IJST (2012) A2: 97-113
Iranian Journal of Science & Technology
http://www.shirazu.ac.ir/en

Regular *AG*-groupoids characterized by $(\in, \in \vee q_k)$-fuzzy ideals

A. Khan[1]*, N. H. Sarmin[2], F. M. Khan[2] and B. Davvaz[3]

[1]*Department of Mathematics, COMSATS Institute of Information Technology Khyber Pukhtoon Khwa, Abbottabad, Pakistan*
[2]*Department of Mathematics, Faculty of Science, University of Technology, Malaysia 81310 UTM Johor Bahru, Johor, Malaysia*
[3]*Department of Mathematics, Yazd University Yazd, Iran*
E-mails: azhar4set@yahoo.com, faiz_zady@yahoo.com & bdavvaz@yahoo.com

Abstract

In this paper, we introduce a considerable machinery which permits us to characterize a number of special (fuzzy) subsets in AG-groupoids. Generalizing the concepts of $(\in, \in \vee q)$-fuzzy bi-ideals (interior ideal), we define $(\in, \in \vee q_k)$-fuzzy bi-ideals, $(\in, \in \vee q_k)$-fuzzy left (right)-ideals and $(\in, \in \vee q_k)$-fuzzy interior ideals in AG-groupoids and discuss some fundamental aspects of these ideals in AG-groupoids. We further define $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior ideals and give some of their basic properties in AG-groupoids. In the last section, we define lower/upper parts of $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals and investigate some characterizations of regular and intera-regular AG-groupoids in terms of the lower parts of $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideal of AG-groupoids.

Keywords: Fuzzy ideals; $(\in, \in \vee q_k)$-fuzzy ideals; $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals; regular AG-groupoids

1. Introduction

Mordeson et al. in [1] presented an up to date account of fuzzy sub-semigroups and fuzzy ideals of a semigroup. The book concentrates on theoretical aspects, but also includes applications in the areas of fuzzy coding theory, fuzzy finite state machines, and fuzzy languages. Basic results on fuzzy subsets, semigroups, codes, finite state machines, and languages are reviewed and introduced, as well as certain fuzzy ideals of a semigroup and advanced characterizations and properties of fuzzy semigroups. Kuroki [2] introduced the notion of fuzzy bi-ideals in semigroups. Kehayopulu applied the fuzzy concept in ordered semigroups and studied some properties of fuzzy left (right) ideals and fuzzy filters in ordered semigroups (see [3]). Fuzzy implicative and Boolean filters of R_0-algebra were initiated by Liu and Li (see [4]).

The idea of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [5, 6], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das (see [5]) gave the concepts of (α, β)-fuzzy subgroups by using the *belongs to* relation ($\in$) and *quasi-coincidence with* relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroup [7]. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. In algebra, the concept of (α, β)-fuzzy sets was introduced by Davvaz in [8], where $(\in, \in \vee q)$-fuzzy subnearrings (ideals) of a nearring were initiated and studied. With this objective in view, Ma et al. in [9], introduced the interval valued $(\in, \in \vee q)$-fuzzy ideals of pseudo-MV algebras and gave some important results of pseudo-MV algebras. Jun and Song (see [10])

*Corresponding author
Received: 23 July 2011 / Accepted: 17 October 2011

discussed general forms of fuzzy interior ideals in semigroups, also see [11]. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [12] and gave some properties of fuzzy bi-ideals in terms of $(\in, \in\vee q)$-fuzzy bi-ideals. Jun et al. [11] gave the concept of a generalized fuzzy bi-ideal in ordered semigroups and characterized regular ordered semigroups in terms of this notion. Davvaz et al. used the idea of generalized fuzzy sets in hyperstructures and introduced different generalized fuzzy subsystems of hyperstructures, e.g., see references. In [13], Ma et al. introduced the concept of a generalized fuzzy filter of R_0-algebra and provided some properties in terms of this notion. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra [14-23]. The concept of an (α, β)-fuzzy interior ideal in ordered semigroups was first introduced by Khan and Shabir in [24], where some basic properties of (α, β)-fuzzy interior ideals were discussed. Generalizing the concept of $(\in, \in\vee q)$-fuzzy subalgebas of BCK/BCI-algebras, Jun [25] introduced the concept of $(\in, \in\vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras and defined $x_\lambda q_k F$, as $F(x)+\lambda+k>1$, where $k\in[0,1)$. In [26], Shabir et al. discussed $(\in, \in\vee q_k)$-fuzzy ideals in semigroups.

On the other hand, the concept of a *left almost semigroup* (LA-semigroup) [27] was first introduced by Kazim and Naseeruddin in 1972. In [28], the same structure was called a *left invertive groupoid*. Protic and Stevanivic [29] called it an *Abel-Grassmann's groupoid* abbreviated as AG-groupoid, a groupoid whose elements satisfy the *left invertive law*: $(ab)c=(cb)a$ for all $a,b,c\in G$. An AG-groupoid is the midway structure between a *commutative semigroup* and a *groupoid* [30]. It is a useful non-associative structure with wide range of applications in the theory of flocks [31]. In an AG-groupoid the *medial law*, $(ab)(cd)=(ac)(bd)$ for all $a,b,c,d\in G$ (see [27]) holds. If there exists an element e in an AG-groupoid G such that $ex = x$ for all $x\in G$ then G is called an *AG-groupoid with left identity e*. If an AG-groupoid G has the *right identity* then G is a *commutative monoid*. If an AG-groupoid G contains left identity then $(ab)(cd)=(dc)(ba)$ holds for all $a,b,c,d\in G$. Also, $a(bc)=b(ac)$ holds for all $a,b,c\in G$.

In this paper, we generalize the concept of $(\in, \in\vee q)$-fuzzy bi-ideals given in [32], $(\in, \in\vee q)$-fuzzy left (right) ideals and $(\in, \in\vee q)$-fuzzy interior ideals given in [33] and define $(\in, \in\vee q_k)$-fuzzy bi-ideals, $(\in, \in\vee q_k)$-fuzzy left (right) ideals and $(\in, \in\vee q_k)$-fuzzy interior ideals in AG-groupoids and discuss some fundamental aspects of these ideals in AG-groupoids. We further define $(\bar{\in}, \bar{\in}\vee \overline{q_k})$-fuzzy bi-ideals and $(\bar{\in}, \bar{\in}\vee \overline{q_k})$-fuzzy interior ideals and give some of their basic properties in AG-groupoids. In the last section, we define upper/lower parts of $(\in, \in\vee q_k)$-fuzzy bi-ideals and characterize regular and intra-regular AG-groupoids in terms of the lower parts of an $(\in, \in\vee q_k)$-fuzzy bi-ideal and $(\in, \in\vee q_k)$-fuzzy left (resp. right) ideals of G.

2. Preliminaries

Throughout this paper, G will denote an AG-groupoid unless otherwise stated. For basic definitions we refer to [34, 35]. For subsets A,B of an AG-groupoid G, we denote by $AB=\{ab\in G \mid a\in A, b\in B\}$. A non-empty subset A of G is called a *left (right) ideal* of G if $GA\subseteq A (AG\subseteq A)$. A nonempty subset A of an AG-groupoid G is called an *AG-subgroupoid* of G if $A^2\subseteq A$. An AG-subgroupoid A of G is called a *bi-ideal* of G if $(AG)A\subseteq A$. An AG-subgroupoid A of G is called an *interior ideal* of G if $(GA)G\subseteq A$.

Now, we give some fuzzy logic concepts.

Let G be an AG-groupoid. By a fuzzy subset F of an AG-groupoid G, we mean a mapping, $F : G\to[0,1]$.

We denote by $\mathbf{F}(G)$ the set of all fuzzy subsets of G. The order relation $\subseteq$ on $\mathbf{F}(G)$ is defined as follows:

$F_1\subseteq F_2$ if and only if $F_1(x)\le F_2(x)$ for all $x\in G$ and for all $F_1, F_2\in \mathbf{F}(G)$.

A fuzzy subset F of G is called a *fuzzy AG-*

subgroupoid if $F(xy) \geq \min\{F(x), F(y)\}$ for all $x, y \in G$. F is called a *fuzzy left (right) ideal* of G if $F(xy) \geq F(y)(F(xy) \geq F(x))$ for all $x, y \in G$. F is called a *fuzzy ideal* of G if it is both a fuzzy left and right ideal of G.

Let G be an AG-groupoid and F a fuzzy subset of G. Then F is called a *fuzzy bi-ideal* of G, if it satisfies the following conditions:

(1) $(\forall x, y \in G)(F(xy) \geq \min\{F(x), F(y)\})$.

(2) $(\forall x, y, z \in G)(F((xy)z) \geq \min\{F(x), F(z)\})$.

A fuzzy subset F of G is called a *fuzzy interior ideal* of G if it satisfies the following conditions:

(1) $(\forall x, y \in G)(F(xy) \geq \min\{F(x), F(y)\})$.

(2) $(\forall x, a, y \in G)(F((xa)y) \geq F(a))$.

Let F be a fuzzy subset of G, and $\phi \neq A \subseteq G$ then the *characteristic function* F_A of A is defined as:

$$F_A : G \to [0,1], a \mapsto F_A(a) := \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ if } a \notin A \end{cases}$$

Let F and H be the two fuzzy subsets of AG-groupoid G. Then the product $F \circ H$ is defined by

$$(F \circ H)(x) = \begin{cases} \bigvee_{x=yz} \min\{F(y), H(z)\} & \text{if for } y, z \in G,\ x = yz, \\ 0 & \text{if } x \neq yz. \end{cases}$$

For an AG-groupoid G, the fuzzy subsets "0" and "1" of G are defined as follows:

$$0 : G \to [0,1] \mid x \to 0(x) = 0,$$
$$1 : G \to [0,1] \mid x \to 1(x) = 1.$$

Clearly, the fuzzy subset 0 (resp. 1) of G is the least (resp. the greatest) element of the AG-groupoid ($\mathbf{F}(G), \circ$) (that is, $0 \leq F$ and $F \leq 1$ for every $F \in \mathbf{F}(G)$). The fuzzy subset 0 is the zero element of ($\mathbf{F}(G), \circ$) (that is, $F \circ 0 = 0 \circ F = 0$ and $0 \leq F$ for every $F \in \mathbf{F}(G)$). Moreover, $F_S = 1$ and $F_\phi = 0$.

2.1. Lemma (cf. [32]). Let G be an AG-groupoid and F a fuzzy subset of G. Then F is a fuzzy bi-ideal of G if and only if F_A is a fuzzy bi-ideal of G.

Let G be an AG-groupoid and F a fuzzy subset of G. Then for every $\lambda \in (0,1]$ the set $U(F;\lambda) := \{x \mid x \in G \text{ and } F(x) \geq \lambda\}$ is called a *level set* of F with support x and value λ.

