Fuzzy soft Γ-hyperrings

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Abstract

Maji et al. introduced the concept of fuzzy soft sets as a generalization of the standard soft sets and presented an application of fuzzy soft sets in a decision making problem. In this paper, we apply fuzzy soft sets to Γ-hyperrings. The concept of $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy soft Γ-hyperideals of Γ-hyperrings is first introduced. Some new characterizations are investigated. In particular, a kind of new Γ-hyperrings by congruence relations is obtained.

Keywords: Soft set; Γ-hyperideal; $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy soft Γ-hyperideal; congruence relation; Γ-hyperring

1. Introduction

Uncertainties, which could be caused by information incompleteness, data randomness limitations of measuring instruments, etc., are pervasive in many complicated problems in biology, engineering, economics, environment, medical science and social science. Alternatively, mathematical theories, such as probability theory, fuzzy set theory [1], vague set theory, rough set theory [2] and interval mathematics, have been proven to be useful mathematical tools for dealing with uncertainties. However, all these theories have their inherent difficulties, as pointed out by Molodtsov in [3]. Nowadays, works on the soft set theory are progressing rapidly. Maji et al. [4, 5] described the application of soft set theory to a decision making problem. Ali et al. [6] proposed some new operations on soft sets. Chen et al. [7] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attribute reduction in rough set theory. In particular, fuzzy soft set theory has been investigated by some researchers, for examples, see [8, 9]. Recently, the algebraic structures of soft sets have been studied increasingly, see [10-18].

On the other hand, the theory of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. In the literature, the theory of hyperstructure was first initiated by Marty in 1934 [19] when he defined the hypergroups and began to investigate their properties

with applications to groups, rational fractions and algebraic functions. Later on, many people observed that the theory of hyperstructures also has many applications in both pure and applied sciences, for example, semi-hypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Some review of the theory of hyperstructures can be found in [20-23], respectively. One well-known type of a hyperring, is called the Krasner hyper-ring [24]. Krasner hyperrings are essentially rings with approximately modified axioms in which addition is a hyperoperation (i.e., a + b is a set). This concept has been studied by a variety of authors, see [25-28]. In particular, the relationships between the fuzzy sets and algebraic hyperstructures have been considered by Ameri, Cristea, Corsini, Davvaz, Leoreanu, Zhan and many other researchers [18, 29-42].

The concept of Γ-rings was introduced by Barnes [43]. After that, this concept was discussed further by some researchers. The notion of fuzzy ideals in a Γ-ring was introduced by Jun and Lee in [44]. They studied some preliminary properties of fuzzy ideals of Γ-rings. Jun [45] defined fuzzy prime ideals of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a fuzzy prime ideal. In particular, Dutta and Chanda [46] studied the structures of the set of fuzzy ideals of a Γ-ring. Ma et al. [47, 48] considered the characterizations of Γ-hemirings and Γ-rings, respectively. Recently, Ameri et al. [4] considered the concept of fuzzy hyperideals of Γ-hyperrings. By a different way of [49], Yin et al. [50]

investigated some new results on Γ-hyperrings. Ma et al. [51] considered the (fuzzy) isomorphism theorems of Γ-hyperrings. At the same time, Davvaz et al. [52] considered the properties of Γ-hypernear-rings and derived some related results.

After the introduction of fuzzy sets by Zadeh [1], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [53] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets. In fact, the $(\in, \in \vee q)$-fuzzysubgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems with other algebraic structures, see [28, 31, 33, 37, 39].

In this paper, we introduce the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft Γ-hyperideals of Γ-hyperrings. Some new characterizations of them are investigated. In particular, a new kind Γ-hyperrings are obtained by congruence relations.

2. Preliminaries

A hypergroupoid is a non-empty set H together with a mapping "$\circ$" $: H \times H \to P^*(H)$, where $P^*(H)$ is the set of all the non-empty subsets of H.

A quasicanonical hypergroup (not necessarily commutative) is an algebraic structure $(H,+)$ satisfying the following conditions:

(i) for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;

(ii) there exists a $0 \in H$ such that $0 + x = x$, for all $x \in H$;

(iii) for every $x \in H$, there exists a unique element $x^{'} \in H$ such that $0 \in (x + x^{'}) \cap (x^{'} + x)$; "(we call the element $-x$ the opposite of x)"

(iv) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

Quasicanonical hypergroups are also called polygroups.

We note that if $x \in H$ and A; B are non-empty subsets in H, then by $A + B$, $A + x$ and $x + B$ we mean that $A + B = \bigcup_{a \in A, b \in B} a + b$, $A + x = A + \{x\}$ and $x + B = \{x\} + B$, respectively. Also, for all $x, y \in H$, we have $-(-x) = x$, $-0 = 0$, where 0 is unique and $-(x + y) = -y - x$.

A sub-hypergroup $A \subset H$ is said to be normal if $x + A - x \subseteq H$ for all $x \in H$.

A normal sub-hypergroup A of H is called left (right) hyperideal of H if $xA \subseteq A$

($Ax \subseteq A$ respectively) for all $x \in H$. Moreover, A is said to be a hyperideal of H if it is both a left and a right hyperideal of H. A canonical hypergroup is a commutative quasicanonical hypergroup.

Definition 2.1. [24] A hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, that is, $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$;

(3) The multiplication is distributive with respect to the hyperoperation "+" that is, $z \cdot (x + y) = z \cdot x + z \cdot y$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.

For the sake of simplicity, we shall omit the symbol "$\cdot$", writing ab for $a \cdot b$ $(a, b \in R)$.

Definition 2.2. [49, 50] Let $(R, \oplus)$ and $(\Gamma, \oplus)$ be two canonical hypergroups. Then R is called a Γ-hyperring, if the following conditions are satisfied for all $x, y, z \in R$ and for all $\alpha, \beta \in \Gamma$,

(1) $x \alpha y \in R$;

(2) $(x \oplus y) \alpha z = x \alpha z \oplus y \alpha z$
$x(\alpha \oplus \beta)y = a \alpha y \oplus x \beta y$,
$x \alpha(y \oplus z) = x \alpha y \oplus x \alpha z$,

(3) $x \alpha(y \beta z) = (x \alpha y) \beta z$.

In the sequel, unless otherwise stated, $(R, \oplus, \Gamma)$ always denotes a Γ-hyperring.

