0.96 \quad 1.0]^T.$$

and the approximate solution of the mixed Volterra-Fredholm integral equation is obtained by

$$u(t) \approx \sum_{i=1}^5 u_i C_i(t),$$

which coincides for 3-digit arithmetic with the exact solution. The method for $N = 10$ gives 11 true digits. The results are reported in Table 1.

Table 1. The infinity norm of error functions for different N .

Example	$N = 5$	$N = 15$	$N = 30$
1	2.05×10^{-2}	7.82×10^{-3}	8.74×10^{-4}
2	1.17×10^{-3}	7.52×10^{-14}	5.05×10^{-36}
3	2.41×10^{-5}	1.04×10^{-17}	8.01×10^{-41}
4	2.01×10^{-52}	3.57×10^{-57}	4.51×10^{-59}
5	3.51×10^{-4}	1.37×10^{-13}	2.91×10^{-30}
6	3.10×10^{-5}	1.94×10^{-17}	2.73×10^{-40}
7	4.51×10^{-5}	1.48×10^{-18}	1.44×10^{-43}
8	4.06×10^{-6}	3.82×10^{-20}	1.12×10^{-48}
9	8.31×10^{-2}	5.59×10^{-5}	3.61×10^{-17}

Example 4. Consider the following mixed Volterra-Fredholm integral equation [37, 24]

$$u(t) = \frac{-1}{30}t^6 + \frac{1}{3}t^4 - t^2 + \frac{5}{3}t - \frac{5}{4} + \int_0^t (t-s)[u(s)]^2 ds + \int_0^1 (t+s)u(s)ds, \quad t, s \in [0, 1],$$

where the exact solution is $u(t) = t^2 - 2$. According to (39)

$$k_1(t, s) = t-s, \quad k_2 = t+s, \quad p = 2, \quad q = 1, \quad \lambda_1 = \lambda_2 = 1 \quad m = 0,$$

$$f(t) = \frac{-1}{30}t^6 + \frac{1}{3}t^4 - t^2 + \frac{5}{3}t - \frac{5}{4}, \quad \alpha_0(t) = 1,$$

and we approximate the solution as follows

$$u(t) \approx U^T \Theta_N(t).$$

Then, for $N = 4$, we obtain the vector and matrices as follows

$$\hat{A}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \tilde{A}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_0 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}, \quad F = \begin{bmatrix} -0.30 \\ -0.43 \\ -0.65 \\ -0.95 \\ -1.21 \end{bmatrix},$$

$$K^f = \begin{bmatrix} 1.95 & 1.77 & 1.48 & 1.18 & 1.0 \\ 1.77 & 1.59 & 1.29 & 1.0 & 0.82 \\ 1.48 & 1.29 & 1.0 & 0.71 & 0.52 \\ 1.18 & 1.0 & 0.71 & 0.41 & 0.23 \\ 1.0 & 0.82 & 0.52 & 0.23 & 0.05 \end{bmatrix}, \\ K^v = \begin{bmatrix} 0.0 & 0.18 & 0.48 & 0.77 & 0.95 \\ -0.18 & 0.0 & 0.29 & 0.59 & 0.77 \\ -0.48 & -0.29 & 0.0 & 0.29 & 0.48 \\ -0.77 & -0.59 & -0.29 & 0.0 & 0.18 \\ -0.95 & -0.77 & -0.48 & -0.18 & 0.0 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.06 & -0.01 & 0.01 & -0.0 & 0.0 \\ 0.27 & 0.13 & -0.02 & 0.01 & -0.0 \\ 0.30 & 0.33 & 0.15 & -0.03 & 0.0 \\ 0.26 & 0.25 & 0.29 & 0.13 & -0.0 \\ 0.08 & 0.09 & 0.08 & 0.10 & 0.03 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.08 & 0 & 0 & 0 & 0 \\ 0 & 0.26 & 0 & 0 & 0 \\ 0 & 0 & 0.31 & 0 & 0 \\ 0 & 0 & 0 & 0.26 & 0 \\ 0 & 0 & 0 & 0 & 0.08 \end{bmatrix},$$