2.2. Theorem (cf. [32]). Let G be an AG-groupoid and F a fuzzy subset of G. Then F is a fuzzy bi-ideal of G if and only if $U(F;\lambda)(\neq \phi)$ is a bi-ideal of G for every $\lambda \in (0,1]$.

2.3. Theorem (cf. [33]). Let G be an AG-groupoid and F a fuzzy subset of G. Then F is a fuzzy interior ideal of G if and only if $U(F;\lambda)(\neq \phi)$ is an interior ideal of G for every $\lambda \in (0,1]$.

Let F be a fuzzy subset of G, then the set of the form

$$F(y) := \begin{cases} \lambda(\neq 0) & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is called a *fuzzy point* with support x and value λ and is denoted by x_λ. A fuzzy point x_λ is said to *belong to* (resp. *quasi-coincidence*) with a fuzzy set F, written as $x_\lambda \in F$ (resp. $x_\lambda q F$) if $F(x) \geq \lambda$ (resp. $F(x) + \lambda > 1$). If $x_\lambda \in F$ or $x_\lambda q F$, then we write $x_\lambda \in \vee q F$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

3. Fuzzy ideals of type $(\in, \in \vee q_k)$.

In what follows let G denote an AG-groupoid and k an arbitrary element of $[0,1)$ unless otherwise specified. Generalizing the concept of $x_\lambda q F$, Jun defined $x_\lambda q_k F$, as $F(x) + \lambda + k > 1$, where $k \in [0,1)$ (see [25]). Generalizing the concepts of [32, 33], here we extend our studies to more general forms of fuzzy bi-deals, fuzzy left (right) ideals and fuzzy interior ideals in AG-groupoids. In this section we define the notions of $(\in, \in \vee q_k)$-fuzzy bi-ideals of an AG-groupoid which is a generalization of $(\in, \in \vee q)$-fuzzy bi-ideals and investigate some of their properties in terms of $(\in, \in \vee q_k)$-fuzzy bi-ideals.

3.1. Definition A fuzzy subset F of G is called an $(\in,\in\vee q_k)$-*fuzzy bi-ideal* of G if it satisfies the conditions:

(B1) $(\forall x,y\in G)(\forall r,t\in(0,1])(x_r\in F, y_t\in F\rightarrow (xy)_{\min\{r,t\}}\in\vee q_k F)$,

(B2) $(\forall x,a,y\in G)(\forall r,t\in(0,1])(x_r\in F, y_t\in F\rightarrow ((xa)y)_{\min\{r,t\}}\in\vee q_k F)$.

3.2. Example Let $S=\{a,b,c,d,e\}$ be an AG-groupoid with the following multiplication table

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

Then $(G,\cdot)$ is an AG-groupoid and $\{a\},\{a,c,d,e\}$ and G are bi-ideals of G. Define a fuzzy subset $F\ :\ S\rightarrow[0,1]$ as follows:

$F(a)=0.8,\ \ F(c)=0.6,\ \ F(d)=0.4,\ \ F(e)=0.3,\ \ F(b)=0.1.$

Then F is an $(\in,\in\vee q_k)$-fuzzy bi-ideal for every $k\in(0,\frac{1-k}{2}]$ when $k=0.4$.

3.3. Proposition Let A be an AG-subgroupoid of G and F a fuzzy subset of G defined as follows:

$$F(x)=\begin{cases}\geq\frac{1-k}{2}\text{ if } x\in A,\\ 0\text{ otherwise.}\end{cases}$$

Then

(1) F is an $(q,\in\vee q_k)$-fuzzy AG-subgroupoid of G.

(2) F is an $(\in,\in\vee q_k)$-fuzzy AG-subgroupoid of G.

Proof: The proof follows from [26].

3.4. Theorem Let F be a fuzzy subset of G. Then F is an $(\in,\in\vee q_k)$-fuzzy AG-subgroupoid of G if and only if $F(xy)\geq\min\{F(x),F(y),\frac{1-k}{2}\}$ for all $x,y\in S$ and $k\in[0,1)$.

Proof: Let F be an $(\in,\in\vee q_k)$-fuzzy AG-subgroupoid of G. If there exist $x,y\in G$ such that $F(xy)<\min\{F(x),F(y),\frac{1-k}{2}\}$. Choose $\lambda\in(0,1]$ such that $F(xy)<\lambda\leq\min\{F(x),F(y),\frac{1-k}{2}\}$. Then $x_\lambda\in F$ and $y_\lambda\in F$ but $F(xy)<\lambda$ and $F(xy)+\lambda+k<\frac{1-k}{2}+\frac{1-k}{2}+k=1,$ so $(xy)_{\min\{\lambda,\lambda\}}=(xy)_\lambda\overline{\in\vee q_k}F$, a contradiction. Hence $F(xy)\geq\min\{F(x),F(y),\frac{1-k}{2}\}$ for all $x,y\in G$. Conversely, assume that $F(xy)\geq\min\{F(x),F(y),\frac{1-k}{2}\}$. Let $x_{\lambda_1}\in F$ and $y_{\lambda_2}\in F$ for $\lambda_1,\lambda_2\in(0,1]$. Then $F(x)\geq\lambda_1$ and $F(y)\geq\lambda_2$ and by hypothesis, $F(xy)\geq\min\{F(x),F(y),\frac{1-k}{2}\}\geq\min\{\lambda_1,\lambda_2,\frac{1-k}{2}\}$.

If $\min\{\lambda_1,\lambda_2\}>\frac{1-k}{2}$, then $F(xy)>\frac{1-k}{2}$ and so $F(xy)+\min\{\lambda_1,\lambda_2\}+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$, it follows that $(xy)_{\min\{\lambda_1,\lambda_2\}}q_kF$. If $F(xy)\leq\frac{1-k}{2}$, then $F(xy)\geq\min\{\lambda_1,\lambda_2\}$ and so $(xy)_{\min\{\lambda_1,\lambda_2\}}\in F$. Thus $(xy)_{\min\{\lambda_1,\lambda_2\}}\in\vee q_kF$ and F is an $(\in,\in\vee q_k)$-fuzzy AG-subgroupoid of G.

3.5. Theorem A fuzzy subset F is an $(\in,\in\vee q_k)$-fuzzy AG-subgroupoid of G if and only if $U(F;\lambda)(\neq\phi)$ is an AG-subgroupoid of G for all $\lambda\in(0,\frac{1-k}{2}]$.

Proof: Suppose that F is an $(\in,\in\vee q_k)$-fuzzy AG-subgroupoid of G and $x,y\in U(F;\lambda)$ for some $\lambda\in(0,\frac{1-k}{2}]$. Then $F(x)\geq\lambda$ and $F(y)\geq\lambda$, and by hypothesis $F(xy)\geq\min\{F(x),F(y),\frac{1-k}{2}\}\geq\min\{\lambda,\lambda,\frac{1-k}{2}\}=\lambda$. Hence $xy\in U(F;\lambda)$, and so $U(F;\lambda)$ is an AG-subgroupoid of G. Conversely, assume that $U(F;\lambda)(\neq\phi)$ is an AG-subgroupoid of G for all $\lambda\in(0,\frac{1-k}{2}]$. If there exist $x,y\in G$ such that $F(xy)<\min\{F(x),F(y),\frac{1-k}{2}\}$, then choose $\lambda\in(0,\frac{1-k}{2}]$ such that $F(xy)<\lambda\leq\min\{F(x),F(y),\frac{1-k}{2}\}$. Thus $x,y\in U(F;\lambda)$ but $xy\notin U(F;\lambda)$, a contradiction. Hence

$F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$ for all $x, y \in G$ and $k \in [0,1)$.

3.6. Proposition Let A be a bi-ideal of G and F a fuzzy subset of G defined as follows:

$$F(x) = \begin{cases} \geq \frac{1-k}{2} \text{ if } x \in A, \\ 0 \text{ otherwise.} \end{cases}$$

Then

(1) F is a $(q, \in \vee q_k)$-fuzzy bi-ideal of G.

(2) F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G.

Proof: The proof follows from [26].

3.7. Theorem Let F be a fuzzy subset of G. Then F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G if and only if the following conditions are satisfied:

$(B3)$ $(\forall x, y \in G)(\forall k \in [0,1))(F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\})$,

$(B4)$ $(\forall x, a, y \in G)(\forall k \in [0,1))(F((xa)y) \geq \min\{F(x), F(y), \frac{1-k}{2}\})$.

Proof: $(B1) \Leftrightarrow (B3)$ follows from Theorem 3.4. $(B2) \Rightarrow (B4)$. If there exist $x, a, y \in G$ such that $F((xa)y) < \min\{F(x), F(y), \frac{1-k}{2}\}$. Choose $\lambda \in (0,1]$ such that $F((xa)y) < \lambda \leq \min\{F(x), F(y), \frac{1-k}{2}\}$. Then $x_\lambda \in F$ and $y_\lambda \in F$ but $F((xa)y) < \lambda$ and $F((xa)y) + \lambda + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $((xa)y)_{\min\{\lambda,\lambda\}} = ((xa)y)_\lambda \overline{\in \vee q_k} F$, a contradiction. Hence $F((xa)y) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$ for all $x, a, y \in G$.

$(B4) \Rightarrow (B2)$. Assume that $F((xa)y) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$. Let $x_{\lambda_1} \in F$ and $y_{\lambda_2} \in F$ for some $\lambda_1, \lambda_2 \in (0,1]$. Then $F(x) \geq \lambda_1$ and $F(y) \geq \lambda_2$ and by hypothesis,

$$F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\} \geq \min\{\lambda_1, \lambda_2, \frac{1-k}{2}\}.$$

If $\min\{\lambda_1, \lambda_2\} > \frac{1-k}{2}$, then $F(xy) > \frac{1-k}{2}$ and so $F(xy) + \min\{\lambda_1, \lambda_2\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, it follows that $(xy)_{\min\{\lambda_1,\lambda_2\}} q_k F$. If $F(xy) \leq \frac{1-k}{2}$, then $F(xy) \geq \min\{\lambda_1, \lambda_2\}$ and so $(xy)_{\min\{\lambda_1,\lambda_2\}} \in F$. Thus $(xy)_{\min\{\lambda_1,\lambda_2\}} \in \vee q_k F$ and F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G.

If we take $k = 0$ in Theorem 3.7, then we have the following corollary:

3.8. Corollary (cf. [32]) Let F be a fuzzy subset of G. Then F is an $(\in, \in \vee q)$-fuzzy bi-ideal of G if and only if the following conditions are satisfied:

(1) $(\forall x, y \in G)(F(xy) \geq \min\{F(x), F(y), 0.5\})$,

(2) $(\forall x, a, y \in G)(F((xa)y) \geq \min\{F(x), F(y), 0.5\})$

3.9. Theorem A fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G if and only if $U(F;\lambda)(\neq \phi)$ is a bi-ideal of G for all $\lambda \in (0, \frac{1-k}{2}]$.

