

## $\frac{\lambda}{2}$ -Legendre curves in 3-dimensional Heisenberg group $IN^3$

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### Abstract

In this study, we focused on  $\frac{\lambda}{2}$ -Legendre curves and non- $\frac{\lambda}{2}$ -Legendre curves in 3-dimensional Heisenberg group  $IN^3$ . Also, we gave some characterizations of these curves.

**Keywords:** Heisenberg group; Sasakian manifold; Legendre curve

### 1. Introduction

In mathematics, the Heisenberg group, named after Werner Heisenberg, is the group of  $3 \times 3$  upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

or its generalizations under the operation of matrix multiplication. In 1987, L. Bianchi classified the homogeneous metrics. L. Bianchi, E. Cartan and G. Vranceanu found the following 2-parameter family of homogeneous Riemannian metrics on the cartesian 3- space  $IR^3(x, y, z)$ :

$$g_{\lambda, \mu} = \frac{dx^2 + dy^2}{\{1 + \mu(x^2 + y^2)\}} + \left\{ dz + \frac{\lambda}{2} \frac{ydx - xdy}{\{1 + \mu(x^2 + y^2)\}} \right\}^2, \forall \lambda, \mu \in IR.$$

In this family, if  $\lambda = \mu = 0$ , the Euclidean metric is obtained, and if  $\lambda \neq 0, \mu = 0$ , the Heisenberg metric is obtained. Inoguchi studied the differential geometry of Heisenberg metric.

The Legendre curves play an important role in the study of contact manifolds. In a 3- dimensional Sasakian manifold, the Legendre curves are studied by Baikousis and Blair who gave the Frenet 3- frame in this space [1]. Yıldırım gave some characterizations of Legendre curves in Homogeneous space [2]. İlarslan gave a characterization of curves on non-Euclidean manifolds [3]. On the other hand, Baikosis and Hirica studied Legendre curves in Riemannian and Lorentzian Sasaki spaces [4]. Also, Legendre

curves in  $\alpha$ - Sasakian spaces are studied by Özgür and Tripathi [5]. In this study, we focused on  $\frac{\lambda}{2}$ -Legendre curves in 3-dimensional Heisenberg group in  $IN^3$  and gave a characterization of these curves. Also, we gave similar results for non- $\frac{\lambda}{2}$ -Legendre curves in 3-dimensional Heisenberg group in  $IN^3$ .

### 2. Preliminaries

In this section, we will give some basic concepts related to Sasakian geometry for later use.

The Heisenberg group  $IN^3$  can be seen as the Euclidean space with the multiplication

$$(x, y, z)(x', y', z') = \left( x + x', y + y', z + z' + \frac{\lambda}{2}(xy' - yx') \right)$$

and with the Riemannian metric

$$g_{\lambda} = dx^2 + dy^2 + \left\{ dz + \frac{\lambda}{2} \frac{ydx - xdy}{\{1 + \mu(x^2 + y^2)\}} \right\}^2, \forall \lambda, \mu \in IR. \quad (1)$$

$IN^3$  is a three dimensional, connected, simply connected and 2-step nilpotent Lie group. The Lie algebra of  $IN^3$  has a basis

$$\begin{cases} e_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z} \\ e_2 = \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \\ e_3 = \frac{\partial}{\partial z} \end{cases} \quad (2)$$

which is dual to

$$\begin{cases} \theta^1 = dx \\ \theta^2 = dy \\ \theta^3 = dz + \frac{\lambda}{2}(ydx - xdy). \end{cases} \quad (3)$$

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For this basis Lie brackets are

$$[e_1, e_2] = e_3, [e_3, e_1] = [e_2, e_3] = 0, [6].$$

To study curves in  $IN^3$ , we shall use their Frenet vector fields and equations. Let  $\gamma: I \rightarrow IN^3$  be a differentiable curve parametrized by arc length and let  $\{V_1, V_2, V_3\}$  be the orthonormal frame field tangent defined as follows: by  $V_1$  we denote  $\dot{\gamma}$  tangent to  $\gamma$ , by  $V_2$  the unit vector field in the direction  $D_{V_1}V_1$  normal to  $\gamma$  and we choose  $V_3 = V_1 \times V_2$ , so that  $\{V_1, V_2, V_3\}$  is a positive oriented orthonormal basis. Thus, we have the following Frenet equations [7]:

$$\begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}V_2 \\ D_{V_1}V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \quad (4)$$

Now, let us consider the odd-dimensional Riemannian manifold  $(M, g)$ . So, the Riemannian manifold  $(M, g)$  is said to be an almost contact metric manifold if there exist on  $M$  a  $(1,1)$ -tensor field  $\varphi$ , a vector field  $\xi$  (called the Reeb vector field) and a 1-form  $\eta$  such that

$$\eta(\xi) = 1, \varphi^2(X) = -X + \eta(X)\xi$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields  $X, Y$  on  $M$ . In particular, in an almost contact metric manifold we also have  $\varphi\xi = 0$  and  $\eta\varphi = 0$ .