$$\tilde{U}_p = \begin{bmatrix} u_1^2 & 0 & 0 & 0 & 0 \\ 0 & u_2^2 & 0 & 0 & 0 \\ 0 & 0 & u_3^2 & 0 & 0 \\ 0 & 0 & 0 & u_4^2 & 0 \\ 0 & 0 & 0 & 0 & u_5^2 \end{bmatrix}, \quad U_q = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix},$$

and \hat{H} , which is constructed by entries of $K^v \tilde{U}_p P$

$$\hat{H}^T = \begin{bmatrix} 0.049u_2^2 + 0.145u_3^2 + 0.203u_4^2 + 0.079u_5^2 \\ 0.002u_1^2 + 0.098u_3^2 + 0.147u_4^2 + 0.067u_5^2 \\ -0.003u_1^2 + 0.007u_2^2 + 0.084u_4^2 + 0.037u_5^2 \\ 0.003u_1^2 - 0.007u_2^2 + 0.008u_3^2 + 0.018u_5^2 \\ 0.001u_2^2 - 0.001u_3^2 + 0.001u_4^2 \end{bmatrix}.$$

Substituting these values in (53) gives the nonlinear system of equations

$$\begin{cases} 0.84u_1 - 0.46u_2 - 0.45u_3 - 0.31u_4 - 0.08u_5 + 0.30 - 0.05u_2^2 - 0.14u_3^2 - 0.20u_4^2 - 0.08u_5^2 = 0, \\ -0.15u_1 + 0.58u_2 - 0.40u_3 - 0.26u_4 - 0.07u_5 + 0.43 - 0.10u_3^2 - 0.15u_4^2 - 0.07u_5^2 = 0, \\ -0.12u_1 - 0.34u_2 + 0.69u_3 - 0.19u_4 - 0.04u_5 + 0.65 - 0.01u_2^2 - 0.08u_4^2 - 0.04u_5^2 = 0, \\ -0.10u_1 - 0.26u_2 - 0.22u_3 + 0.89u_4 - 0.02u_5 + 0.95 + 0.01u_2^2 - 0.01u_3^2 - 0.02u_5^2 = 0, \\ -0.08u_1 - 0.22u_2 - 0.16u_3 - 0.06u_4 + 1.0u_5 + 1.21 = 0. \end{cases}$$

From this system, the coefficients u_i , $i = 1, 2, \dots, 5$ are computed as

$$U = [-1.05 \quad -1.37 \quad -1.75 \quad -1.96 \quad -2.0]^T,$$

and the approximate solution of the mixed Volterra-Fredholm integral equation is obtained by

$$u(t) \approx \sum_{i=1}^5 u_i C_i(t).$$

The numerical results are reported in Table 1.

Example 5. Consider the following initial value problem [38, 39]

$$u'(t) + u(t) = 1 + 2t + \int_0^t (1+2t)e^{s(t-s)} u(s) ds, \quad u(0) = 1, \quad t \in [0, 1],$$

where $u(0) = 1$ and the exact solution is $u(t) = e^{t^2}$. With respect to (39),

$$k_1 = t(1+2t)e^{s(t-s)}, \quad p = 1, \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad m = 1,$$

$$f(t) = 1 + 2t, \quad \alpha_0(t) = 1, \quad \alpha_1(t) = 1,$$

and we approximate the solution as follows:

$$u'(t) \approx U^T \Theta_N(t).$$

Then, for $N = 4$, we obtain the vector and matrices as follows:

$$\hat{A}_0 = \hat{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \tilde{A}_0 = \tilde{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 2.95 \\ 2.59 \\ 2.0 \\ 1.41 \\ 1.05 \end{bmatrix}, \quad e_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$L_0^T = U^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}, \quad L_1^T = U^T P + e_0 = \begin{bmatrix} 0.06u_1 + 0.27u_2 + 0.30u_3 + 0.26u_4 + 0.08u_5 \\ -0.01u_1 + 0.13u_2 + 0.33u_3 + 0.25u_4 + 0.09u_5 \\ 0.01u_1 - 0.02u_2 + 0.15u_3 + 0.29u_4 + 0.08u_5 \\ 0.01u_2 - 0.03u_3 + 0.13u_4 + 0.10u_5 \\ 0.03u_5 \end{bmatrix},$$