A subset A in R is said to be a left (right) Γ-hyperideal of R if it satisfies the following conditions:

(1) $(A, \oplus)$ is a normal sub-hypergroup of $(R, \oplus)$;

(2) $x \alpha y \in A$ ($y \alpha x \in A$ respectively) for all $x \in R$, $y \in A$ and $\alpha \in \Gamma$.

A is said to be a Γ-hyperideal of R if it is both a left and a right Γ-hyperideal of R.

Definition 2.3. [51] A fuzzy set μ of a Γ-hyperring R is called a fuzzy Γ-hyperideal of R if the following conditions hold:

(1) $\min\{\mu(x), \mu(y)\} \leq \inf_{z=x+y} \mu(z)$ for all $x, y \in R$;

(2) $\mu(x) \leq \mu(-x)$ for all $x \in R$;

(3) $\max\{\mu(x), \mu(y)\} \leq \mu(x + y)$ for all $x, y \in R$ and for all $\alpha \in \Gamma$

(4) $\mu(x) \leq \inf_{z=-y+x+y} \mu(z)$ for all $x, y \in R$.

Example 1. [51] Let R be a Γ-ring such that $x(-\alpha)y = -x \alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Denote $\bar{R} = \{\bar{x} = \{x, -x\} | x \in R\}$ and $\bar{\Gamma} = \{\bar{\alpha} = \{\alpha, -\alpha\} | \alpha \in \Gamma\}$.

Define the hyperoperations on $\bar{R}$ and $\bar{\Gamma}$ as follows: $\bar{x} \oplus \bar{y} = \{\overline{x+y}, \overline{x-y}\}$, $\bar{\alpha} \oplus \bar{\beta} = \{\overline{\alpha+\beta}, \overline{\alpha-\beta}\}$ and $\overline{x}\bar{\alpha}\overline{y} = \overline{x\alpha y}$ for all $x, y \in R$ and $\alpha, \beta \in \Gamma$. Then $(\bar{R}, \oplus, \bar{\Gamma})$ is a $\bar{\Gamma}$-hyperring.

Example 2. [51] Let $(G, \cdot)$ be a group and $\Gamma = G$. Denote $G^0 = \Gamma^0 = G \cup \{0\}$ and define $x\alpha y = x \cdot \alpha \cdot y$ for all $x, y \in G$ and $\alpha \in \Gamma$. Then $(G^0, \oplus, \Gamma^0)$ is a Γ^0-hyperring with respect to the hyperoperation "$\oplus$" on G^0 and Γ^0, defined by
$x \oplus 0 = 0 \oplus x = \{x\}$ for all $x \in G^0$,
$x \oplus x = G^0 \setminus \{0\}$ for all $x \in G^0 \setminus \{0\}$,
$x \oplus y = \{x, y\}$ for all $x, y \in G^0 \setminus \{0\}$ with $x \neq y$, and
$\alpha \oplus 0 = 0 \oplus \alpha = \{\alpha\}$ for all $\alpha \in \Gamma^0$,
$\alpha \oplus \alpha = \Gamma^0 \setminus \{0\}$ for all $\varepsilon \in \Gamma^0 \setminus \{0\}$,
$\alpha \oplus \beta = \{\alpha, \beta\}$ for all $\alpha, \beta \in \Gamma^0 \setminus \{0\}$ with $\alpha \neq \beta$, respectively.

Definition 2.4. [51] If R and R' are Γ-hyperrings, then a mapping $f : R \to R'$ such that
$f(x \oplus y) = f(x) \oplus f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$
for all $x, y \in R$ and $\alpha \in \Gamma$, is called a Γ-hyperring homomorphism.

Clearly, a Γ-hyperring homomorphism f is an isomorphism if f is injective and surjective. We write $R \cong R'$ if R is isomorphic to R'.

If N is a Γ-hyperideal of R, then we define the relation N^* by
x congruent $y \Leftrightarrow (x - y) \cap N \neq \varnothing$.

This is a congruence relation on R.

Let N be a Γ-hyperideal of R. Then, for $x, y \in N$, the following are equivalent:
(1) $(x - y) \cap N \neq \varnothing$,
(2) $x - y \subseteq N$,
(3) $y \in x + N$.

The class $x + N$ is represented by x and we denote it with $N^*(x)$. Moreover,
$N^*(x) = N^*(y)$ if and only if $x \equiv y (\mathrm{mod}\, N)$. We can define R / N as follows:

$$R / N = \{N^*(x) | x \in R\}.$$

Define a hyperoperation $\oplus$ and an operation $\odot_\alpha$ on R / N by

$N^*(x) \oplus N^*(y) = \{N^*(z) | z \in N^*(x) \oplus N^*(y)\}$;
$$N^*(x) \odot_\alpha N^*(y) = N^*(x\alpha y)$$

for all $N^*(x), N^*(y) \in R / N$.
Then, $(R / N, \oplus, \odot_\alpha)$ is a Γ-hyperring, see [51].

3. Fuzzy soft sets

A fuzzy set μ of R of the form

$$\mu(y) = \begin{cases} t(\neq 0) & if \quad y = x, \\ 0 & if \quad y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. A fuzzy point x_t is said to be "belong to" (resp.," $quasi - coincident$ with") a fuzzy set μ, written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q\mu$, then we write $x_t \in \vee q\mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$), then we say that $x_t \bar{\in} \mu$ (resp., $x_t \bar{q}\mu$). We note here that the symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For a fuzzy point x_r and μ of X, we say
(1) $x_r \in_\gamma \mu$ if $\mu(x) \geq r > \gamma$.
(2) $x_r q_\delta \mu$ if $\mu(x) + y > 2\delta$.
(3) $x_r \in_\gamma \vee q_\delta \mu$ if $x_r \in_\gamma \mu$ or $x_r q_\delta \mu$.
(4) $x_r \bar{\in}_\gamma \vee \bar{q}_\delta \mu$ if $x_r \bar{\in}_\gamma \mu$ or $x_r \bar{q}_\delta \mu$.

Molodtsov [41] defined the soft set in the following way: let U be an initial universe set, E a set of parameters and $A \subseteq E$.

A pair (F, A) is called a soft set over U, where F is a mapping given by

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

Definition 3.1. [8] Let U be an initial universe set, E a set of parameters and $A \subseteq E$. Then $(\overline{F}, A)$ is called a fuzzy soft set over U, where $\overline{F}$ is a mapping given by $\overline{F} : A \to P(U)$.