Such a manifold is said to be contact metric manifold, if  $d\eta = \Phi$ , where  $\Phi(X, Y) = g(X, \Phi Y)$  is called the fundamental 2-form of  $M$ . If  $\xi$  is a Killing vector field, then  $M$  is said to be a  $K$ -contact manifold, we have  $(D_X\varphi)Y = R(\xi, X)Y$  for any  $X, Y \in M$ .

On the other hand, the almost contact metric structure of  $M$  is said to be normal if  $[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$ , [8, 9].

A normal contact metric manifold is called a Sasakian Manifold. It can be proved that a Sasakian manifold is  $K$ -contact, and that an almost contact metric manifold is Sasakian if and only if

$$(D_X\varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for any  $X, Y$ . Furthermore, assuming that  $\eta = \theta^3$ ,  $\xi = e_3$  and defining

$$\begin{aligned} \varphi: \chi(IN^3) &\rightarrow \chi(IN^3), \varphi(X) \\ &= -a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} \\ &\quad + \frac{\lambda}{2} (x_1 a_1 + x_2 a_2) \frac{\partial}{\partial x_3} \end{aligned}$$

where  $\sum_{i=1}^3 a_i \frac{\partial}{\partial x_i} \in \chi(IN^3)$ , it can be easily seen that  $IN^3$  is a Sasakian space. Since all computations have  $\frac{\lambda}{2}$  coefficients, we have denoted  $IN^3$  as  $\frac{\lambda}{2}$ -Sasakian space. We need the following Lemma for later use:

**Lemma:** Let  $X$  and  $Y$  be two vector fields in  $\chi(IN^3)$ ,  $D$  and  $\tilde{D}$  be Riemannian connections on  $IN^3$  and  $IE^3$ , respectively. Thus,

$$D_X Y = \frac{\lambda}{2} X \wedge Y - g_\lambda([e_1, e_2], X)\varphi Y + \tilde{D}_X^Y. \quad (5)$$

On the other hand, if  $D$  is the contact distribution in a contact manifold  $(M, \varphi, \xi, \eta)$ , defined by the subspaces  $D_m = \{X \in T_m M \mid \eta(X) = 0\}$ , then a one-dimensional integral submanifold of  $D$  will be called a Legendre curve. A curve  $\gamma: I \rightarrow M$ , parametrized by its arc length is a Legendre curve if and only if  $\eta(\dot{\gamma}) = 0$ , [8, 9].

### 3. $\frac{\lambda}{2}$ -Legendre Curves in $IN^3$

**Theorem 3.1.** Let  $\gamma: I \rightarrow IN^3$  be a non-geodesic  $\frac{\lambda}{2}$ -Legendre curve. The Frenet frame of  $\gamma$  is  $\{V_1, \varphi V_1, \xi\}$  and the Frenet formulas are

$$\begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}\varphi V_1 \\ D_{V_1}\xi \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ \varphi V_1 \\ \xi \end{bmatrix}. \quad (6)$$

**Proof:** Let  $\gamma: I \rightarrow IN^3$  be a curve with arc length parameter and the Frenet frame of  $\gamma$  be  $\{V_1, V_2, V_3\}$ . Assume that  $\eta(\dot{\gamma}) = \sigma \neq 0$ . In this case, an orthonormal basis of  $\frac{\lambda}{2}$ -Sasakian space is  $\left\{V_1, \frac{\varphi V_1}{\sqrt{1-\sigma^2}}, \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}\right\}$ . From here, we get

$$D_{V_1}V_1 = \alpha \frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \quad \alpha, \beta \in C^\infty(\mathbb{N}^3, \mathbb{R}).$$