$$K^v = \begin{bmatrix} 2.88 & 3.33 & 3.65 & 3.37 & 2.95 \\ 1.72 & 2.05 & 2.38 & 2.32 & 2.09 \\ 0.63 & 0.79 & 1.0 & 1.06 & 1.01 \\ 0.14 & 0.18 & 0.25 & 0.29 & 0.29 \\ 0.01 & 0.01 & 0.02 & 0.02 & 0.03 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.06 & -0.01 & 0.01 & -0.0 & 0.0 \\ 0.27 & 0.13 & -0.02 & 0.01 & -0.0 \\ 0.30 & 0.33 & 0.15 & -0.03 & 0.0 \\ 0.26 & 0.25 & 0.29 & 0.13 & -0.0 \\ 0.08 & 0.09 & 0.08 & 0.10 & 0.03 \end{bmatrix},$$

$$U_p = \begin{bmatrix} 0.06u_1 + 0.27u_2 + 0.30u_3 + 0.26u_4 + 0.08u_5 + 1 \\ -0.01u_1 + 0.13u_2 + 0.33u_3 + 0.25u_4 + 0.09u_5 + 1 \\ 0.01u_1 - 0.02u_2 + 0.15u_3 + 0.29u_4 + 0.08u_5 + 1 \\ 0.01u_2 - 0.03u_3 + 0.13u_4 + 0.10u_5 + 1 \\ 0.03u_5 + 1 \end{bmatrix},$$

$$\tilde{U}_p = \text{diagonal}((U_p)_{11}, (U_p)_{22}, (U_p)_{33}, (U_p)_{44}, (U_p)_{55}),$$

and \hat{H} , which is constructed by entries of $K^v \tilde{U}_p P$

$$\hat{H}^T = \begin{bmatrix} 3.30 + 0.15u_2 + 0.49u_3 + 0.70u_4 + 0.27u_5 \\ 0.02u_2 + 0.19u_3 + 0.36u_4 + 0.15u_5 + 1.81 \\ 0.52 + 0.01u_3 + 0.08u_4 + 0.04u_5 \\ 0.06 \\ 0.0 \end{bmatrix}.$$

Substituting these values in (53) gives the linear system of equations

$$\begin{cases} 1.06u_1 + 0.12u_2 - 0.19u_3 - 0.43u_4 - 0.19u_5 - 5.25 = 0, \\ -0.01u_1 + 1.11u_2 + 0.14u_3 - 0.11u_4 - 0.06u_5 - 3.40 = 0, \\ 0.01u_1 - 0.02u_2 + 1.14u_3 + 0.21u_4 + 0.04u_5 - 1.52 = 0, \\ 0.01u_2 - 0.03u_3 + 1.12u_4 + 0.09u_5 - 0.47 = 0, \\ 1.03u_5 - 0.05 = 0. \end{cases}$$

From this system, the coefficients u_i , $i = 1, 2, \dots, 5$, are computed as

$$U = [5.05 \quad 2.98 \quad 1.28 \quad 0.43 \quad 0.05]^T,$$

and the approximate solution of the mixed Volterra-Fredholm integral equation is obtained by

$$u(t) \approx \sum_{i=1}^5 u_i C_i(t),$$

which coincides for 2-digit arithmetic with the exact solution. Also, the norm infinity of error function for $N = 10$ shows 8 true digits with respect to the exact solution.

We are reminded that we used the 64 digits for solving the examples, however, we show the results with 2 digits. Compare the last equation in (54) with the obtained value for u_5 . It confirms digits can have a serious effect on the results. The results are reported in Table 1.

Example 6. Consider the following nonlinear Volterra integral equation of the second kind [40, 24]

$$u(t) = 1 + \sin^2 t - 3 \int_0^t \sin(t-s)[u(s)]^2 ds, \quad t \in [0, 1],$$

where the exact solution is $u(t) = \cos t$. Corresponding to (39)

$$k_1(t, s) = \sin(t-s), \quad p = 2, \quad \lambda_1 = -3, \quad \lambda_2 = 0, \quad m = 0,$$

$$f(t) = 1 + \sin^2 t, \quad \alpha_0(t) = 1,$$

and we approximate the solution as follows

$$u(t) \approx U^T \Theta_N(t).$$

The results are reported in Table 1.