Proof: Suppose that F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G and let $x, a, y \in G$ be such that $x, y \in U(F;\lambda)$ for some $\lambda \in (0, \frac{1-k}{2}]$. Then $F(x) \geq \lambda$ and $F(y) \geq \lambda$ and by hypothesis

$$F((xa)y) \geq \min\{F(x), F(y), \frac{1-k}{2}\} \geq \min\{\lambda, \lambda, \frac{1-k}{2}\} = \lambda.$$

Hence $(xa)y \in U(F;\lambda)$. Conversely, assume that $U(F;\lambda)(\neq \phi)$ is a bi-ideal of G for all $\lambda \in (0, \frac{1-k}{2}]$. If there exist $x, y \in G$ such that $F(xy) < \min\{F(x), F(y), \frac{1-k}{2}\}$. Then choose $\lambda \in (0, \frac{1-k}{2}]$ such that $F((xa)y) < \lambda \leq \min\{F(x), F(y), \frac{1-k}{2}\}$. Thus $x, y \in U(F;\lambda)$ but $(xa)y \notin U(F;\lambda)$, a contradiction. Hence $F((xa)y) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$ for all $x, a, y \in G$ and $k \in [0,1)$. The remaining proof is a consequence of the Theorem 3.5. Therefore F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G.

If we take $k = 0$ in Theorem 3.9, then we have the following corollary:

3.10. Corollary (cf. [32]) A fuzzy subset F is an $(\in, \in \vee q)$-fuzzy bi-ideal of G if and only if $U(F;\lambda)(\neq \phi)$ is a bi-ideal of G for all $\lambda \in (0, 0.5]$.

3.11. Remark A fuzzy subset F of an AG-groupoid G is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G if and only if it satisfies conditions $(B3)$ and $(B4)$ of Theorem 3.7.

3.12. Remark By Remark 3.11, every fuzzy bi-ideal of an AG-groupoid G is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of G. However, the converse is not true, in general.

3.13. Example Consider the AG-groupoid as given in Example 3.2, and define a fuzzy subset F as follows:

$$F(a)=0.8,\ \ F(c)=0.6,\ \ F(d)=0.4,\ \ F(e)=0.3,\ \ F(b)=0.1.$$

Then F is an $(\in,\in\vee q_{0.4})$-fuzzy bi-ideal of G. But

i) F is not an $(\in,q_{0.4})$-fuzzy bi-ideal of G, since $d_{0.28}\in F$ but $(dd)_{0.28\wedge 0.28}=e_{0.28}\overline{q_{0.4}}F.$

ii) F is not an $(q_{0.4},\in)$-fuzzy bi-ideal of G, since $c_{0.58}q_{0.4}F$ and $e_{0.88}q_{0.4}F$ but $(ce)_{0.58\wedge 0.88}=d_{0.58}\overline{\in}F.$

iii) F is not an $(\in\vee q_{0.4},q_{0.4})$-fuzzy bi-ideal of G, since $c_{0.28}\in\vee q_{0.4}F$ but $(cc)_{0.28\wedge 0.28}=e_{0.28}\overline{\in\vee q_{0.4}}F.$

Thus F is not an (α,β)-fuzzy bi-ideal of G, for every $\alpha,\beta\in\{\in,q_{0.4},\in\vee q_{0.4}\}$.

3.14. Proposition Every $(\in,\in)$-fuzzy bi-ideal of G is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of G.

Proof: It is straightforward, since for $x_\lambda\in F$ we have $x_\lambda\in\vee q_kF$ for all $x\in G$.

The converse of Proposition 3.14, is not true in general, as shown in the following example:

3.15. Example Consider an AG-groupoid G and a fuzzy subset F as defined in Example 3.2, then F is an $(\in,\in\vee q_{0.4})$-fuzzy bi-ideal of G but F is not an $(\in,\in)$-fuzzy bi-ideal of G, since $d_{0.38}\in F$ but $(dd)_{0.38\wedge 0.38}=e_{0.38}\overline{\in}F.$

3.16. Proposition Every $(\in\vee q_k,\in\vee q_k)$-fuzzy bi-ideal G is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of G.

Proof: The proof is straightforward.
The converse of Proposition 3.16, is not true in general, as given in the following example.

3.17. Example Consider the AG-groupoid G as given in Example 3.2, and define a fuzzy subset F as follows:

$$F(a)=0.8,\ \ F(c)=0.6,\ \ F(d)=0.4,\ \ F(e)=0.2,\ \ F(b)=0.1.$$

Then F is an $(\in,\in\vee q_{0.4})$-fuzzy bi-ideal of G but F is not an $(\in\vee q_{0.4},\in\vee q_{0.4})$-fuzzy bi-ideal of G, since $d_{0.26}\in\vee q_{0.4}F$ but $(dd)_{0.26\wedge 0.26}=e_{0.26}\overline{\in\vee q_{0.4}}F.$

3.18. Definition A fuzzy subset F of G is called an $(\in,\in\vee q_k)$-*fuzzy left* (resp. *right*) *ideal* of G if the following condition holds:

$$(\forall x,y\in S)(\forall k\in[0,1))(y_\lambda\in F\rightarrow(xy)_\lambda\in\vee q_kF(\text{resp. }(yx)_\lambda\in\vee q_kF).$$

3.19. Theorem Let A be a left (resp. right) ideal of G and F a fuzzy subset of G defined as follows:

$$F(x)=\begin{cases}\geq\frac{1-k}{2} & \text{if } x\in A,\\ 0 & \text{otherwise.}\end{cases}$$

Then

(1) F is an $(q,\in\vee q_k)$-fuzzy left (resp. right) ideal of G.

(2) F is an $(\in,\in\vee q_k)$-fuzzy left (resp. right) ideal of G.

Proof: (1) Let $x,y\in G$ and $\lambda\in(0,1]$ be such that $y_\lambda qF$, then $y\in A$, $\lambda(y)+\lambda>1$. Since A is a left ideal of G, we have $xy\in A$. Thus $F(xy)\geq\frac{1-k}{2}$. If $\lambda\leq\frac{1-k}{2}$, then $F(xy)\geq\lambda$ and so $(xy)_\lambda\in F$. If $\lambda>\frac{1-k}{2}$, then $F(xy)+\lambda+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ and hence $(xy)_\lambda q_kF$. Therefore $(xy)_\lambda\in\vee q_kF$.

(2) Let $x,y\in G$ and $\lambda\in(0,1]$ be such that $y_\lambda\in F$. Then $F(y)\geq\lambda>0$. Thus $F(y)\geq\frac{1-k}{2}$ and so $y\in A$. It follows that $xy\in A$. Thus $F(xy)\geq\frac{1-k}{2}$. If $\lambda\leq\frac{1-k}{2}$, then $F(xy)\geq\lambda$ and so $(xy)_\lambda\in F$. If $\lambda>\frac{1-k}{2}$, then $F(xy)+\lambda+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ and hence $(xy)_\lambda q_kF$. Therefore $(xy)_\lambda\in\vee q_kF$.

3.20. Proposition If $\{F_i\}_{i\in I}$ is a family of $(\in,\in\vee q_k)$-fuzzy bi-ideals of an AG-groupoid

G, then $\bigcap_{i\in I} F_i$ is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of G.

Proof: Let $\{F_i\}_{i\in I}$ be a family of $(\in,\in\vee q_k)$-fuzzy bi-ideals of G. Let $x, y\in G$. Then

$$\left(\bigcap_{i\in I} F_i\right)(xy) = \bigwedge_{i\in I} F_i(xy) \geq \bigwedge_{i\in I}(F_i(x)\wedge F_i(y)\wedge \frac{1-k}{2})$$
$$=\left(\bigwedge_{i\in I}\left(F_i(x)\wedge \frac{1-k}{2}\right)\wedge \bigwedge_{i\in I}\left(F_i(y)\wedge\frac{1-k}{2}\right)\right)$$
$$=\left(\bigcap_{i\in I} F_i\right)(x)\wedge\left(\bigcap_{i\in I} F_i\right)(y)\wedge\frac{1-k}{2}.$$

Let $x, y, z\in G$. Then

$$\left(\bigcap_{i\in I} F_i\right)((xy)z) = \bigwedge_{i\in I} F_i((xy)z) \geq \bigwedge_{i\in I}(F_i(x)\wedge F_i(z)\wedge \frac{1-k}{2})$$
$$=\left(\bigwedge_{i\in I}\left(F_i(x)\wedge \frac{1-k}{2}\right)\wedge \bigwedge_{i\in I}\left(F_i(z)\wedge\frac{1-k}{2}\right)\right)$$
$$=\left(\bigcap_{i\in I} F_i\right)(x)\wedge\left(\bigcap_{i\in I} F_i\right)(z)\wedge\frac{1-k}{2}.$$

Thus $\bigcap_{i\in I} F_i$ is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of G.

3.21. Lemma The intersection of any family $(\in,\in\vee q_k)$-fuzzy left (resp. right) ideals of an AG-groupoid G is an $(\in,\in\vee q_k)$-fuzzy left (resp. right) ideal of G.

Proof: Follows from Proposition 3.20.

3.22. Lemma The union of any family $(\in,\in\vee q_k)$-fuzzy left (resp. right) ideals of an AG-groupoid G is an $(\in,\in\vee q_k)$-fuzzy left (resp. right) ideal of G.

3.23. Definition A fuzzy subset F of an AG-groupoid G is called an $(\in,\in\vee q_k)$-*fuzzy interior ideal* of G if it satisfies the following conditions:

$(I1)$ $(\forall x, y\in S)(\forall \lambda_1,\lambda_2\in(0,1])(x_{\lambda_1}\in F$ and $y_{\lambda_2}\in F\rightarrow (xy)_{\min\{\lambda_1,\lambda_2\}}\in\vee q_k F)$,

$(I2)$ $(\forall x, a, y\in S)(\forall\lambda\in(0,1])(a_\lambda\in F\rightarrow ((xa)y)_\lambda\in\vee q_k F)$.

3.24. Theorem. Let A be an interior ideal and F a fuzzy subset of G defined as follows:

$$F(x)=\begin{cases}\geq \frac{1-k}{2} \text{ if } x\in A,\\ 0 \text{ otherwise.}\end{cases}$$

Then

(1) F is an $(q,\in\vee q_k)$-fuzzy interior ideal of G.

(2) F is an $(\in,\in\vee q_k)$-fuzzy interior ideal of G.