In general, for every $x \in A$, $\overline{F}(x)$ is a fuzzy set in U and it is called fuzzy value set of parameter

x. If for every $x \in A$, $\overline{F}(x)$ is a crisp subset of U,

Then $(\overline{F}, A)$ is degenerated to be the standard soft set. Thus, from the above definition, it is clear that fuzzy soft sets are a generalization of standard soft sets.

Let $A \subseteq E$ and $F \in F(U)$. Let $(\overline{F}, A)$ be a fuzzy soft set by $\overline{F}(\alpha) = F$ for all $\alpha \in A$. For any fuzzy point x_r in U, define $(\overline{x_r}, A)$ as a fuzzy soft set by $\overline{x_r}[\alpha] = x_r$ for all $\alpha \in A$.

The notions of AND, OR and bi-intersection operations of fuzzy soft sets can be found in [3, 4].

Now, we introduce a new ordered relation; " $\subseteq \lor q_{(\gamma,\delta)}$ " on the set of all fuzzy soft sets over U.

For two fuzzy soft sets $(\overline{F}, A)$ and $(\overline{G}, B)$ over U, by $(\overline{F}, A) \subseteq \lor q_{(\gamma,\delta)} (\overline{G}, B)$ we mean that $A \subseteq B$ and $x_r \in_\gamma \overline{F}[\alpha] \Rightarrow x_r \in_\gamma \lor q_\delta \overline{G}[\alpha]$ for all $\alpha \in A$, $x \in U$ and $r \in (\gamma, 1]$.

Definition 3.2. For two fuzzy soft sets $(\overline{F}, A)$ and $(\overline{G}, B)$ over U, we say that $(\overline{F}, A)$ is an $(\in_\gamma, x_r \in_\gamma \lor q_\delta)$ -fuzzy soft subset of $(\overline{G}, B)$, if $(\overline{F}, A) \subseteq \lor q_{(\gamma,\delta)} (\overline{G}, B)$.

$(\overline{F}, A)$ and $(\overline{G}, B)$ are said to be $(\in_\gamma, x_r \in_\gamma \lor q_\delta)$ -equal if $(\overline{F}, A) \subseteq \lor q_{(\gamma,\delta)} (\overline{G}, B)$ and $(\overline{G}, B) \subseteq \lor q_{(\gamma,\delta)} (\overline{F}, A)$. This is denoted by $(\overline{F}, A) =_{(\gamma,\delta)} (\overline{G}, B)$

The following two lemmas are obvious.

Lemma 3.3. Let $(\overline{F}, A)$ and $(\overline{G}, B)$ be two fuzzy soft sets over U. Then $(\overline{F}, A) \subseteq \lor q_{(\gamma,\delta)} (\overline{G}, B)$ if and only if $\max \{\overline{G}[\alpha](x), \gamma\} \geq \min \{\overline{F}[\alpha](x), \delta\}$ for all $\alpha \in A$, $x \in U$.

Lemma 3.4. Let $(\overline{F}, A)$ $(\overline{G}, B)$ and $(\overline{H}, C)$ be fuzzy soft sets over U such that $(\overline{F}, A) \subseteq \lor q_{(\gamma,\delta)} (\overline{G}, B)$ and $(\overline{G}, B) \subseteq \lor q_{(\gamma,\delta)} (\overline{H}, C)$. Then $(\overline{F}, A) \subseteq \lor q_{(\gamma,\delta)} (\overline{H}, C)$.

It follows from Lemmas 3.1 and 3.2 that " $=_{(\gamma,\delta)}$ " is an equivalence relation on the set of all fuzzy soft sets over U.

Now, let us define some operations of fuzzy subsets of R.

Definition 3.5. Let $\mu, \nu \in F(R)$ and $\alpha \in \Gamma$. We define fuzzy subsets $-\mu, \mu \oplus \nu$ and $\mu @ \nu$ by

$$-\mu(x) = \mu(-x),$$
$$(\mu \oplus \nu)(x) = \sup_{x = y \oplus z} \min\{\mu(y), \mu(z)\}$$

and

$$(\mu @ \nu)(x) = \begin{cases} \sup_{x = y\alpha z} \min\{\mu(y), \mu(z)\}, & if \exists y, z \in R, x = y\alpha z \\ 0, & otherwise \end{cases}$$

respectively, for all $x \in R$ and $\alpha \in \Gamma$.

Definition 3.6. Let $(\overline{F}, A)$ and $(\overline{G}, B)$ be two fuzzy soft sets over R. The sum of $(\overline{F}, A)$ and $(\overline{G}, B)$, denoted by $(\overline{F}, A) \oplus (\overline{G}, B)$, is defined to be the fuzzy soft set $(\overline{F}, A) \oplus (\overline{G}, B) = (\overline{F} \oplus \overline{G}, C)$ over R, where $C = A \cup B$ and

$$(\overline{F} \oplus \overline{G})[\alpha](x) = \begin{cases} \overline{F}[\alpha](x) & if \quad \alpha \in A - B, \\ \overline{G}[\alpha](x) & if \quad \alpha \in B - A, \\ (\overline{F}[\alpha] \oplus \overline{G}[\alpha])(x) & if \quad \alpha \in A \cap B, \end{cases}$$

for all $\alpha \in C \in C$ and $x \in R$.

Definition 3.7. Let $(\overline{F}, A)$ and $(\overline{G}, B)$ be two fuzzy soft sets over R. The α -product of $(\overline{F}, A)$ and $(\overline{G}, B)$, denoted by $(\overline{F}, A) @ (\overline{G}, B)$, is defined to be the fuzzy soft set $(\overline{F}, A) @ (\overline{G}, B) = (\overline{F} @ \overline{G}, C)$ over R, where $C = A \cup B$ and

$$(\overline{F} @ \overline{G})[\alpha](x) = \begin{cases} \overline{F}[\alpha](x) & if \quad \alpha \in A - B, \\ \overline{G}[\alpha](x) & if \quad \alpha \in B - A, \\ (\overline{F}[\alpha] @ \overline{G}[\alpha])(x) & if \quad \alpha \in A \cap B, \end{cases}$$

for all $\alpha \in C \in C$ and $x \in R$.

4. $(\in_\gamma, \in_\gamma \lor q_\delta)$ -fuzzy soft Γ -hyperideals

In this section, we concentrate our study on the $(\in_\gamma, \in_\gamma \lor q_\delta)$ -fuzzy soft Γ -hyperideals of Γ -hyperrings.