Example 7. Consider the Fredholm integro-differential equation as follows [41]:

$$\begin{aligned} u''(t) + tu'(t) - tu(t) &= e^t - 2 \sin t \\ &+ \int_{-1}^1 \sin(s) e^{-s} u(s) ds, \quad t \in [-1, 1], \\ u(0) &= 1, \quad u'(0) = 1, \end{aligned}$$

where the exact solution is $u(t) = e^t$. According to (39)

$$\begin{aligned} k_1(t, s) &= \sin(s) e^{-s}, \quad q = 1, \quad \lambda_1 = 0, \quad \lambda_2 = 1, \quad m = 2, \\ f(t) &= e^t - 2 \sin t, \quad \alpha_0(t) = -t, \quad \alpha_1(t) = t, \quad \alpha_2(t) = 1, \end{aligned}$$

and we approximate the solution as follows

$$u''(t) \approx U^T \Theta_N(t).$$

The results are shown in Table 1.

Example 8. Consider the Volterra integro-differential equation as follows [42]

$$\begin{aligned} u''(t) &= \sinh t + \frac{1}{2} \cosh t \sinh t - \frac{1}{2} t - \int_0^t u^2(s) ds, \quad t \in [0, 1], \\ u(0) &= 0, \quad u'(0) = 1, \end{aligned}$$

where the exact solution is $u(t) = \sinh t$. According to (39)

$$\begin{aligned} k_1(t, s) &= 1, \quad q = 2, \quad \lambda_1 = -1, \quad \lambda_2 = 0, \quad m = 2, \\ f(t) &= \sinh t + \frac{1}{2} \cosh t \sinh t - \frac{1}{2} t, \\ \alpha_0(t) &= 0, \quad \alpha_1(t) = 0, \quad \alpha_2(t) = 1, \end{aligned}$$

and we approximate the solution as follows

$$u''(t) \approx U^T \Theta_N(t).$$

The results are reported in Table 1.

Example 9. Consider the integral equation of the first kind as follows [17]

$$\frac{4\pi \cos(4\pi t) + \sin(4\pi t) - 4\pi e^t}{(1+t^2)(1+16\pi^2)} - \int_0^t \frac{e^{t-s}}{1+s^2} u(s) ds = 0,$$

where the exact solution is $u(t) = \sin(4\pi t)$. According to (39)

$$k_1(t, s) = \frac{e^{t-s}}{1+s^2}, \quad q=1, \quad \lambda_1 = -1, \quad \lambda_2 = 0,$$

$$f(t) = \frac{4\pi \cos(4\pi t) + \sin(4\pi t) - 4\pi e^t}{(1+t^2)(1+16\pi^2)},$$

and we approximate the solution as follows:

$$u(t) \approx U^T \Theta_N(t).$$

We illustrate the numerical example in Table 1.

Conclusion

The Chebyshev cardinal functions and the associated operational matrices of integration P and product \tilde{F} are applied to solve the general type of linear and nonlinear integral equations. Moreover, the new defined matrix operations utilize the computations so that the proposed method can reduce the integral equation to an algebraic system without using the collocation scheme. The obtained results showed that this approach can solve the problem effectively with simple computations.

There are some notable points in the numerical results which we investigate in detail. Table 1 shows the maximum errors for $N = 5, 15$ and 30 . Looking carefully at the results shows the different behavior of error functions. In particular, Examples 1 and 4 have an irregular rate of convergence. Experimental results show the Fredholm integral equations of the first kind have the most ill-conditioned systems, which lead to weak accurate. Also, the best candidate integral equations for this method have the solution in the polynomial forms, even if the problem includes many terms or the higher order of differentiations.

The method of Chebyshev cardinal functions proposed in this paper can be extended to solve the more general type as follows

$$\sum_{j=0}^m \alpha_j(t) u^{(j)}(t) = f(t) + \lambda_1 \int_a^t k_1(t, s) F(t, s, u(s)) ds$$

$$+ \lambda_2 \int_a^b k_2(t, s) G(t, s, u(s)) ds,$$

under the initial conditions

$$u^{(j)}(a) = \mu_j, \quad j = 0, 1, \dots, m-1.$$

Here we can use Taylor series of F and G .