Proof: (1) Let $x, a, y\in G$ and $\lambda\in(0,1]$ be such that $a_\lambda qF$. Then $a\in A$ and $F(a)+\lambda+k>1$. Since A is an interior ideal of G, we have $(xa)y\in A$. Thus $F((xa)y)\geq\frac{1-k}{2}$. If $\lambda\leq\frac{1-k}{2}$, then $F((xa)y)\geq\lambda$ and so $((xa)y)_\lambda\in F$. If $\lambda>\frac{1-k}{2}$, then $F((xa)y)+\lambda+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ and so $((xa)y)_\lambda q_k F$. Therefore $((xa)y)_\lambda\in\vee q_k F$.

(2) Let $x, a, y\in G$ and $\lambda\in(0,1]$ be such that $a_\lambda\in F$. Then $a\in A$ and $F(a)+\lambda+k>1$. Since A is an interior ideal of G, we have $(xa)y\in A$. Thus $F((xa)y)\geq\frac{1-k}{2}$. If $\lambda\leq\frac{1-k}{2}$, then $F((xa)y)\geq\lambda$ and so $((xa)y)_\lambda\in F$. If $\lambda>\frac{1-k}{2}$, then $F((xa)y)+\lambda+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ and so $((xa)y)_\lambda q_k F$. Therefore $((xa)y)_\lambda\in\vee q_k F$. The remaining proof is a consequence of the proof of Proposition 3.3.

If we take $k=0$ in Theorem 3.24, then we have the following corollary:

3.25. Corollary [33] Let A be an interior ideal and F a fuzzy subset of G defined as follows:

$$F(x)=\begin{cases}\geq 0.5 \text{ if } x\in A,\\ 0 \text{ otherwise.}\end{cases}$$

Then

(1) F is an $(q,\in\vee q)$-fuzzy interior ideal of G.

(2) F is an $(\in,\in\vee q)$-fuzzy interior ideal of G.

3.26. Theorem Let F be a fuzzy subset of G. Then F is an $(\in,\in\vee q_k)$-fuzzy interior ideal of G if and only if the following conditions are satisfied:

$(I3)$ $(\forall x, y\in G)(\forall k\in[0,1))(F(xy)\geq\min\{F(x), F(y), \frac{1-k}{2}\})$,

$(I4)$ $(\forall x, a, y\in G)(\forall k\in[0,1))(F((xa)y)\geq\min\{F(a), \frac{1-k}{2}\})$.

Proof: $(I1)\Leftrightarrow(I3)$ follows from Theorem 3.4.

$(I2) \Rightarrow (I4)$. If there exist $x, a, y \in G$ such that $F((xa)y) < \min\{F(a), \frac{1-k}{2}\}$. Choose $\lambda \in (0,1]$ such that $F((xa)y) < \lambda \leq \min\{F(a), \frac{1-k}{2}\}$. Then $a_\lambda \in F$ but $F((xa)y) < \lambda$ and $F((xa)y) + \lambda + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $((xa)y)_{\min\{\lambda,\lambda\}} = ((xa)y)_\lambda \overline{\in \vee q_k} F$, a contradiction. Hence $F((xa)y) \geq \min\{F(a), \frac{1-k}{2}\}$ for all $x, a, y \in G$.

$(I4) \Rightarrow (I2)$ Assume that $F((xa)y) \geq \min\{F(a), \frac{1-k}{2}\}$. Let $a_\lambda \in F$ for $\lambda \in (0,1]$. Then $F(a) \geq \lambda$ and by hypothesis, $F((xa)y) \geq \min\{F(a), \frac{1-k}{2}\} \geq \min\{\lambda, \frac{1-k}{2}\}$. If $\lambda > \frac{1-k}{2}$, then $F((xa)y) > \frac{1-k}{2}$ and so $F((xa)y) + \lambda + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, it follows that $((xa)y)_\lambda q_k F$. If $F((xa)y) \leq \frac{1-k}{2}$, then $F((xa)y) \geq \lambda$ and so $((xa)y)_\lambda \in F$. Thus $((xa)y) \in \vee q_k F$ and F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of G.

If we take $k = 0$ in Theorem 3.26, then we have the following corollary:

3.27. Corollary (cf. [33]) Let F be a fuzzy subset of G. Then F is an $(\in, \in \vee q)$ -fuzzy interior ideal of G if and only if the following conditions are satisfied:

(1) $(\forall x, y \in G)(F(xy) \geq \min\{F(x), F(y), 0.5\})$,

(2) $(\forall x, a, y \in G)(F((xa)y) \geq \min\{F(a), 0.5\})$.

3.28. Theorem A fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of G if and only if $U(F;\lambda)(\neq \phi)$ is an interior ideal of G for all $\lambda \in (0, \frac{1-k}{2}]$.

Proof: Suppose that F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of G and let $x, a, y \in G$ be such that $a \in U(F;\lambda)$ for some $\lambda \in (0, \frac{1-k}{2}]$. Then $F(a) \geq \lambda$ and by hypothesis $F((xa)y) \geq \min\{F(a), \frac{1-k}{2}\} \geq \min\{\lambda, \frac{1-k}{2}\} = \lambda$. Hence $(xa)y \in U(F;\lambda)$. Conversely, assume that $U(F;\lambda)(\neq \phi)$ is an interior ideal of G for all $\lambda \in (0, \frac{1-k}{2}]$. If $x, a, y \in G$ such that $F((xa)y) < \min\{F(a), \frac{1-k}{2}\}$, choose $\lambda \in (0, \frac{1-k}{2}]$ such that $F((xa)y) < \lambda \leq \min\{F(a), \frac{1-k}{2}\}$. Thus $a \in U(F;\lambda)$ but $(xa)y \notin U(F;\lambda)$, a contradiction. Hence $F((xa)y) \geq \min\{F(a), \frac{1-k}{2}\}$ for all $x, a, y \in G$ and $k \in [0,1)$. The remaining proof is a consequence of the Theorem 3.5. Therefore F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of G.

If we take $k = 0$ in Theorem 3.28, then we have the following corollary:

3.29. Corollary (cf. [33]) A fuzzy subset F is an $(\in, \in \vee q)$-fuzzy interior ideal of G if and only if $U(F;\lambda)(\neq \phi)$ is an interior ideal of G for all $\lambda \in (0, 0.5]$.

3.30. Remark A fuzzy subset F of an AG-groupoid G is an $(\in, \in \vee q_k)$-fuzzy interior ideal of G if and only if it satisfies conditions $(I3)$ and $(I4)$ of Theorem 3.26.

3.31. Proposition Every $(\in, \in)$-fuzzy interior ideal of G is an $(\in, \in \vee q_k)$-fuzzy interior ideal of G.

Proof: It is straightforward.

3.32. Proposition Every $(\in \vee q_k, \in \vee q_k)$-fuzzy interior ideal of G is an $(\in, \in \vee q_k)$-fuzzy interior ideal of G.

Proof: The proof is straightforward.

3.33. Lemma A non-empty subset A of G is a generalized bi-ideal if and only if the characteristic function F_A of A is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of G.

Proof: The proof is straightforward.

3.34. Lemma A non-empty subset A of G is an interior ideal if and only if the characteristic function F_A of A is an $(\in,\in\vee q_k)$-fuzzy interior ideal of G.

Proof: The proof is straightforward.

Note that, every $(\in,\in\vee q_k)$-fuzzy ideal of G is an $(\in,\in\vee q_k)$-fuzzy interior ideal. Similarly, every $(\in,\in\vee q_k)$-fuzzy bi-ideal is an $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal of G. In the following propositions, we prove that the converses of the above statements are true.

3.35. Proposition If G is an intra-regular AG-groupoid, then every $(\in,\in\vee q_k)$-fuzzy interior ideal is an $(\in,\in\vee q_k)$-fuzzy ideal of G.

Proof: Let F be an $(\in,\in\vee q_k)$-fuzzy interior ideal of G. Let $a,b\in G$. Since G is an intra regular, there exist $x,y\in G$ such that $a=(xa^2)y$. Thus,

$$\begin{aligned}F(ab)&=F(((xa^2)y)b)=F((by)(xa^2))=F((by)(x(aa)))\\&=F((by)(a(xa))=F((by)a(xa)))\geq F(a)\wedge\frac{1-k}{2}.\end{aligned}$$

Similarly, we can show that $F(ab)\geq F(b)\wedge\frac{1-k}{2}$ and hence, F is an $(\in,\in\vee q_k)$-fuzzy ideal of G.

3.36. Proposition If G is an intra-regular AG-groupoid. Then every $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of G.

Proof: Let F be an $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal of G. Let $a,b\in G$. Since G is intra-regular, there exist $x,y\in G$ such that $a=(xa^2)y$. Thus,

$$\begin{aligned}F(ab)&=F(((xa^2)y)b)=F(((xa^2)(ey))b)\\&=F(((ye)(a^2x)b)=F((a^2((ye)x))b)\\&=F(((aa)((ye)x))b)=F(((x(ye))(aa))b)\\&=F((a((x(ye)a))b)\geq F(a)\wedge F(b)\wedge\frac{1-k}{2}.\end{aligned}$$

Thus, F is an $(\in,\in\vee q_k)$-fuzzy AG-subgroupoid of G and consequently, F is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of G.

In Propositions 3.35 and 3.36, if we take G as regular, then the concepts of $(\in,\in\vee q_k)$-fuzzy ideals, $(\in,\in\vee q_k)$-fuzzy interior ideals, $(\in,\in\vee q_k)$-fuzzy generalized bi-ideals and $(\in,\in\vee q_k)$-fuzzy bi-ideals coincide.

4. Upper and lower parts of fuzzy ideals of type $(\in,\in\vee q_k)$.

In this section, we define the upper/lower parts of an $(\in,\in\vee q_k)$-fuzzy left (resp. right) ideal and $(\in,\in\vee q_k)$-fuzzy bi-ideals and characterize regular and intra-regular AG-groupoids in terms of lower parts of $(\in,\in\vee q_k)$-fuzzy left (resp. right) ideal and $(\in,\in\vee q_k)$-fuzzy right ideals and $(\in,\in\vee q_k)$-fuzzy left ideals.