Definition 4.1. A fuzzy soft set $(\overline{F}, A)$ over R is called an $(\in_\gamma, \in_\gamma \lor q_\delta)$ -fuzzysoft left (resp., right) Γ -hyperideal of R if for all $\alpha \in A$, $r, s \in (\gamma, 1]$ and $x, y \in R$:

(F1a) $x_r \in_\gamma \overline{F}[\alpha]$ and $y_s \in_\gamma \overline{F}[\alpha]$ imply $z_{\min\{r,s\}} \in_\gamma \vee q_\delta \overline{F}[\alpha]$ for all $z = x \oplus y$;

(F2a) $x_r \in_\gamma \overline{F}[\alpha]$ implies $(-x)_r \in_\gamma \vee q_\delta \overline{F}[\alpha]$;

(F3a) $x_r \in_\gamma \overline{F}[\alpha]$ implies $(y\alpha x)_r \in_\gamma \vee q_\delta \overline{F}[\alpha]$ (resp., $(x\alpha y)_r \in_\gamma \vee q_\delta \overline{F}[\alpha]$);

(F4a) $x_r \in_\gamma \overline{F}[\alpha]$ implies $z_r \in_\gamma \vee q_\delta \overline{F}[\alpha]$ for all $z = -y \oplus x \oplus y$.

A fuzzy soft set $(\overline{F}, A)$ of R is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$)-fuzzy soft Γ-hyperideal of R if it is both an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right Γ-hyperideal of R.

Example 3. Let $R = \Gamma = \{0, 1, 2\}$ be two canonical hypergroups with hyperoperation $\oplus$ as follows:

$\oplus$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

Define a mapping $R \times \Gamma \times R \to R$ $a\gamma b = a \cdot \gamma \cdot b$ by a for all $a, b \in R$ and $\gamma \in \Gamma$, where "·" is the following multiplication.

·	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

So it can be easily verified that $(R, \oplus, \Gamma)$ is a Γ-hyperring.

Let $E = \{e_1, e_2\}$ be a set of parameters. Define a fuzzy soft set $(\overline{F}, E)$ over R by

$$\overline{F}(e_1) = \frac{0.6}{0} + \frac{0.7}{1} + \frac{0.3}{2}$$

and

$$\overline{F}(e_2) = \frac{0.5}{0} + \frac{0.5}{1} + \frac{0.3}{2}$$

Then $(\overline{F}, E)$ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-fuzzy soft Γ-hyperideal of R.

Lemma 4.2. Let $(\overline{F}, A)$ be a fuzzy soft set over R. Then (F1a) holds if and only if one of the following conditions hold: for all $\alpha \in A$ and $x, y \in R$,

(F1b)
$$\max\left\{\inf_{z = x \oplus y} \overline{F}[\alpha](z), \gamma\right\} \geq \min\left\{\overline{F}[\alpha](x), \overline{F}[\alpha](y), \delta\right\};$$

(F1c) $(\overline{F}, A) \oplus (\overline{F}, A) \subseteq \vee q_{(\gamma, \delta)}(\overline{F}, A)$.

Proof: (F1a) $\Rightarrow$ (F1b) Let $\alpha \in A$ and $x, y \in R$. Suppose if possible that $z \in H$ be such that $z \in x \oplus y$ and $\max\{\overline{F}[\alpha](z), \gamma\} < r = \min\{\overline{F}[\alpha](x), \overline{F}[\alpha](y), \delta\}$. Then $\overline{F}[\alpha](x) \geq r$, $\overline{F}[\alpha](y) \geq r$, $\overline{F}[\alpha](z) < r \leq \delta$, hence, $x_r, y_r \in_\gamma \overline{F}[\alpha]$ and $z_r \overline{\in}_\gamma \vee q_\delta \overline{F}[\alpha]$, a contradiction. Hence (F1b) is valid.

(F1b) $\Rightarrow$ (F1c) Let $\alpha \in A$, $r \in (\gamma, 1]$ and $x \in R$ be such that $x_r \in_\lambda (\overline{F} \oplus \overline{F})[\alpha]$ Suppose if possible, that $x_r \overline{\in}_\gamma \vee q_\delta \overline{F}[\alpha]$. Then $x_r \overline{\in}_\gamma \overline{F}[\alpha]$ and $x_r \overline{q}_\delta \overline{F}[\alpha]$, i.e., $\overline{F}[\alpha](x) < r$ and $\overline{F}[\alpha](x) + r < 2\delta$ which implies that $\overline{F}[\alpha](x) < \delta$. If $x \in y \oplus z$ for some $y, z \in R$, by (F1b), we have

$$\max\{\overline{F}[\alpha](x), \gamma\} \geq \min\{\overline{F}[\alpha](y), \overline{F}[\alpha](z), \delta\},$$

it follows from $\overline{F}[\alpha](x) < \delta$ that

$$\max\{\overline{F}[\alpha](x), \gamma\} \geq \min\{\overline{F}[\alpha](y), \overline{F}[\alpha](z)\}.$$

Hence we have

$$r < (\overline{F} \oplus \overline{F})[\alpha](x) = \sup_{x = a \oplus b} \min\{\overline{F}[\alpha](a), \overline{F}[\alpha](b)\}$$
$$\leq \sup_{x = a \oplus b} \max\{\overline{F}[\alpha](x), \gamma\} = \max\{\overline{F}[\alpha](x), \gamma\}$$

a contradiction. Hence (F1c) is satisfied.

(F1c) $\Rightarrow$ (F1a) Let $\alpha \in A$, $r, s \in (\gamma, 1]$ and $x, y \in R$ be such that $x_r, y_s \in_\lambda \overline{F}[\alpha]$. Then for any $z \in x \oplus y$, we have,

$$(\overline{F} \oplus \overline{F})[\alpha](z) = \sup_{x = a \oplus b} \min\{\overline{F}[\alpha](a), \overline{F}[\alpha](b)\}$$
$$\geq \min_{x = a \oplus b} \{\overline{F}[\alpha](x), \overline{F}[\alpha](y)\gamma\} \geq \min\{r, s\} \geq \gamma.$$

Hence

$$z_{\min\{r,s\}} \in \vee q \overline{F}[\alpha]$$

and so

$$z_{\min\{r,s\}} \in \vee q \overline{F}[\alpha],$$

by (F1c). Hence (F1a) holds.

For any fuzzy soft set $(\overline{F}, A)$ over R, denote by $(\overline{F}^{-1}, A)$ the fuzzy soft set defined by $\overline{F}^{-1}[\alpha](x) = \overline{F}[\alpha](-x)$ for all $\alpha \in A$ and $x \in R$.