On the other hand, derivating  $\sigma$  we obtain

$$\begin{aligned} \dot{\sigma} &= D_{V_1}\sigma \\ &= D_{V_1}g_\lambda(V_1, \xi) \\ &= g_\lambda(D_{V_1}V_1, \xi) + g_\lambda(V_1, D_{V_1}\xi) \\ &= g_\lambda\left(\alpha \frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \xi\right) + g_\lambda(V_1, -\frac{\lambda}{2}\varphi V_1) \\ &= \beta\sqrt{1-\sigma^2}. \end{aligned}$$

From here, we say that

$$\beta = \dot{\sigma} \frac{1}{\sqrt{1-\sigma^2}}.$$

Since  $\gamma$  is a  $\frac{\lambda}{2}$ -Legendre curve, we can easily see that  $\beta = 0$ . Moreover, from (4) we get  $\alpha = \kappa$ ,

$$V_2 = \varphi V_1, D_{V_1} V_1 = \kappa \varphi V_1 \text{ and}$$

$$\begin{aligned} D_{V_1} V_2 &= \varphi D_{V_1} V_1 + (D_{V_1} \varphi) V_1 \\ &= \varphi(\kappa \varphi V_1) + \frac{\lambda}{2} \{g_\lambda(V_1, V_1) \xi - \eta(V_1) V_1\} \\ &= -\kappa V_1 + \frac{\lambda}{2} \xi. \end{aligned}$$

From (4), we get  $V_3 = \xi$ ,  $\tau = -\frac{\lambda}{2}$ . Hence, the Serret-Frenet formulas are

$$\begin{bmatrix} D_{V_1} V_1 \\ D_{V_1} \varphi V_1 \\ D_{V_1} \xi \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ \varphi V_1 \\ \xi \end{bmatrix}.$$

**Theorem 3.2:** Let  $\gamma: I \rightarrow IN^3$  be a non-geodesic  $\frac{\lambda}{2}$ -Legendre curve and  $0 < |\eta(\dot{\gamma})| < 1$ . The curvature and the torsion of  $\gamma$  are

$$\kappa = \sqrt{\alpha^2 + \beta^2}, \alpha, \beta \in C^\infty(\mathbb{N}^3, \mathbb{R}) \quad (7)$$

and

$$\tau = \frac{\lambda}{2} + \frac{\alpha\beta - \alpha\dot{\beta}}{\alpha^2 + \beta^2} + \frac{\alpha\dot{\sigma}}{\sqrt{1-\sigma^2}}, \quad (8)$$

respectively.

**Proof:** Let  $\gamma: I \rightarrow IN^3$  be a curve with arc length parameter and the Frenet frame of  $\gamma$  be  $\{V_1, V_2, V_3\}$ . Assume that  $\eta(\dot{\gamma}) = \sigma \neq 0$ . In this case, an orthonormal basis of  $\frac{\lambda}{2}$ -Sasakian space is  $\left\{V_1, \frac{\varphi V_1}{\sqrt{1-\sigma^2}}, \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}\right\}$ . From here we get

$$D_{V_1} V_1 = \alpha \frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \alpha, \beta \in C^\infty(\mathbb{N}^3, \mathbb{R}).$$

So, we obtain

$$\kappa = \|D_{V_1} V_1\| = \sqrt{\alpha^2 + \beta^2}, \alpha, \beta \in C^\infty(\mathbb{N}^3, \mathbb{R})$$

and

$$V_2 = \frac{1}{\kappa} D_{V_1} V_1.$$

On the other hand, derivating  $\varphi V_1$ , we have

$$\begin{aligned} D_{V_1} \varphi V_1 &= \varphi D_{V_1} V_1 + (D_{V_1} \varphi) V_1 \\ &= \varphi \left( \alpha \frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}} \right) + \frac{\lambda}{2} (\xi - \sigma V_1) \\ &= -\frac{\alpha}{\sqrt{1-\sigma^2}} V_1 + \frac{\alpha\dot{\sigma}}{\sqrt{1-\sigma^2}} \xi - \frac{\beta\sigma}{\sqrt{1-\sigma^2}} \varphi V_1 + \\ &\frac{\lambda}{2} (\xi - \sigma V_1). \end{aligned} \quad (9)$$

Similarly, derivating  $\xi - \sigma V_1$  we get,

$$D_{V_1} (\xi - \sigma V_1) = D_{V_1} \xi - \dot{\sigma} V_1 - \sigma D_{V_1} V_1$$

$$= -\frac{\lambda}{2} \varphi V_1 - \dot{\sigma} V_1 - \sigma \alpha \frac{\varphi V_1}{\sqrt{1-\sigma^2}} - \sigma \beta \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}. \quad (10)$$