References

- [1] Atkinson, K. E. (1997). *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge University Press.
- [2] Jerri, A. J. (2004). *Introduction to Integral Equations with Applications*. John Wiley and Sons, INC.
- [3] Chari, M. V. K. & Salon, S. J. (2000). *Numerical Methods in Electromagnetism*. Academic Press.
- [4] Cheng, Z. (2002). Quantum effects of thermal radiation in a Kerr nonlinear blackbody. *J. Opt. Soc. Am. B.*, 1692-1705.
- [5] Chew, W. C., Tong, M. S. & Hu, B. (2009). integral equation methods for electromagnetic and elastic waves. Morgan & Claypool.
- [6] Liu, Y. & Ichiye, T. (1999). Integral equation theories for predicting water structure around molecules. *Biophys. Chem.*, 97-111.
- [7] Mirceski, V. & Tomovski, Z. (2008). Analytical solutions of integral equations for modelling of reversible electrode processes under voltammetric conditions. *J. Electroanal. Chem.*, 164-168.
- [8] Tang, Q. & Waxman, D. (2003). An integral equation describing an asexual population in a changing environment. *Nonlinear Analysis.*, 683-699.
- [9] Warnick, K. F. (2008). *Numerical Analysis for Electromagnetic Integral Equations*. Artech House.
- [10] Polyanin, A. D. & Manzhirov, A. V. (2008). *Handbook of Integral Equations*, Chapman & Hall/CRC.
- [11] Brunner, H. (2004). *Collocation Methods for Volterra Integral and Related Functional Equations*. Cambridge University Press.
- [12] Wazwaz, A. M. (1997). *A First Course in Integral Equations*. New Jersey, World Scientific.
- [13] Abbasbandy, S. (2007). Application of He's homotopy perturbation method to functional integral equations. *Chaos & Solitons Fractals.*, 1243-1247.
- [14] Yousefi, S. A., Lotfi, A. & Dehghan, M. (2009). He's variational iteration method for solving nonlinear mixed Volterra-Fredholm integral equations. *Comput. Math. Appl.*, 2172-2176.
- [15] Belbas, S. A. (2007). A new method for optimal control of Volterra integral equations. *Appl. Math. Comput.*, 1902-1915.
- [16] Mahmoudi, Y. (2005). Wavelet Galerkin method for numerical solution of nonlinear integral equation. *Appl. Math. Comput.*, 1119-1129.
- [17] Maleknejad, K., Mollapourasl, R. & Alizadeh, M. (2007). Numerical solution of Volterra type integral equation of the first kind with wavelet basis. *Appl. Math. Comput.*, 400-405.
- [18] Vainikko, G., Kivinukk, A. & Lippus, J. (2005). Fast solvers of integral equations of the second kind: wavelet methods. *J. Complexity.*, 243-273.
- [19] Yousefi, S. & Razzaghi, M. (2005). Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations. *Math. Comput. Simul.*, 1, 8.
- [20] Golbabai, A., Mammadov, M. & Seifollahi, S. (2009). Solving a system of nonlinear integral equations by an RBF network. *Comput. Math Appl.*, 1651-1658.
- [21] Kalos, M. H. & Whitlock, P. A. (2004). *Monte Carlo Methods*. Wiley-VCH.
- [22] Maleknejad, K., Shahrezaee, M. & Khatami, H.