Lemma 4.3. Let $(\overline{F}, A)$ be a fuzzy soft set over R. Then (F2a) holds if and only if one of the following conditions hold: for all $\alpha \in A$ and $x, y \in R$,

(F2b) $\max\{\overline{F}[\alpha](-x), \gamma\} \geq \min\{\overline{F}[\alpha](x), \delta\}$;

(F2c) $(\overline{F}, A) \subseteq \vee q_{(\gamma, \delta)}(\overline{F}^{-1}, A)$.

Proof: It is similar to the proof of Lemma 4.2.

Lemma 4.4. Let $(\overline{F}, A)$ be the fuzzy soft set over R. Then (F3a) hold if and only if one of the following conditions hold: for all $\alpha \in A$ and $x, y \in R$,

(F3b) $\max\{\max \overline{F}[\alpha](y \alpha x), \gamma\} \geq \min\{\overline{F}[\alpha](x), \delta\}$

(resp., $\{\max \overline{F}[\alpha](y \alpha x), \gamma\} \geq \min\{\overline{F}[\alpha](x), \delta\}$);

(F3c) $\left(\chi_{R}^{\sim}, A\right) @ \left(\overline{F}, A\right) \subseteq \vee q_{(\gamma, \delta)}(\overline{F}, A)$

(resp., $\left(\overline{F}, A\right) @ \left(\overline{F}, A\right) \subseteq \vee q_{(\gamma, \delta)}(\chi_{R}^{\sim}, A)$).

Proof. It is similar to the proof of Lemma 4.2.

Lemma 4.5. Let $(\overline{F}, A)$ be a fuzzy soft set over R. Then (F4a) holds if and only if one of the following conditions hold: for all $\alpha \in A$ and $x, y \in R$,

(F4b) $\max\{\inf_{z \in -y \oplus x \oplus y} \overline{F}[\alpha](z), \gamma\}$
$\geq \min\{\overline{F}[\alpha](x)\}, \overline{F}[\alpha(y), \delta]$;

(F4c) $(x_1, A) \oplus (\overline{F}, A) \oplus (x_1, A) \subseteq \vee q_{(\lambda, \delta)}(\overline{F}, A)$

Proof: It is similar to the proof of Lemma 4.2.

From the above discussion, we can immediately get the following two theorems:

Theorem 4.6. A fuzzy soft set $(\overline{F}, A)$ over R is an $(\in \gamma, \in \gamma \vee q_{\delta})$ -fuzzy soft Γ-hyperideal of R if and only if it satisfies (F1b), (F2b), (F3b) and (F4b).

Theorem 4.7. A fuzzy soft set $(\overline{F}, A)$ over R is an $(\in \gamma, \in \gamma \vee q_{\delta})$ -fuzzy softΓ-hyperideal of R if and only if it satisfies (F1c), (F2c), (F3c) and (F4c).

For any fuzzy soft set $(\overline{F}, A)$ over R, $\alpha \in A$ and $\gamma \in [0, 1]$, we define

$\overline{F}(\alpha)_r = \{x \in R \mid x_r \in {}_{\gamma}\overline{F}[\alpha]\}$,

$\overline{F}[\alpha]_r^{\delta} = \{x \in R \mid x_r q_{\delta}\overline{F}[\alpha]\}$,

and $[\overline{F}[\alpha]]_r^{\delta} = \{x \in R \mid x_r \in \vee q\overline{F}[\alpha]\}$.

It is clear that

$$[\overline{F}[\alpha]]_r^{\delta} = \overline{F}[\alpha]_r^{\delta} \cup \overline{F}[\alpha]_r.$$

The next theorem provides the relationships between $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy softΓ-hyperideal of R and crisp Γ-hyperideals of R.

Theorem 4.8. Let $(\overline{F}, A)$ be a fuzzy soft set over R. Then

(1) $(\overline{F}, A)$ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft Γ-hyperideal of R .if and only if non-empty set $\overline{F}[\alpha]_r$. is a Γ-hyperideal of R .for all $\alpha \in A$ and $r \in (\gamma, \delta)$.

(2) If $2\delta = 1 + \gamma$, then $(\overline{F}, A)$ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy softΓ-hyperideal of R and only if non-empty set $\overline{F}[\alpha]_r^{\delta}$ is a Γ-hyperideals of R . for all $\alpha \in A$ and $r \in (\delta, 1]$.

(3) If $2\delta = 1 + \gamma$, then $(\overline{F}, A)$ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy softΓ-hyperideal of R and only if non-empty set $\overline{F}[\alpha]_r^{\delta}$ is a Γ-hyperideals of R . for all $\alpha \in A$ and $r \in (\delta, 1]$.

Proof: We only show (3). The proofs of (1) and (2) are similar. Let $(\overline{F}, A)$ be an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft Γ-hyperideal of R . and $x, y \in \overline{F}[\alpha]_r^{\delta}$ for some $\alpha \in A$ and $r \in (\delta, 1]$

Then $x_r \in \gamma \vee q_{\delta}\overline{F}[\alpha]$ and $y_r \in \gamma \vee q_{\delta}\overline{F}[\alpha]$,

i.e, , $\overline{F}[\alpha](x) \geq r$

or $\overline{F}[\alpha](x) \geq 2\delta - r \geq 2\delta - 1 = \gamma$,

and $\overline{F}[\alpha](y) \geq r$ or $\overline{F}[\alpha](y) \geq 2\delta - 1 = \gamma$.

Since $(\overline{F}, A)$ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy softΓ-hyperideal of R and $\min\{\overline{F}[\alpha](x)\}, \overline{F}[\alpha(y), \delta] \geq r$,

we have $\overline{F}[\alpha](Z) \geq \min\{\overline{F}[\alpha](x)\}, \overline{F}(\alpha(y), \delta]$ for all $z \in x \oplus y$. We consider the following cases:

Case 1: $r \in (\gamma, \delta)$. Since $r \in (\gamma, \delta)$, we have $2\delta \geq \delta \geq r$.

Case 1a: $\overline{F}[\alpha](x) \geq r$ or $\overline{F}[\alpha](y) \geq r$. Then

$$\overline{F}[\alpha](z) \geq \min\{\overline{F}[\alpha](x), \overline{F}[\alpha](y), \delta\} \geq r$$

Hence

$$z^r \in_\gamma \overline{F}[\alpha]$$
.