On the other hand, derivating  $\sigma$  we have

$$\begin{aligned} \dot{\sigma} &= D_{V_1} \sigma \\ &= D_{V_1} g_\lambda(V_1, \xi) \\ &= g_\lambda(D_{V_1} V_1, \xi) + g_\lambda(V_1, D_{V_1} \xi) \\ &= g_\lambda \left( \alpha \frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \xi \right) + g_\lambda(V_1, -\frac{\lambda}{2} \varphi V_1) \\ &= \beta \sqrt{1-\sigma^2}. \end{aligned}$$

From here, we see that

$$\beta = \dot{\sigma} \frac{1}{\sqrt{1-\sigma^2}}.$$

Similarly, derivating  $\frac{\alpha}{\sqrt{1-\sigma^2}}$  and  $\frac{\beta}{\sqrt{1-\sigma^2}}$  we obtain

$$D_{V_1} \left( \frac{\alpha}{\sqrt{1-\sigma^2}} \right) = \dot{\alpha} \frac{1}{\sqrt{1-\sigma^2}} + \alpha \beta \sigma \frac{1}{1-\sigma^2} \quad (11)$$

and

$$D_{V_1} \left( \frac{\beta}{\sqrt{1-\sigma^2}} \right) = \dot{\beta} \frac{1}{\sqrt{1-\sigma^2}} + \beta^2 \sigma \frac{1}{1-\sigma^2} \quad (12)$$

respectively. Furthermore,

$$\begin{aligned} D_{V_1} V_2 &= D_{V_1} \left( \frac{1}{\kappa} D_{V_1} V_1 \right) \\ &= -\frac{\dot{\kappa}}{\kappa^2} D_{V_1} V_1 + \frac{1}{\kappa} D_{V_1} D_{V_1} V_1 \\ &= -\frac{\dot{\kappa}}{\kappa^2} D_{V_1} V_1 + \frac{1}{\kappa} D_{V_1} \left( \frac{\alpha}{\sqrt{1-\sigma^2}} \right) \varphi V_1 \\ &\quad + \frac{1}{\kappa} \left( \frac{\alpha}{\sqrt{1-\sigma^2}} \right) D_{V_1} \varphi V_1 \\ &\quad + \frac{1}{\kappa} D_{V_1} \left( \frac{\beta}{\sqrt{1-\sigma^2}} \right) (\xi - \sigma V_1) + \\ &\frac{1}{\kappa} \left( \frac{\beta}{\sqrt{1-\sigma^2}} \right) D_{V_1} (\xi - \sigma V_1). \end{aligned}$$

Using (9), (10), (11) and (12), we get

$$\begin{aligned} D_{V_1} V_2 &= -\kappa V_1 - \left( -\frac{\alpha\dot{\kappa}}{\kappa^2} + \frac{\dot{\alpha}}{\kappa} - \frac{\lambda\beta}{2\kappa} \right. \\ &\quad \left. - \frac{\alpha\beta\sigma}{\kappa\sqrt{1-\sigma^2}} \right) \frac{\varphi V_1}{\sqrt{1-\sigma^2}} \\ &\quad + \left( -\frac{\beta\dot{\kappa}}{\kappa^2} + \frac{\dot{\beta}}{\kappa} - \frac{\lambda\alpha}{2\kappa} - \frac{\alpha^2\sigma}{\kappa\sqrt{1-\sigma^2}} \right) \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}. \end{aligned}$$

From (6), it can be easily seen that

$$\begin{aligned} \tau V_3 &= \left( -\frac{\alpha\dot{\kappa}}{\kappa^2} + \frac{\dot{\alpha}}{\kappa} - \frac{\lambda\beta}{2\kappa} - \frac{\alpha\beta\sigma}{\kappa\sqrt{1-\sigma^2}} \right) \frac{\varphi V_1}{\sqrt{1-\sigma^2}} \\ &\quad + \left( -\frac{\beta\dot{\kappa}}{\kappa^2} + \frac{\dot{\beta}}{\kappa} - \frac{\lambda\alpha}{2\kappa} - \frac{\alpha^2\sigma}{\kappa\sqrt{1-\sigma^2}} \right) \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}. \end{aligned}$$