4.1. Definition Let F and H be a fuzzy subset of G. Then the fuzzy subsets $\overline{F}^k$, $(F\wedge^k H)^-,(F\vee^k H)^-,(F\circ^k H)^-,\overset{+}{F}^k$, $(F\wedge^k H)^+,(F\vee^k H)^+$ and $(F\circ^k H)^+$ of G are defined as follows:

$$\overline{F}^k:G\to[0,1]\mid x\mapsto F^k(x)=F(x)\wedge\frac{1-k}{2},$$
$$(F\wedge^k H)^-:G\to[0,1]\mid x\mapsto(F\wedge^k H)(x)=(F\wedge H)(x)\wedge\frac{1-k}{2},$$
$$(F\vee^k H)^-:G\to[0,1]\mid x\mapsto(F\vee^k H)(x)=(F\vee H)(x)\wedge\frac{1-k}{2},$$
$$(F\circ^k H)^-:G\to[0,1]\mid x\mapsto(F\circ^k H)(x)=(F\circ H)(x)\wedge\frac{1-k}{2},$$

and

$$\overset{+}{F}^k:G\to[0,1]\mid x\mapsto F^k(x)=F(x)\vee\frac{1-k}{2},$$
$$(F\wedge^k H)^+:G\to[0,1]\mid x\mapsto(F\wedge^k H)(x)=(F\wedge H)(x)\vee\frac{1-k}{2},$$
$$(F\vee^k H)^+:G\to[0,1]\mid x\mapsto(F\vee^k H)(x)=(F\vee H)(x)\vee\frac{1-k}{2},$$
$$(F\circ^k H)^+:G\to[0,1]\mid x\mapsto(F\circ^k H)(x)=(F\circ H)(x)\vee\frac{1-k}{2},$$

for all $x\in G$.

4.2. Lemma Let F and H be fuzzy subsets of

G. Then the following hold:

(i) $\left(F \wedge^k H\right)^- = \left(\overline{F}^k \wedge \overline{H}^k\right)$,

(ii) $\left(F \vee^k H\right)^- = \left(\overline{F}^k \vee \overline{H}^k\right)$,

(iii) $\left(F \circ^k H\right)^- = \left(\overline{F}^k \circ \overline{H}^k\right)$.

Proof: The proof follows from [26].

4.3. Lemma Let F and H be fuzzy subsets of G. Then the following hold:

(i) $\left(F \wedge^k H\right)^+ = \left(\overset{+}{F}^k \wedge \overset{+}{H}^k\right)$,

(ii) $\left(F \vee^k H\right)^+ = \left(\overset{+}{F}^k \vee \overset{+}{H}^k\right)$,

(iii) $\left(F \circ^k H\right)^+ \geq \left(\overset{+}{F}^k \circ \overset{+}{H}^k\right)$ if $x \neq yz$ and $\left(F \circ^k H\right)^+ = \left(\overset{+}{F}^k \circ \overset{+}{H}^k\right)$ if $x = yz$.

Proof: The proof follows from [26].

Let A be a non-empty subset of G, then the upper and lower parts of the characteristic function F_A^k of A are defined as follows:

$$\overline{F}_A^k : G \to [0,1] \mid x \mapsto \overline{F}_A^k(x) = \begin{cases} \frac{1-k}{2} & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

$$\overset{+}{F}_A^k : G \to [0,1] \mid x \mapsto \overset{+}{F}_A^k(x) = \begin{cases} 1 & \text{if } x \in A, \\ \frac{1-k}{2} & \text{otherwise.} \end{cases}$$

4.4. Lemma Let A and B be non-empty subsets of G. Then the following hold:

(1) $(F_A \wedge^k F_B)^- = \overline{F}_{A\cap B}^k$,

(2) $(F_A \vee^k F_B)^- = \overline{F}_{A\cup B}^k$,

(3) $\left(F_A \circ^k F_B\right)^- = \overline{F}_{AB}^k$.

Proof: The proofs of (1) and (2) are obvious.

(3) Let $x \in AB$. Then $F_{AB}(x) = 1$ and hence $\overline{F}_{AB}^k(x) = 1 \wedge \frac{1-k}{2} = \frac{1-k}{2}$. Since $x \in AB$, we have $x = ab$ for some $a \in A$ and $b \in B$. Thus

$$\left(F_A \circ^k F_B\right)^-(x) = (F_A \circ F_B)(x) \wedge \frac{1-k}{2} = \left[\bigvee_{x=yz}\left(F_A(y) \wedge F_B(z)\right)\right] \wedge \frac{1-k}{2} \geq \left(F_A(a) \wedge F_B(b)\right) \wedge \frac{1-k}{2}.$$

Since $a \in A$ and $b \in B$, we have $F_A(a) = 1$ and $F_B(b) = 1$ and so

$$(F_A \circ^k F_B)^-(x) \geq \left(F_A(a) \wedge F_B(b)\right) \wedge \frac{1-k}{2} = (1 \wedge 1) \wedge \frac{1-k}{2} = \frac{1-k}{2}.$$

Thus, $(F_A \circ^k F_B)^-(x) = \frac{1-k}{2} = \overline{F}_{AB}^k(x)$. Let $x \notin AB$, then $F_{AB}(x) = 0$ and hence, $\overline{F}_{AB}^k(x) = 0 \wedge \frac{1-k}{2} = 0$. Let $x = yz$, then

$$(F_A \circ^k F_B)^-(x) = (F_A \circ F_B)(x) \wedge \frac{1-k}{2} = \left[\bigvee_{x=yz}\left(F_A(y) \wedge F_B(z)\right)\right] \wedge \frac{1-k}{2}.$$

Since $x = yz$, if $y \in A$ and $z \in B$, then $yz \in AB$ and so $x \in AB$. This is a contradiction. If $y \notin A$ and $z \in B$, then

$$\left[\bigvee_{x=yz}\left(F_A(y) \wedge F_B(z)\right)\right] \wedge \frac{1-k}{2} = \left[\bigvee_{x=yz}(0 \wedge 1)\right] \wedge \frac{1-k}{2} = 0.$$

Hence, $\overline{F}_{AB}^k(x) = 0 = (F_A \circ^k F_B)^-(x)$. Similarly, for $y \in A$ and $z \notin B$, we have $\overline{F}_{AB}^k(x) = 0 = (F_A \circ^k F_B)^-(x)$.

4.5. Lemma The lower part $\overline{F}_A^k$ of the characteristic function F_A^k of A is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G if and only if A is a bi-ideal of G.

Proof: Let A be a bi-ideal of G. Then, by Theorem 2.1 and 3.7, $\overline{F}_A^k$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G . Conversely, assume that $\overline{F}_A^k$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G. Let $x, y \in G$. If $x, y \in A$, then $\overline{F}_A^k(x) = \frac{1-k}{2}$ and $\overline{F}_A^k(y) = \frac{1-k}{2}$. Since $\overline{F}_A^k$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G, we have $\overline{F}_A^k(xy) \geq \overline{F}_A^k(x) \wedge \overline{F}_A^k(y) \wedge \frac{1-k}{2} = \frac{1-k}{2}$. It follows that $\overline{F}_A^k(xy) = \frac{1-k}{2}$ and so $xy \in A$. Let $x, z \in A$ and $a \in G$. Then, $\overline{F}_A^k(x) = \frac{1-k}{2}$ and $\overline{F}_A^k(z) = \frac{1-k}{2}$. Now,

$$\overline{F}_A^k((xa)z) \geq \overline{F}_A^k(x) \wedge \overline{F}_A^k(z) \wedge \frac{1-k}{2} = \frac{1-k}{2}.$$

Hence $\overline{F}_A^k((xa)z) = \frac{1-k}{2}$ and so $(xa)z \in A$. Therefore A is a bi-ideal of G.

4.6. Lemma The lower part $\overline{F}_A^k$ of the characteristic function F_A^k of A is an $(\in, \in \vee q_k)$-fuzzy left (resp. right)-ideal of G if and only if A is a left (resp. right)-ideal of G.

Proof: The proof follows from Lemma 4.5.
In the following proposition, we show that, if F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G, then $\overline{F}^k$ is a fuzzy bi-ideal of G.

4.7. Proposition If F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G, then $\overline{F}^k$ is a fuzzy bi-ideal of G.

Proof: Let $x, y \in G$. Since F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of G, we have $F(xy) \geq F(x) \wedge F(y) \wedge \frac{1-k}{2}$. It follows that $F(xy) \wedge \frac{1-k}{2} \geq F(x) \wedge F(y) \wedge \frac{1-k}{2} = \left(F(x) \wedge \frac{1-k}{2}\right) \wedge \left(F(y) \wedge \frac{1-k}{2}\right)$, and hence $\overline{F}^k(xy) \geq \overline{F}^k(x) \wedge \overline{F}^k(y)$. For $x, a, z \in G$, we have $F((xa)z) \geq F(x) \wedge F(z) \wedge \frac{1-k}{2}$. Then $F((xa)z) \wedge \frac{1-k}{2} \geq F(x) \wedge F(z) \wedge \frac{1-k}{2} = \left(F(x) \wedge \frac{1-k}{2}\right) \wedge \left(F(z) \wedge \frac{1-k}{2}\right)$, and so $\overline{F}^k((xa)z) \geq \overline{F}^k(x) \wedge \overline{F}^k(z)$. Consequently, $\overline{F}^k$ is a fuzzy bi-ideal of G.

4.8. Lemma [34] Let G be an *AG*-groupoid such that $(xe)G = xG$ for all $x \in G$. Then the following are equivalent:
(1) G is regular,
(2) $R \cap L = RL$ for every right ideal R and left ideal L of G.

4.9. Lemma Let G be an AG-groupoid, F an $(\in, \in \vee q_k)$-fuzzy right ideal, and H an $(\in, \in \vee q_k)$-fuzzy left ideal of G. Then $(F \circ^k H)^- \leq (F \wedge^k H)^-$.

Proof: Let $x \in G$. Then $(F \circ^k H)^-(x) = (F \circ H)(x) \wedge \frac{1-k}{2} = 0 \wedge \frac{1-k}{2} = 0$, if $x \neq yz$ and hence $(F \circ^k H)^-(x) = 0 \leq (F \wedge^k H)^-(x)$. Otherwise,

$$\begin{aligned}(F \circ^k H)^-(x) &= (F \circ H)(x) \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{x=yz} (F(y) \wedge H(z))\right] \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{x=yz} \left\{\left(F(y) \wedge \frac{1-k}{2}\right) \wedge \left(H(z) \wedge \frac{1-k}{2}\right)\right\}\right] \wedge \frac{1-k}{2}.\end{aligned}$$

Since F is an $(\in, \in \vee q_k)$-fuzzy right ideal and H an $(\in, \in \vee q_k)$-fuzzy left ideal of G, and $x = yz$, we have $F(x) = F(yz) \geq F(y) \wedge \frac{1-k}{2}$ and $H(x) = H(yz) \geq H(z) \wedge \frac{1-k}{2}$. Hence,

$$\begin{aligned}&\left[\bigvee_{x=yz} \left\{\left(F(y) \wedge \frac{1-k}{2}\right) \wedge \left(H(z) \wedge \frac{1-k}{2}\right)\right\}\right] \wedge \frac{1-k}{2} \\ &\leq \left[\bigvee_{x=yz} F(x) \wedge H(x)\right] \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{x=yz} (F \wedge H)(x)\right] \wedge \frac{1-k}{2} \\ &= (F \wedge H)(x) \wedge \frac{1-k}{2} = (F \wedge^k H)^-(x).\end{aligned}$$

Thus, $(F \circ^k H)^- \leq (F \wedge^k H)^-$.