Case 1b: $\overline{F}[\alpha](x) \geq 2\delta - r$ and $\overline{F}[\alpha](y) \geq 2\delta - r$ then

$$\overline{F}[\alpha](z) \geq \min\{\overline{F}[\alpha](x), \overline{F}[\alpha](y), \delta\} = \delta$$

and so

$$f(z) + r \geq r + \delta > 2\delta.$$

Hence

$$z_r q_\delta \overline{F}[\alpha]$$
.

Case 2. $r \in (\delta, 1]$. Since $r \in (\delta, 1]$, we have $2\delta - r < \delta < r$.

Case 2a: $\overline{F}[\alpha](x) \geq r$ and $\overline{F}[\alpha](y) \geq r$. Then $\overline{F}[\alpha](z) \geq \min\{\overline{F}[\alpha](x), \overline{F}[\alpha](y), \delta\} = \delta$, and so $f(z) + r \geq r + \delta > 2\delta$.

Hence $z_r q_\delta \overline{F}[\alpha]$.

Case 2b: $\overline{F}[\alpha](x) > 2\delta - r$ or $\overline{F}[\alpha](y) > 2\delta - r$. Then $\overline{F}[\alpha](z) \geq \min\{\overline{F}[\alpha](x), \overline{F}[\alpha](y), \delta\} > 2\delta - r$

Hence $z_r q_\delta \overline{F}[\alpha]$. Thus in any case, $z_r \in_\gamma \vee q_\delta \overline{F}[\alpha]$, i.e., $z \in \overline{F}[\alpha]]_r^\delta$ for all $z \in x \oplus y$. Similarly we can show that the other conditions hold.

Therefore, $\overline{F}[\alpha]]_r^\delta$ is a Γ-hyperideal of R.

Conversely, assume that the given condition holds. Let $\alpha \in A$ and $x, y \in R$. If there exists $z \in R$ such that $z \in x \oplus y$ and $\max\{\overline{F}[\alpha](z), \gamma\} < r = \min\{\overline{F}[\alpha](x), \overline{F}[\alpha](y), \delta\}$, then $\overline{F}[\alpha](x) \geq r$, $\overline{F}[\alpha](y) \geq r$, $\overline{F}[\alpha](z) < r \leq \delta$, i.e., $x_r, y_r \in \overline{F}[\alpha]$ but $z_r \overline{\in_\gamma} \vee q_\delta \overline{F}[\alpha]$. Hence $x, y \in \overline{F}[\alpha]]_r^\delta$ but $z \overline{\in} \overline{F}[\alpha]]_r^\delta$, a contradiction. Therefore, $\max\{\overline{F}[\alpha](z), \gamma\} \geq \min\{\overline{F}[\alpha](x), \overline{F}[\alpha](y), \delta\}$

for all $z \in x \oplus y$. This proves (F1b) holds. Similarly we can show that (F2b), (F3b) and (F4b) hold. Therefore $(\overline{F}, A)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft Γ-hyperideal of R by Theorem 4.7.

Remark 4.9. For any $(\in \gamma, \in \gamma \vee q_\delta)$-fuzzy soft Γ-hyperideal of R, we can conclude that

(1) $\overline{F}[\alpha]$ is an $(\in \gamma, \in \gamma \vee q_\delta)$-fuzzy soft Γ-hyperideal of R, for all $\alpha \in A$. When $\gamma = 0$ and $\delta = 1$ (see[51]);

(2) $(\overline{F}, A)$ is an $(\in, \in \vee q)$-fuzzy soft Γ-hyperideal of R. When $\gamma = 0$, $\delta = 0.5$.

Theorem 4.10. Let $(\overline{F}, A)$ and $(\overline{G}, B)$ be two $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft Γ-hyperideals of R. Then $(\overline{F}, A) \oplus (\overline{G}, B)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft Γ-hyperideal of R if and only if $(\overline{F}, A) \oplus (\overline{G}, B) =_{(\gamma, \delta)} (\overline{G}, B) \oplus (\overline{F}, A)$.

Proof: If $(\overline{F}, A) \oplus (\overline{G}, B) =_{(\gamma, \delta)} (\overline{G}, B) \oplus (\overline{F}, A)$. then $((\overline{F}, A) \oplus (\overline{G}, B)) \oplus ((\overline{F}, A) \oplus (\overline{G}, B))$

$= (\overline{F}, A) \oplus ((\overline{G}, B) \oplus (\overline{F}, A)) \oplus (\overline{G}, B)$

$=^{(\gamma, \delta)} (\overline{F}, A) \oplus ((\overline{F}, A) \oplus (\overline{G}, B)) \oplus (\overline{G}, B)$

$= ((\overline{F}, A) \oplus (\overline{F}, A)) \oplus ((\overline{G}, B) \oplus (\overline{G}, B))$

$\subseteq \vee q_{(\gamma, \delta)} (\overline{G}, B) \oplus (\overline{F}, A)$.

Hence $((\overline{F}, A) \oplus (\overline{G}, B)) \oplus ((\overline{F}, A) \oplus (\overline{G}, B))$

$\subseteq \vee q_{(\gamma, \delta)} (\overline{F}, A)) \oplus (\overline{G}, B)$
.

This proves that (F1c) holds.

Now for any $\alpha \in A \bigcup B$ and $x \in R$,

Case 1. $\alpha \in A - B$. Then

$\max\{(\overline{F} \oplus \overline{G})[\alpha](-x), \gamma\} = \max\{\overline{F}[\alpha](-x), \gamma\}$

$\geq \min\{\overline{F}[\alpha](x), \delta\} = \min\{(\overline{F} \oplus \overline{G})[\alpha](x), \delta\}$

Case 2. $\alpha \in B - A$. Analogous to case 1, we have

$\max\{(\overline{F} \oplus \overline{G})[\alpha](-x), \gamma\}$

$\geq \min\{(\overline{F} \oplus \overline{G})[\alpha](x), \delta\}$

Case 3. $\alpha \in B \bigcap A$.