Taking the norm of the last equation, we have

$$\tau = \frac{\lambda}{2} + \frac{\alpha\beta - \alpha'\beta'}{\alpha^2 + \beta^2} + \frac{\alpha\sigma}{\sqrt{1 - \sigma^2}}$$

**Lemma 3.1.** Let  $\gamma: I \rightarrow IN^3$  be a curve with arc length parameter and  $\{V_1, V_2, V_3\}$  be the Frenet frame of  $\gamma$ . Then, the following equation is obtained:

$$D_{V_1}^3 V_1 - 2 \frac{\dot{\kappa}}{\kappa} D_{V_1}^2 V_1 + \left( 2 \frac{\dot{\kappa}}{\kappa} - \frac{\ddot{\kappa}}{\kappa} + \kappa^2 + \frac{\lambda^2}{4} \right) D_{V_1} V_1 + \kappa \dot{\kappa} V_1 = 0. \tag{13}$$

**Proof:** From (6), we know that

$$D_{V_1} \varphi V_1 = -\kappa V_1 + \frac{\lambda}{2} \xi$$

and

$$D_{V_1} V_1 = -\kappa \varphi V_1.$$

From here,

$$\begin{aligned} D_{V_1} \frac{1}{\kappa} D_{V_1} V_1 &= -\kappa V_1 + \frac{\lambda}{2} \xi \\ \Rightarrow \left( \frac{1}{\kappa} \right)' D_{V_1} V_1 + \frac{1}{\kappa} D_{V_1}^2 V_1 &= -\kappa V_1 + \frac{\lambda}{2} \xi. \end{aligned}$$

Differentiating the last equation, we have

$$\begin{aligned} \frac{1}{\kappa} D_{V_1}^3 V_1 + 2 \left( \frac{1}{\kappa} \right)' D_{V_1}^2 V_1 + \left( \left( \frac{1}{\kappa} \right)'' + \kappa + \frac{\lambda^2}{4} \frac{1}{\kappa} \right) D_{V_1} V_1 \\ + \dot{\kappa} V_1 = 0. \end{aligned}$$

Considering the last equation, we get

$$D_{V_1}^3 V_1 - 2 \frac{\dot{\kappa}}{\kappa} D_{V_1}^2 V_1 + \left( 2 \frac{\dot{\kappa}}{\kappa} - \frac{\ddot{\kappa}}{\kappa} + \kappa^2 + \frac{\lambda^2}{4} \right) D_{V_1} V_1 + \kappa \dot{\kappa} V_1 = 0.$$

**Theorem 3.3.** Let  $\gamma: I \rightarrow IN^3$ ,  $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ , be a  $\frac{\lambda}{2}$ -Legendre curve in  $IN^3$  and  $\alpha$  be the projection curve of  $\gamma$  on  $z = 0$  plane. Then, the curvature of  $\gamma$  is the curvature of  $\alpha$ .

**Proof:** The tangent vector field of  $\gamma$  is

$$\dot{\gamma}(t) = \dot{\gamma}_1(t)e_1 + \dot{\gamma}_2(t)e_2 + \dot{\gamma}_3(t)e_3.$$

We can choose the parameter of  $\gamma$  as  $\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2 = 1$ . Then, if we choose  $\gamma_1(t)$  and  $\gamma_2(t)$  as  $\gamma_1(t) = -\sin \theta(t)$ ,  $\gamma_2(t) = \cos \theta(t)$ , respectively, we obtain

$$D_{\dot{\gamma}(t)} \dot{\gamma}(t) = \ddot{\gamma}_1(t)e_1 + \ddot{\gamma}_2(t)e_2$$

and

$$\|D_{\dot{\gamma}(t)} \dot{\gamma}(t)\| = \frac{1}{2} \sqrt{\ddot{\gamma}_1(t)^2 + \ddot{\gamma}_2(t)^2}$$

$$\kappa = \dot{\theta}(t).$$

On the other hand, the projection curve  $\alpha$  of  $\gamma$  on

$z = 0$  plane is  $\alpha(t) = (\gamma_1(t), \gamma_2(t))$ . Thus, it can be easily seen that  $\alpha$  is a unit speed curve. The curvature of  $\alpha$  is

$$\kappa_\alpha = \frac{|\dot{\gamma}_1(t)\dot{\gamma}_2(t) - \gamma_1(t)\dot{\gamma}_2(t)|}{\sqrt{(\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2)^2}}$$

From here,

$$\kappa = \kappa_\alpha.$$

**Corollary 3.1.** Let  $\gamma$  be a non-geodesic Legendre curve in  $IN^3$ . Then,

- i)  $\gamma$  is not a circle.
- ii) If  $\gamma$  is a helix, it satisfies the following equation:  
 $\Delta H = \left( \kappa^2 + \frac{\lambda^2}{4} \right) H$ .
- iii) If  $\gamma$  is a line,  
 $g_\lambda(D_{V_1} V_1, \varphi V_1) = 0$ .
- iv)  $\gamma$  is not a planar curve.