- (2005). Numerical solution of integral equations system of the second kind by block-pulse functions. *Appl. Math. Comput.*, 15, 24.
- [23] Avazzadeh, Z., Heydari, M. & Loghmani, G. B. (2010). Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem. *Appl. Math. Modelling.*, 2374-2383.
- [24] Avazzadeh, Z., Heydari, M. & Loghmani, G. B. Solution of Volterra-Fredholm-Hammerstein integral equations using radial basis functions. Information Science, Submitted.
- [25] Maleknejad, K. & Derili, H. (2006). Numerical solution of integral equations by using combination of spline-collocation method and Lagrange interpolation. *Appl. Math. Comput.*, 1235-1244.
- [26] Mammen, E. & Yu, K. (2009). Nonparametric estimation of noisy integral equations of the second kind. *J. Korean Statist. Soc.*, 99-110.
- [27] Pathak, H. K., Khan, M. S. & Tiwari, R. (2007). A common fixed point theorem and its application to nonlinear integral equations. *Comput. Math. Appl.*, 961-971.
- [28] Boyd, J. P. (2000). *Chebyshev and Fourier Spectral Methods*. Dover Publications, Inc.
- [29] Babolian, E. & Fattahzadeh, F. (2007). Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration. *AMC.*, 1016-1022.
- [30] Babolian, E., Masouri, Z. & Varmazyar, S. H. (2008). New direct method to solve nonlinear Volterra-Fredholm integral and integro-differential equations using operational matrix with block-pulse functions. *Progress In Electromagnetics Research B.*, 59-76.
- [31] Lakestani, M. & Dehghan, M. (2010). The use of Chebyshev cardinal functions for the solution of a partial differential equation with an unknown time-dependent coefficient subject to an extra measurement. *J. Comput. Appl. Math.*, 669-678.
- [32] Lakestani, M. & Dehghan, M. (2010). Numerical solution of Riccati equation using the cubic B-spline scaling functions and Chebyshev cardinal functions. *Comput. Phys. Commun.*, 957-966.
- [33] Kythe, P. K. & Schäferkötter, M. R. (2005). *Handbook of Computational Method for Integration*. Chapman and Hall/CRC Press.
- [34] Shang, X. & Han, D. (2007). Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets. *Appl. Math. Comput.*, 440-444.
- [35] Blyth, W. F., May, R. L. & Widyaningsih, P. (2004). Volterra integral equations solved in Fredholm form using Walsh functions. *Anziam J.*, 269-282.
- [36] Reihani, M. H. & Abadi, Z. (2007). Rationalized Haar functions method for solving Fredholm and Volterra integral equations. *J. Comput. Appl. Math.*, 12, 20.
- [37] Ordokhani, Y. & Razzaghi, M. (2008). Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via a collocation method and rationalized Haar functions. *Appl. math. Lett.*, 4, 9.
- [38] Makroglou, A. (1980). Convergence of a block-by-block method for nonlinear Volterra integro-differential equations. *Math. Comp.*, 783-796.
- [39] Shahmorad, S. (2005). Numerical solution of the general form linear Fredholm-Volterra integro-differential equations by the Tau method with an error estimation. *Appl. Math. Comput.*, 1418-1429.
- [40] Conte, D. & Paternoster, B. (2009). Multistep collocation methods for Volterra Integral Equations. *Appl. Numer. Math.*, 1721-1736.
- [41] Yalçınbas, S. & Sezer, M. (2000). The approximate solution of high-order linear Volterra Fredholm integro-differential equations in terms of Taylor polynomials. *Appl. Math. Comput.*, 291-308.
- [42] Saberi-Nadjafi, J. & Ghorbani, A. (2009). He's homotopy perturbation method: An effective tool for solving nonlinear integral and integro-differential equations. *Comput. Math. Appl.*, In press, doi: 10.1016/j.camwa.2009.03.032.

به کارگیری توابع کاردینال چبیشف برای حل معادلات انتگرال-دیفرانسیل ولترا-فردهلم به وسیله ماتریس‌های عملیاتی

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چکیده:

در این مقاله، یک روش مستقیم موثر برای تعیین جواب تقریبی معادلات انتگرال-دیفرانسیل ولترا و فردهلم خطی و غیر خطی ارائه شده است. اساس این روش گسترش جواب تقریبی بر حسب توابع کاردینال چبیشف می‌باشد. ماتریس‌های عملیاتی برای انتگرال و ضرب با جزئیات کامل شرح داده شده است. این ماتریس‌ها در تبدیل معادله انتگرال به یک دستگاه معادلات جبری نقش مهمی را ایفا می‌کنند. مثال‌های روشنگر، نشان دهنده‌ی قابلیت اطمینان و کارآمدی روش ارائه شده است.

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