$\max\{(\overline{F} \oplus \overline{G})[\alpha](-x), \gamma\}$

$= \max\{\sup_{-x \in a \oplus b} \min \overline{F}[\alpha](a), \overline{G}[\alpha](b), \gamma\}$

$$\geq \sup_{x\in(-a)\oplus(-b)} \min\{\min\{\overline{F}[\alpha](a),\delta\},$$

$$\min\{\overline{G}[\alpha](b),\delta\}\}$$

$$= \min\{\sup_{x\in(-a)\oplus(-b)} \min\{\overline{F}[\alpha](a),\overline{G}[\alpha](b)\},\delta\}$$

$$= \min\{(\overline{F}\oplus\overline{G})[\alpha],\delta\}.$$

Thus, in any case,

$$\max\{\overline{F}[\alpha](-x),\gamma\}$$

$$\geq \min\{(\overline{F}\oplus\overline{G})[\alpha](x),\delta\}.$$

This proves that (F2b) holds. Similarly, we can prove that (F3b) and (F4b) hold. Therefore, $(\overline{F},A)\oplus(\overline{G},B)$ is an $(\in\gamma,\in\gamma\vee q_\delta)$-fuzzy soft Γ-hyperideal of R. Then for any $\alpha\in B\bigcup A$

Case 1. $\alpha\in A-B$. Then

$$(\overline{F}\oplus\overline{G})[\alpha] = \overline{F}[\alpha] = (\overline{G}\oplus\overline{F})[\alpha].$$

Case 2: $\alpha\in B-A$. Then

$$(\overline{F}\oplus\overline{G})[\alpha] = \overline{G}[\alpha] = (\overline{G}\oplus\overline{F})[\alpha].$$

Case 3. $\alpha\in A\cap B$. Then for any $x\in R$.

$$\max\{(\overline{F}\oplus\overline{G})[\alpha](x),\gamma\}$$

$$\geq \max\{\min\{(\overline{F}\oplus\overline{G})[\alpha],\delta\},\gamma\}$$

$$= \min\{\max\{(\overline{F}\oplus\overline{G})[\alpha](-x),\gamma\},\delta\}$$

$$= \min\left\{\max\left\{\sup_{-x\in a\oplus b}\min\left\{\begin{matrix}\overline{F}[\alpha](a),\\\overline{G}[\alpha](b)\end{matrix}\right\},\gamma\right\},\delta\right\}$$

$$= \min\left\{\sup_{x\in -b\oplus -a}\min\left\{\begin{matrix}\max\{\overline{F}[\alpha](a),\gamma\},\\\max\{\overline{G}[\alpha](b),\gamma\}\end{matrix}\right\},\delta\right\}$$

$$\geq \min\left\{\sup_{x\in -b\oplus -a}\min\left\{\begin{matrix}\min\{\overline{F}[\alpha](-a),\delta\},\\\min\{\overline{G}[\alpha](-b),\delta\}\end{matrix}\right\},\delta\right\}$$

$$= \min\left\{\sup_{x\in -b\oplus -a}\min\{\overline{F}[\alpha](-a),\overline{G}[\alpha](-b)\},\delta\right\}$$

$$= \min\{(\overline{G}\oplus\overline{F})[\alpha](x),\delta\}$$

It follows that

$$\max\{\min\{\overline{F}\oplus\overline{G}[\alpha](x),\delta\},\gamma\}$$

$$\geq \max\{\min\{(\overline{G}\oplus\overline{F})[\alpha](x),\delta\},\gamma\}.$$

In a similar way, we have

$$\max\{\min\{(\overline{G}\oplus\overline{F})[\alpha](x),\delta\},\gamma\}$$

$$\geq \max\{\min\{(\overline{F}\oplus\overline{G})[\alpha](x),\delta\},\gamma\}.$$

Thus, in any case, we have

$(\overline{F},A)\oplus(\overline{G},B) =_{(\gamma,\delta)} (\overline{G},B)\oplus(\overline{F},A)$, as required.

The following proposition is obvious.

Proposition 4.11. Let $(\overline{F},A)$ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft Γ-hyperideal of R, then $(\overline{F},A)\oplus(\overline{x}_1,A) =_{(\gamma,\delta)} (\overline{x}_1,A)\oplus(\overline{F},A)$ for all $x\in R$.

Let ρ be an equivalence relation on a Γ-hyperring R, then we say that $A\overline{\rho}B$ if it satisfies the following conditions:
(1) $\forall x\in A,\exists y\in B$, such that $x\rho y$;
(2) $\forall b\in B,\exists a\in A$, such that $a\rho b$.

Definition 4.12. An equivalence relation on R is said to be a congruence relation on R if for any $x,y,z\in R$ and $\alpha\in\Gamma$,
$x\rho y\Rightarrow x\oplus z\ \overline{\rho}\ y\oplus z, z\oplus x\ \overline{\rho}z\oplus y, z\alpha x\ \overline{\rho}z\alpha y$ and $x\alpha z\overline{\rho}y\alpha z$.

Theorem 4.13. Let $(\overline{F},A)$ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft Γ-hyperideal of R, $\alpha\in A$ and $r\in[\gamma,\min\{\overline{F}[\alpha](0),\delta\}]$. Define a relation $\overline{F}[\alpha]^*_r$ on R by

$$x\overline{F}[\alpha]^*_r y\Leftrightarrow \min\{(\overline{x}_1\oplus\overline{F})[\alpha](y),\delta\}>r,$$

for all $x,y\in R$. Then $\overline{F}[\alpha]^*_r$ is a congruence relation on R.

Proof: Let $(\overline{F},A)$ be an $(\in_\gamma,\in\vee q)$-fuzzy soft Γ-hyperideal of R, $\alpha\in A$ and $r\in[\gamma,\min\{\overline{F}[\alpha](0),\delta\}]$. Then we have
(1) $\overline{F}[\alpha]^*_r$ is reflexive on R. In fact, for any $x\in R$, we have

$$\min\{(\overline{x}_1\oplus\overline{F})[\alpha](y),\delta\}$$

$$= \min\{\sup_{x\in x\oplus a}\overline{F}[\alpha],\delta\}$$

$$\geq \min\{\overline{F}[\alpha](0),\delta\}>r.$$

Hence $x\overline{F}[\alpha]^*_r$, and so $\overline{F}[\alpha]^*_r$ is reflexive.
(2) $\overline{F}[\alpha]^*_r$ is symmetric on R by Proposition 4.11.
(3) $(\overline{F},A)$ is transitive on R. In fact, let $x,y,z\in R$ be such that $x\overline{F}[\alpha]^*_r y$ and $x\overline{F}[\alpha]^*_r z$. Then $\min\{(\overline{x}_1\oplus\overline{F})[\alpha](y),\delta\}>r$ and $\min\{(\overline{x}_1\oplus\overline{F})[\alpha](y),\delta\}>r$.