4.10. Theorem Let G be an AG-groupoid such that $(xe)G = xG$ for all $x \in G$. Then G is regular if and only if for every $(\in, \in \vee q_k)$-fuzzy right ideal F, and $(\in, \in \vee q_k)$-fuzzy left ideal H of G, we have $(F \wedge^k H)^- = (F \circ^k H)^-$.

Proof: Let $a \in G$ and F be an $(\in, \in \vee q_k)$-fuzzy right ideal and H an $(\in, \in \vee q_k)$-fuzzy left ideal of G. Since G is regular, there exists $x \in G$ such that $a = (ax)a$. Then

$$\begin{aligned}\left(F \circ^k H\right)^-(a) &= (F \circ H)(a) \wedge \frac{1-k}{2} = \left[\bigvee_{a=yz} (F(y) \wedge H(z))\right] \wedge \frac{1-k}{2} \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &\geq (F(ax) \wedge H(a)) \wedge \frac{1-k}{2} \geq (F(a) \wedge H(a)) \wedge \frac{1-k}{2}.\end{aligned}$$

Hence, $(F \wedge^k H)^- \leq \left(F \circ^k H\right)^-$. On the other hand, $\left(F \circ^k H\right)^- \leq (F \wedge^k H)^-$. Thus, $(F \wedge^k H)^- = \left(F \circ^k H\right)^-$.
Conversely, assume that for every $(\in, \in \vee q_k)$-fuzzy right ideal F and $(\in, \in \vee q_k)$-fuzzy left

ideal H of G, we have $(F \wedge^k H)^- = \left(F \circ^k H\right)^-$. To prove that G is regular, by Lemma 4.8, it is enough to prove that $R \cap L = RL$ for every right ideal R and left ideal L of G. Let $x \in R \cap L$. Then $x \in R$ and $x \in L$. Since R is a right ideal and L a left ideal of G. By Lemma 4.6, $\overline{F}^k_R$ is an $(\in, \in \vee q_k)$-fuzzy right ideal and $\overline{F}^k_L$ an $(\in, \in \vee q_k)$-fuzzy left ideal of G. By hypothesis, we have

$$\left(F_R \circ^k F_L\right)^-(x) = \left(F_R \wedge^k F_L\right)^-(x) = \left(\overline{F}^k_R(x) \wedge \overline{F}^k_L(x)\right).$$

Since $x \in R$ and $x \in L$, we have $\overline{F}^k_R(x) = \frac{1-k}{2}$ and $\overline{F}^k_L(x) = \frac{1-k}{2}$. It follows that, $\left(F_R \circ^k F_L\right)^-(x) = \frac{1-k}{2}$ and by Lemma 4.4, (3), we have $\left(F_R \circ^k F_L\right)^- = \overline{F}^k_{RL}$. Hence, we have $\overline{F}^k_{RL}(x) = \frac{1-k}{2}$ and it follows that $x \in RL$ and so $R \cap L \subseteq RL$. On the other hand, $RL \subseteq R \cap L$. Thus, $R \cap L = RL$, and consequently, G is regular.

4.11. Lemma Let G be an AG-groupoid and F (resp. H) an $(\in, \in \vee q_k)$-fuzzy right (resp. left) ideal of G. Then $(F \circ^k 1)^- \leq \overline{F}^k$ (resp. $(1 \circ^k H)^- \leq \overline{H}^k$).

Proof: Let F be an $(\in, \in \vee q_k)$-fuzzy right ideal of G and let $x \in G$. Then

$$(F \circ^k 1)^-(x) = (F \circ 1)(x) \wedge \frac{1-k}{2} = 0 \wedge \frac{1-k}{2} = 0 \leq \overline{F}^k(x), \text{ if } x \neq yz.$$

Otherwise, $(F \circ^k 1)^-(x) = (F \circ 1)(x) \wedge \frac{1-k}{2} = \left[\bigvee_{x=yz} (F(y) \wedge 1(z))\right] \wedge \frac{1-k}{2}$. Since F is an $(\in, \in \vee q_k)$-fuzzy right ideal of G and $x = yz$, we have $F(x) = F(yz) \geq F(y) \wedge \frac{1-k}{2}$ and $1(z) = 1$ for all $z \in G$. Thus,

$$\left[\bigvee_{x=yz} (F(y) \wedge 1(z))\right] \wedge \frac{1-k}{2} \leq F(x) \wedge \frac{1-k}{2} = \overline{F}^k(x).$$

Hence, $(F \circ^k 1)^- \leq \overline{F}^k$. Similarly, for an $(\in, \in \vee q_k)$-fuzzy left ideal H of G, we have $(1 \circ^k H)^- \leq \overline{H}^k$.

4.12. Lemma Let G be an AG-groupoid and F (resp. H) an $(\in, \in \vee q_k)$-fuzzy right (resp. left) ideal of G. Then $(F \circ^k F)^- \leq \overline{F}^k$ (resp. $(H \circ^k H)^- \leq \overline{H}^k$).

Proof: Let F be an $(\in, \in \vee q_k)$-fuzzy right ideal of G. Since $F \leq 1$ and $F \leq F$, Lemma 4.9, we have $(F \circ^k F)^- \leq (F \circ^k 1)^-$. Since F is an $(\in, \in \vee q_k)$-fuzzy right ideal of G, by Lemma 4.11 we have, $(F \circ^k 1)^- \leq \overline{F}^k$. Thus, $(F \circ^k F)^- \leq \overline{F}^k$. In a similar way, for an $(\in, \in \vee q_k)$-fuzzy left ideal H of G we have, $(H \circ^k H)^- \leq \overline{H}^k$.

4.13. Proposition If G is a regular AG-groupoid, then for every $(\in, \in \vee q_k)$-fuzzy right ideal F (resp. left ideal H) of G, we have, $(F \circ^k F)^- = \overline{F}^k$ (resp. $(H \circ^k H)^- = \overline{H}^k$).

Proof: Let $a \in G$. Since G is regular, there exists x such that $a = (ax)a$. Then

$$\begin{aligned}(F \circ^k F)^-(a) &= (F \circ F)(a) \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{a=yz} (F(y) \wedge F(z))\right] \wedge \frac{1-k}{2} \\ &\geq (F(ax) \wedge F(a)) \wedge \frac{1-k}{2} \\ &= \left(F(ax) \wedge \frac{1-k}{2}\right) \wedge \left(F(a) \wedge \frac{1-k}{2}\right).\end{aligned}$$

Since F is an $(\in, \in \vee q_k)$-fuzzy right ideal of G and $a = (ax)a$, we have

$$F(a) = F((ax)a) \geq F(ax) \wedge \tfrac{1-k}{2} \geq F(a) \wedge \tfrac{1-k}{2}.$$

Thus, $F(a) \wedge \frac{1-k}{2} = F(ax) \wedge \frac{1-k}{2}$.

Therefore, $(F \circ^k F)^-(a) \geq \overline{F}^k(a)$. On the other hand, by Lemma 4.12 we have $(F \circ^k F)^-(a) \leq \overline{F}^k(a)$ and consequently, we have $(F \circ^k F)^-(a) = \overline{F}^k(a)$. In a similar way, for an $(\in, \in \vee q_k)$-fuzzy left ideal H of a regular AG-groupoid G, we have, $(H \circ^k H)^-(a) = \overline{H}^k(a)$.

4.14. Corollary In regular AG-groupoids, for an $(\in, \in \vee q_k)$-fuzzy ideal F of G, we have $(F \circ^k F)^- = \overline{F}^k$.

4.15. Lemma [34] Let G be an AG-groupoid such that $(xe)G = xG$ for all $x \in G$. Then the following statements are equivalent:
(1) G is intra-regular,
(2) $R \cap L \subseteq LR$ for every right ideal R and left ideal L of G.

4.16. Theorem Let G be an *AG*-groupoid such that $(xe)G = xG$ for all $x \in G$. Then G is intra-regular if and only if for every $(\in, \in \vee q_k)$-fuzzy right ideal F and $(\in, \in \vee q_k)$-fuzzy left ideal H of G, we have $(F \wedge^k H)^- \leq (H \circ^k F)^-$.

Proof Let $a \in G$ and F be an $(\in, \in \vee q_k)$-fuzzy right ideal and H an $(\in, \in \vee q_k)$-fuzzy left ideal of G. Since G is intra-regular, there exist $x, y \in G$ such that $a = (xa^2)y = (xa)(ay)$. Then

$$\begin{aligned}(H \circ^k F)^-(a) &= (H \circ F)(a) \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{a=yz} (H(y) \wedge F(z))\right] \wedge \frac{1-k}{2} \\ &\geq (H(xa) \wedge F(ay)) \wedge \frac{1-k}{2} \\ &\geq (H(a) \wedge F(a)) \wedge \frac{1-k}{2}.\end{aligned}$$

Hence, $(F \wedge^k H)^- \leq (H \circ^k F)^-$.
Conversely, assume that for every $(\in, \in \vee q_k)$-fuzzy right ideal F and $(\in, \in \vee q_k)$-fuzzy left ideal H of G, we have $(F \wedge^k H)^- \leq (F \circ^k H)^-$. To prove that G is intra-regular, by Lemma 4.15, it is enough to prove that

$R \cap L \subseteq LR$ for every right ideal R and left ideal L of G.

Let $x \in R \cap L$. Then $x \in R$ and $x \in L$. Since R is a right ideal and L a left ideal of G by Lemma 4.6, $\overline{F}^k_R$ is an $(\in, \in \vee q_k)$-fuzzy right ideal and $\overline{F}^k_L$ an $(\in, \in \vee q_k)$-fuzzy left ideal of G. By hypothesis, we have

$$(F_L \circ^k F_R)^-(x) \leq (F_R \wedge^k F_L)^-(x) = \left(\overline{F}^k_R(x) \wedge \overline{F}^k_L(x)\right)$$

Since $x \in R$ and $x \in L$, we have $\overline{F}^k_R(x) = \frac{1-k}{2}$ and $\overline{F}^k_L(x) = \frac{1-k}{2}$. It follows that, $(F_L \circ^k F_R)^-(x) = \frac{1-k}{2}$ and by Lemma 4.4, (3), we have $(F_L \circ^k F_R)^- = \overline{F}^k_{LR}$. Hence, we have $\overline{F}^k_{LR}(x) = \frac{1-k}{2}$ and it follows that $x \in LR$ and so $R \cap L \subseteq LR$. Thus, G is intra-regular.