From $(\overline{F}, A) \oplus (\overline{F}, A) \subseteq \vee q_{(r,\delta)} (\overline{F}, A)$,   we   have

$(\overline{x}_1, A) \oplus (\overline{F}, A) \oplus (\overline{F}, A) \subseteq \vee q_{(r,\delta)} (\overline{x}_1, A) \oplus (\overline{F}, A)$,

and so

$\min\left\{(\overline{x}_1 \oplus \overline{F})[\alpha](y), \delta\right\}$

$\geq \min\left\{(\overline{x}_1 \oplus \overline{F} \oplus \overline{F})[\alpha](y), \delta\right\}$

$= \min\left\{\sup_{a \in R} \min \begin{Bmatrix} (\overline{x}_1 \oplus \overline{F})[\alpha](a), \\ (\overline{x}_1 \oplus \overline{F})[\alpha](z) \end{Bmatrix}, \delta\right\}$

$\geq \min\left\{(\overline{x}_1 \oplus \overline{F})[\alpha](y), (\overline{y}_1 \oplus \overline{F})[\alpha](z), \delta\right\} > r.$

It follows that $\min\left\{(\overline{x}_1 \oplus \overline{F})[\alpha](z), \delta\right\} > r$ since

$r \geq \gamma$, i.e., $x\overline{F}[\alpha]_r^* z$. Hence $\overline{F}[\alpha]_r^*$ is transitive.

Summing   the   above   arguments, $\overline{F}[\alpha]_r^*$ is   an

equivalence relation on R . Next let $x, y, z, a \in R$ be

such   that $x\overline{F}[\alpha]_r^* y$ and $a \in x \oplus z.$ Then

$\min\left\{(\overline{y}_1 \oplus \overline{F})[\alpha](x), \delta\right\} > r.$

From Proposition 4.11, we have

$\min\left\{((y \oplus z) \oplus \overline{F})[\alpha](y), \delta\right\}$

$= \min\left\{(\overline{y} \oplus \overline{z}_1 \oplus \overline{F})[\alpha](a), \delta\right\}$

$= \min\left\{(\overline{y} \oplus \overline{z}_1 \oplus \overline{F})[\alpha](a), \delta\right\}$

$= \min\left\{\sup_{b \in r} \min \begin{Bmatrix} (\overline{y}_1 \oplus \overline{F})[\alpha](b), \\ (\overline{b}_1 \oplus \overline{z}_1)[\alpha](a) \end{Bmatrix}, \delta\right\}$

$\geq \min\left\{(\overline{y} \oplus \overline{F})[\alpha](x), (\overline{x}_1 \oplus \overline{z}_1)[\alpha](a), \delta\right\}$

$= \min\left\{(\overline{y}_1 \oplus \overline{F})[\alpha](x), \delta\right\} > r.$

Hence there exists $c \in R$ such that

$c \in y \oplus z$ and $\min\left\{(\overline{c}_1 \oplus \overline{F})[\alpha](y), \delta\right\} > r$, i.e.,

$a\overline{F}[\alpha]_r^* c$. Similarly, if $d \in y \oplus z$ for some $d \in R$,

then   there   exists $e \in R$ such   that $d \in x \oplus z$

and $d\overline{F}[\alpha]_r^* e$, Hence $x \oplus z\overline{F}[\alpha]_r^* y \oplus z$. In a

similar way, we have $z \oplus x\overline{F}[\alpha]_r^* z \oplus y$. Finally, let

$x, y, z \in R$ and $\alpha \in \Gamma$ be such that $x\overline{F}[\alpha]_r^* y.$ Then

$\min\left\{(\overline{y}_1 \oplus \overline{F})(x), \delta\right\} > r.$

From Proposition 4.11, we have

$\min\left\{(\overline{x}_1 \oplus \overline{F})[\alpha](a), \delta\right\}$

$= \min\left\{\sup_{y \in x \oplus a} \min\{\overline{F}[\alpha](a)\}, \delta\right\}$

$= \min\left\{\sup_{b \in R} \min\{\overline{F}[\alpha](b)\}, \delta\right\} > r$

and so ,

$\min\left\{((z\alpha x)_1 \oplus \overline{F})[\alpha](z\alpha y), \delta\right\}$

$= \min\left\{\sup_{z\alpha y \in z\alpha x \oplus a} \min\{\overline{F}[\alpha](a)\}, \delta\right\}$

$= \min\left\{\overline{F}[\alpha](z\alpha b), \delta\right\}$

$= \min\left\{\min\{\overline{F}[\alpha](b)\}, \delta\right\} > r$

that is, $z\alpha z\overline{\overline{F}[\alpha]_r^*} z\alpha y$. In a similar way, we have

$x\alpha z\overline{\overline{F}[\alpha]_r^*} y\alpha z$. Therefore, $\overline{F}[\alpha]_r^*$ is a congruence

relation on R . This completes the proof.

Let $\overline{F}[\alpha]_r^*[x]$ be   the   equivalence   class

containing the element x . We denote by $R / \overline{F}[\alpha]_r^*$

the   set   of   all   equivalence   classes,   i.e.,

$R / \overline{F}[\alpha]_r^* = \left\{\overline{F}[\alpha]_r^*[x] | x \in R\right\}$.   Since $\overline{F}[\alpha]_r^*$ is a

congruence relation on R , we can easily deduce the

following theorem.

Theorem 4.14. Let $(\overline{F}, A)$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy

soft Γ -hyperideal   of R , $\alpha \in A$ and

$r \in \left[\gamma, \min\left\{\overline{F}[\alpha](0), \delta\right\}\right]$. Then $\left(R / \overline{F}[\alpha]_r^*, \oplus, \Gamma\right)$

is   a Γ -hyperideal   with   respect   to   the

hyperoperation $\oplus$ and   the   operation   @:

$\overline{F}[\alpha]_r^*[x] \oplus \overline{F}[\alpha]_r^*[y] = \left\{\overline{F}[\alpha]_r^*[z] | z \in x \oplus y\right\}$

and $\overline{F}[\alpha]_r^*[x] @ \overline{F}[\alpha]_r^*[y] = \overline{F}[\alpha]_r^*[x\alpha y],$ for   all

$x, y \in R$ and $\alpha \in \Gamma$.

5. Conclusion

In this paper,we consider the $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy

soft Γ -hyperideal of Γ -hyperideal. In particular,

we   obtain   a   kind   of   new Γ -hyperideal   by

congruence relations. In our future study of fuzzy

structure of Γ -hyperideal,   the   following topics

could be considered:

1. To consider the (m,n)- Γ -hyperideals;
2. To establish three (fuzzy) isomorphism theorems
   of (m,n)- Γ -hyperideals;
3. To describe the fuzzy soft (m,n)- Γ -hyperideals
   and their applications.

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