4.17. Lemma [34] Let G be an AG-groupoid such that $(xe)G = xG$ for all $x \in G$. Then the following statements are equivalent:
(1) G is regular and intra-regular,
(2) $R \cap L \subseteq RL \cap LR$ for every right ideal R and left ideal L of G.

4.18. Theorem Let G be an AG-groupoid such that $(xe)G = xG$ for all $x \in G$. Then G is regular and intra-regular if and only if for every $(\in, \in \vee q_k)$-fuzzy right ideal F and $(\in, \in \vee q_k)$-fuzzy left ideal H of G, we have

$$(F \wedge^k H)^- \leq \left(F \circ^k H \wedge^k H \circ^k F\right)^-.$$

Proof: The proof is an immediate consequence of Theorems 4.10 and 4.16.

5. Fuzzy ideals of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$

In this section, we define $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals (resp. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior ideals) in AG-groupoids and give some of their related properties.

5.1. Definition A fuzzy subset F of G is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-*fuzzy bi-ideal* of G if it satisfies the following conditions:
(B5)

$(\forall x, y\in G)(\forall\lambda_1,\lambda_2\in(0,1])((xy)_{\min\{\lambda_1,\lambda_2\}}\overline{\in}F \to x_{\lambda_1}\overline{\in}\vee\overline{q_k}F$ or $y_{\lambda_2}\overline{\in}\vee\overline{q_k}F)$,

(B6) $(\forall x, a, y\in G)(\forall\lambda_1,\lambda_2\in(0,1])(((xa)y)_{\min\{\lambda_1,\lambda_2\}}\overline{\in}F \to x_{\lambda_1}\overline{\in}\vee\overline{q_k}F$ or $y_{\lambda_2}\overline{\in}\vee\overline{q_k}F)$.

5.2. Theorem A fuzzy subset F of G is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal of G if and only if it satisfies the following conditions:

$(B7)$ $(\forall x, y\in G)(\max\{F(xy),\frac{1-k}{2}\}\geq\min\{F(x),F(y)\})$,

$(B8)$ $(\forall x, a, z\in G)(\max\{F((xa)z),\frac{1-k}{2}\}\geq\min\{F(x),F(z)\})$.

Proof: $(B5)\Rightarrow(B7)$. If there exist $x, y\in G$ and $k\in[0,1)$ be such that $\max\{F(xy),\frac{1-k}{2}\}<\min\{F(x),F(y)\}$. Choose $\frac{1-k}{2}<\lambda\leq 1$ such that $\max\{F(xy),\frac{1-k}{2}\}<\lambda\leq\min\{F(x),F(y)\}$, then $(xy)_\lambda\overline{\in}F$ but $x_\lambda\in F$ and $y_\lambda\in F$. By $(B5)$, we have $x_\lambda\overline{q_k}F$ and $y_\lambda\overline{q_k}F$. Then $(F(x)\geq\lambda$ and $\lambda+F(x)\leq 1-k)$ or $(F(y)\geq\lambda$ and $\lambda+F(y)\leq 1-k)$, which implies that $\lambda\leq\frac{1-k}{2}$, a contradiction.

$(B7)\Rightarrow(B5)$. Let $(xy)_{\min\{\lambda_1,\lambda_2\}}\overline{\in}F$, then $F(xy)<\min\{\lambda_1,\lambda_2\}$.

i) If $F(xy)\geq\min\{F(x),F(y)\}$, then $\min\{F(x),F(y)\}<\min\{\lambda_1,\lambda_2\}$ and consequently, $F(x)<\lambda_1$ or $F(y)<\lambda_2$. It follows that $x_{\lambda_1}\overline{\in}F$ or $y_{\lambda_2}\overline{\in}F$.

ii) If $F(xy)<\min\{F(x),F(y)\}$, then by $(B7)$, we have $\frac{1-k}{2}\geq\min\{F(x),F(y)\}$. Let $x_{\lambda_1}\in F$ or $y_{\lambda_2}\in F$, then $\lambda_1\leq F(x)\leq\frac{1-k}{2}$ or $\lambda_2\leq F(y)\leq\frac{1-k}{2}$. It follows that $x_{\lambda_1}\overline{\in}\vee\overline{q_k}F$ or $y_{\lambda_2}\overline{\in}\vee\overline{q_k}F$.

$(B6)\Rightarrow(B8)$. If there exist $x, a, y\in G$, $k\in[0,1)$ be such that $\max\{F((xa)y),\frac{1-k}{2}\}<\min\{F(x),F(y)\}$. Choose $\frac{1-k}{2}<\lambda\leq 1$ such that $\max\{F((xa)y),\frac{1-k}{2}\}<\lambda\leq\min\{F(x),F(y)\}$, then $((xa)y)_\lambda\overline{\in}F$ but $x_\lambda\in F$ and $y_\lambda\in F$. By $(B6)$, we have $x_\lambda\overline{q_k}F$ and $y_\lambda\overline{q_k}F$. Then $(F(x)\geq\lambda$ and $\lambda+F(x)\leq 1-k)$ or $(F(y)\geq\lambda$ and $\lambda+F(y)\leq 1-k)$, which implies that $\lambda\leq\frac{1-k}{2}$, a contradiction.

$(B8)\Rightarrow(B6)$. Let $((xa)y)_{\min\{\lambda_1,\lambda_2\}}\overline{\in}F$, then $F((xa)y)<\min\{\lambda_1,\lambda_2\}$.

i) If $F((xa)y)\geq\min\{F(x),F(y)\}$, then $\min\{F(x),F(y)\}<\min\{\lambda_1,\lambda_2\}$ and consequently, $F(x)<\lambda_1$ or $F(y)<\lambda_2$. It follows that $x_{\lambda_1}\overline{\in}F$ or $y_{\lambda_2}\overline{\in}F$.

ii) If $F((xa)y)<\min\{F(x),F(y)\}$, then by $(B8)$, we have $\frac{1-k}{2}\geq\min\{F(x),F(y)\}$. Let $x_{\lambda_1}\in F$ or $y_{\lambda_2}\in F$, then $\lambda_1\leq F(x)\leq\frac{1-k}{2}$ or $\lambda_2\leq F(y)\leq\frac{1-k}{2}$. It follows that $x_{\lambda_1}\overline{\in}\vee\overline{q_k}F$ or $y_{\lambda_2}\overline{\in}\vee\overline{q_k}F$.

If we take $k=0$ in Theorem 5.2, then we have the following corollary:

5.3. Corollary [32] A fuzzy subset F of G is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of G if and only if it satisfies the following conditions:

(1) $(\forall x, y\in G)(\max\{F(xy),0.5\}\geq\min\{F(x),F(y)\})$,

(2) $(\forall x, a, z\in G)(\max\{F((xa)z),0.5\}\geq\min\{F(x),F(z)\})$.

5.4. Definition A fuzzy subset F of G is called an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-*fuzzy interior ideal* of G if it satisfies the following conditions:

$(I5)$ $(\forall x, y\in G)(\forall\lambda_1,\lambda_2\in(0,1])((xy)_{\min\{\lambda_1,\lambda_2\}}\overline{\in}F\to x_{\lambda_1}\overline{\in}\vee\overline{q_k}F$ or $y_{\lambda_2}\overline{\in}\vee\overline{q_k}F)$,

$(I6)$ $(\forall x, a, z\in G)(\forall\lambda_1,\lambda_2\in(0,1])(((xa)z)_\lambda\overline{\in}F\to a_\lambda\overline{\in}\vee\overline{q_k}F)$.

5.5. Theorem A fuzzy subset F of G is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior ideal of G if and only if it satisfies the following conditions:

$(I7)$ $(\forall x, y\in G)(\max\{F(xy),\frac{1-k}{2}\}\geq\min\{F(x),F(y)\})$,

$(I8)$ $(\forall x, a, z\in G)(\max\{F((xa)z),\frac{1-k}{2}\}\geq F(a))$.

Proof: It is an immediate consequence of Theorem 5.2.

5.6. Lemma Let F be a fuzzy subset of G. Then $U(F;\lambda)$ is a bi-ideal of G for all $\lambda\in(\frac{1-k}{2},1]$ if and only if F is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal.

Proof: Assume that $U(F;\lambda)$ is a bi-ideal of G for all $\lambda\in(\frac{1-k}{2},1]$. If there exist $x,y\in G$ such that $\max\{F(xy),\frac{1-k}{2}\}<\min\{F(x),F(y),\frac{1-k}{2}\}=\lambda_1$, then $\lambda_1\in(\frac{1-k}{2},1]$, $x,y\in U(F;\lambda_1)$. But $F(xy)<\lambda_1$ implies $xy\notin U(F;\lambda_1)$, a contradiction. Hence condition $(B7)$ is valid.
If there exist $x,y,z\in G$ such that $\max\{F((xy)z),\frac{1-k}{2}\}<\min\{F(x),F(z),\frac{1-k}{2}\}=\lambda_2$, then $\lambda_2\in(\frac{1-k}{2},1]$, $x,z\in U(F;\lambda_2)$. But $F((xy)z)<\lambda_2$ implies $(xy)z\notin U(F;\lambda_2)$, a contradiction. Hence condition $(B8)$ is valid.
Conversely, suppose that F satisfies conditions $(B7)$ and $(B8)$. For $x,y\in U(F;\lambda)$, by $(B7)$ we get $\max\{F(xy),\frac{1-k}{2}\}\geq\min\{F(x),F(y)\}\geq\lambda>\frac{1-k}{2}$, and so $F(xy)\geq\lambda$. It follows that $xy\in U(F;\lambda)$. Let $x,z\in U(F;\lambda)$, then $F(x)\geq\lambda$ and $F(z)\geq\lambda$. By $(B8)$, we get $\max\{F((xy)z),\frac{1-k}{2}\}\geq\min\{F(x),F(z)\}\geq\lambda>\frac{1-k}{2}$, and so $F((xy)z)\geq\lambda$. It follows that $(xy)z\in U(F;\lambda)$. Thus $U(F;\lambda)$ is a bi-ideal of G for all $\lambda\in(\frac{1-k}{2},1]$.

5.7. Theorem A fuzzy subset F of G is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal of G if and only if $U(F;\lambda)(\neq\phi)$ is a bi-ideal of G, for all $\lambda\in(\frac{1-k}{2},1]$.

Proof: It is an immediate consequence of Theorem 5.2 and Lemma 5.6.

5.8. Lemma Let F be a fuzzy subset of G. Then $U(F;\lambda)$ is an interior ideal of G for all $\lambda\in(\frac{1-k}{2},1]$ if and only if F is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior ideal.

Proof: The proof follows from Lemma 5.6.

5.9. Theorem A fuzzy subset F of G is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior ideal of G if and only if $U(F;\lambda)(\neq\phi)$ is an interior ideal of G for all $\lambda\in(\frac{1-k}{2},1]$.

Proof: The proof follows from Theorem 5.5 and Lemma 5.8.

5.10. Definition [32] Let $\lambda_1,\lambda_2\in(0,1]$ and $\lambda_1<\lambda_2$, then a fuzzy subset F of G is called a fuzzy bi-ideal with thresholds $(\lambda_1,\lambda_2]$ *of* G *if it* satisfies the following conditions:

$(B9)(\forall x,y\in G)(\max\{F(xy),\lambda_1\}\geq\min\{F(x),F(y),\lambda_2\})$,

$(B10)(\forall x,y,z\in G)(\max\{F((xy)z),\lambda_1\}\geq\min\{F(x),F(z),\lambda_2\})$.

5.11. Theorem [32] A fuzzy subset F of G is a fuzzy bi-ideal with thresholds $(\lambda_1,\lambda_2]$ of G if and only if $U(F;\lambda)(\neq\phi)$ is a bi-ideal of G for all $\lambda_1<\lambda\leq\lambda_2$.

5.12. Definition [33] Let $\lambda_1,\lambda_2\in(0,1]$ and $\lambda_1<\lambda_2$, then a fuzzy subset F of G is called a *fuzzy interior ideal with thresholds* $(\lambda_1,\lambda_2]$ of G if it satisfies the following conditions:

$(I9)(\forall x,y\in G)(\max\{F(xy),\lambda_1\}\geq\min\{F(x),F(y),\lambda_2\})$,

$(I10)(\forall x,a,y\in G)(\max\{F((xa)y),\lambda_1\}\geq\min\{F(a),\lambda_2\})$.

5.13. Theorem [33] A fuzzy subset F of G is a fuzzy interior ideal with thresholds $(\lambda_1,\lambda_2]$ of G if and only if $U(F;\lambda)(\neq\phi)$ is an interior ideal of G for all $\lambda_1<\lambda\leq\lambda_2$.

5.14. Remark By Definition 5.10, we have the following result: if F is a fuzzy bi-ideal with thresholds $(\lambda_1,\lambda_2]$ of G then we conclude the following:

(i) F is an ordinary fuzzy bi-ideal (or fuzzy interior ideal) of G when $\lambda_1=0,\lambda_2=1$;

(ii) F is an $(\in,\in\vee q)$-fuzzy bi-ideal (or $(\in,\in\vee q)$-fuzzy interior ideal) of G when $\lambda_1=0,\lambda_2=0.5$;

(iii) F is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal (or $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior ideal) of G when $\lambda_1=0.5,\lambda_2=1$;

(iv) F is an $(\in,\in\vee q_k)$-fuzzy bi-ideal (or $(\in,\in\vee q_k)$-fuzzy interior ideal) of G when $\lambda_1=0,\lambda_2=\frac{1-k}{2}$;

(v) F is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal (or $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior ideal) of G when $\lambda_1=\frac{1-k}{2},\lambda_2=1.$

6. Concluding remarks

In this paper, we study generalized fuzzy bi-ideals (resp. ideals and interior ideals) and give different characterization theorems of AG-groupoids in terms of these notions. In particular, if $J=\{\lambda \mid \lambda\in(0,1]$ and $U(F;\lambda)$ is an empty set or a bi-ideal (resp. an ideal or an interior ideal) of $G\}$, we get the answer to the following questions:

(1) If $J=(0,0.5]$, what kind of fuzzy bi-ideals (resp. fuzzy ideals or fuzzy interior ideals) of G will F be?

(2) If $J=(0.5,1]$, what kind of fuzzy bi-ideals (resp. fuzzy ideals or fuzzy interior ideals) of G will F be?

(3) If $J=(\lambda_1,\lambda_2]$, $(\lambda_1,\lambda_2\in(0,1])$ what kind of fuzzy bi-ideals (resp. fuzzy ideals or fuzzy interior ideals) of G will F be?

(4) If $J=(0,\frac{1-k}{2}]$, what kind of fuzzy bi-ideals (resp. fuzzy ideals or fuzzy interior ideals) of G will F be?

(5) If $J=(\frac{1-k}{2},1]$, what kind of fuzzy bi-ideals (resp. fuzzy ideals or fuzzy interior ideals) of G will F be?

In our future work, we want to study those AG-groupoids for which each generalized fuzzy bi-ideal (resp. each generalized fuzzy ideal or interior ideal) is idempotent. We will also define prime (α,β)-fuzzy bi-ideals (resp. prime (α,β)-fuzzy ideals or (α,β)-fuzzy interior ideals) and establish an (α,β)-fuzzy spectrum of AG-groupoids, where $\alpha,\beta\in\{\in,q,\in\vee q,\in\wedge q\}$.

References

[1] Mordeson, J. N., Malik, D. S. & Kuroki, N. (2003). *Fuzzy Semigroups*. Studies in Fuzziness and Soft Computing *Vol. 131*, Berlin, Springer-Verlag.

[2] Kuroki, N. (1981). On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy Sets and Systems, 5*, 203-215.

[3] Kehayopulu, N. & Tsingelis, M. (2002). Fuzzy sets in ordered groupoids. *Semigroup Forum, 65*, 128-132.

[4] Liu, L. & Li, K. (2005). Fuzzy implicative and Boolean filters of R_0-algebras. *Inform. Sci., 171*, 61-71.

[5] Bhakat, S. K. & Das, P. (1996). $(\in,\in\vee q)$-fuzzy subgroups. *Fuzzy Sets and Systems, 80*, 359-368.

[6] Bhakat, S. K. & Das, P. (1996). Fuzzy subrings and ideals redefined. *Fuzzy Sets and Systems, 81*, 383-393.

[7] Rosenfeld, A. (1971). Fuzzy groups. *J. Math. Anal. Appl., 35*, 512-517.

[8] Davvaz, B. (2006). $(\in,\in\vee q)$-fuzzy subnear-rings and ideals. *Soft Computing, 10*, 206-211.

[9] Ma, X., Zhan, J. & Jun, Y. B. (2008). *Interval valued* $(\in,\in\vee q)$-fuzzy ideals of pseudo-MV algebras. *International Journal of Fuzzy Systems, 10*(2), 84-91.

[10] Jun, Y. B. & Song, S. Z. (2006). Generalized fuzzy interior ideals in semigroups. *Inform. Sci., 176*, 3079-3093.

[11] Jun, Y. B., Khan, A. & Shabir, M. (2009). Ordered semigroups characterized by their $(\in,\in\vee q)$-fuzzy bi-ideals. *Bull. Malays. Math. Sci. Soc., 32*(3), 391-408.

[12] Kazanci, O. & Yamak, S. (2008). Generalized fuzzy bi-ideals of semigroup. *Soft Computing, 12*, 1119-1124.

[13] Ma, X., Zhan, J. & Jun, Y. B. (2009). On $(\in,\in\vee q)$-fuzzy filters of R_0-algebras. *Math. Log. Quart., 55*, 493-508.

[14] Yin, Y. & Zhan, J. (2010). New types of fuzzy filters of BL-algebras. *Computer Mathematics with Applications, 60*(7), 2115-2125.

[15] Yuan, X., Zhang, C. & Ren, Y. (2003). Generalized fuzzy groups and many-valued implications. *Fuzzy Sets and Systems, 138*, 205-211.

[16] Zhan, J. & Davvaz, B. (2010). Study of fuzzy algebraic hypersystems from a general viewpoint. *International Journal of Fuzzy Systems, 12*, 73-79.

[17] Zhan, J., Davvaz, B. & K. P. Shum, (2009). A new view of fuzzy hypernear-rings. *Inf. Sci., 178*, 425-438.

[18] Zhan, J., Davvaz, B. & K. P. Shum, (2009). A new view of fuzzy hyperquasigroups. *Journal of Intelligent and Fuzzy Systems, 20*, 147-157.

[19] Zhan, J., Davvaz, B. & K. P. Shum, (2010). On probabilistic n-ary hypergroups. *Inf. Sci., 180*(7), 1159-1166.

[20] Zhan, J. & Jun, Y. B. (2011). On $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideals of BCI-algebra. *Neural Comput & Appl., 20*, 319-328.

[21] Zhan, J. & Jun, Y. B. (2011). Soft BL-algebras based on fuzzy sets. *Computers & Mathematics with Applications, 59*(6), 2037-2046.

[22] Zhan, J. & Shum, K. P. (2011). On fuzzy h-ideals in hemirings. *Neural Comput & Appl., 20*, 495-505.

[23] Zhan, J. & Yang, X. (2008). Some types of generalized fuzzy filters of BL-algebras. *Computers & Mathematics with Applications, 56*, 1604-1616.

[24] Khan, A. & Shabir, M. (2009). (α,β)-fuzzy interior ideals in ordered semigroups. *Lobachevskii Journal of Mathematics, 30*(1), 30-39.

[25] Jun, Y. B. (2009). Generalizations of $(\in,\in\vee q)$-fuzzy subalgebras in BCK/BCI-algebras. *Comput.*

Math. Applic., 58(7), 1383-1390.
[26] Shabir, M., Jun, Y. B. & Nawaz, Y. (2010). Semigroups characterized by $(\in, \in \vee q_k)$ -fuzzy ideals. *Computer and Mathematics with Applications, 60*, 1473-1493.
[27] Kazim, M. A. & Naseeruddin, M. (1972). On LA-semigroups. *Allig. Bull. Math., 8*, 1-7.
[28] Holgate, P. (1992). Groupoids satisfying a simple invertive law. *The Math. Stud., 1-4*(61), 101-106.
[29] Protic, P. V. & Stevanovic, N. (1995). AG-test and some general properties of Abel-Grassmann's groupoids. *PU. M. A., 4*(6), 371-383.
[30] Mushtaq, Q. & Yousuf, S. M. (1988). On LA-semigroups defined by a commutative inverse semigroup. *Math. Bech., 40*, 59-62.
[31] Naseeruddin, M. (1970). Some studies in almost semigroups and flocks. Ph. D., Thesis, Alig. Muslim University, Aligarh, India,
[32] Khan, A., Shabir, M. & Jun, Y. B. (2010). Generalized fuzzy *Abel Grassmann's* groupoids. *Int. J. Fuzzy Syst., 12*(4), 340-349.
[33] Khan, A., Jun, Y. B. & Mahmood, T. (2010). Generalized fuzzy interior ideals of Abel Grassmann's groupoids. *International J. Math. Math. Sci.*, 2010, Article ID 838392, 14 pages.
[34] Mushtaq, Q. & Khan, M. (2006). Ideals in AG-band and AG^*-groupoids. *Quasigroups and Related Systems, 14*, 207-215.
[35] Khan, M., Nouman, M. & Khan, A. (2009). *Fuzzy Abel Grassmann's groupoids.* arXiv:0904.007vl [math. GR], 1 Apr, pp. 9.