**Proof:** i) Since  $\gamma$  is a  $\frac{\lambda}{2}$ -Legendre curve, the torsion of  $\gamma$  is  $-\frac{\lambda}{2}$ . So, it can be easily seen that  $\gamma$  is not a circle.

ii) If  $\gamma$  is helix,  $\frac{\kappa}{\tau}$  is constant. Also, on the ground that the torsion of  $\gamma$  is  $-\frac{\lambda}{2}$ ,  $\kappa$  must be constant. So,  $\dot{\kappa}, \ddot{\kappa} = 0$ .

From (13), we obtain

$$D_{V_1}^3 V_1 = - \left( \kappa^2 + \frac{\lambda^2}{4} \right) D_{V_1} V_1.$$

Using  $V_1 = \dot{\gamma}$ ,  $\Delta = -D_{V_1} D_{V_1} V_1$  and  $H = D_{V_1} V_1$  we have

$$\Delta H = \left( \kappa^2 + \frac{\lambda^2}{4} \right) H.$$

iii) If  $\gamma$  is a line, the curvature of  $\gamma$  is zero. Also,  $D_{V_1} V_1 = \kappa \varphi V_1$ .

From here, we get

$$g_\lambda(D_{V_1} V_1, \varphi V_1) = 0.$$

iv) Since  $\gamma$  is a  $\frac{\lambda}{2}$ -Legendre curve, the torsion of  $\gamma$  is not zero. So, it is said that  $\gamma$  is not a planar curve.

**Example 3.1.**

$\gamma: I \rightarrow \mathbb{N}^3, \gamma(t) = \left( r \cos t, r \sin t, \frac{\lambda}{2} r^2 t \right)$  is a curve in  $IN^3$ . If we assume that

$$x = r \cos t$$

$$y = r \sin t$$

$$z = \frac{\lambda}{2} r^2 t$$

we get

$$\dot{\gamma}(t) = \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)_{\gamma(t)}.$$

Thus, using (1.3), we get

$$\begin{cases} \theta^1(\dot{\gamma}(t)) = -y \\ \theta^2(\dot{\gamma}(t)) = x \\ \theta^3(\dot{\gamma}(t)) = 0. \end{cases} \quad (14)$$

From (14), we can say that  $\gamma$  is a  $\frac{\lambda}{2}$ -Legendre curve. On the other hand, we obtain

$$\begin{aligned} \|\dot{\gamma}(t)\| &= \sqrt{[\theta^1(\dot{\gamma}(t))]^2 + [\theta^2(\dot{\gamma}(t))]^2 + [\theta^3(\dot{\gamma}(t))]^2} \\ &= |r|, \\ V_1 &= \mp \frac{y}{r} e_1 \mp \frac{x}{r} e_2 \end{aligned}$$

and

$$\varphi V_1 = \mp \frac{x}{r} e_1 \mp \frac{y}{r} e_2.$$

Moreover, from (5) we have

$$\begin{aligned} D_{V_1} V_1 &= \frac{\lambda}{2} V_1 \wedge V_1 - g_\lambda([e_1, e_2], V_1) \varphi V_1 + \tilde{D}_{V_1}^{V_1} \\ &= -g_\lambda([e_1, e_2], V_1) \varphi V_1 + \tilde{D}_{V_1}^{V_1} \\ &= \mp \frac{1}{r} \varphi V_1. \end{aligned}$$

Namely, we see that

$$\kappa = \mp \frac{1}{r}$$

where  $\kappa$  is the curvature of  $\gamma$ . Also, we know that  $\tau = -\frac{\lambda}{2}$  for a non-geodesic  $\frac{\lambda}{2}$ -Legendre curve in  $\mathbb{N}^3$ . As a result,  $\kappa$  and  $\tau$  are non-zero constants. So,  $\gamma$  is a helix.

**Result 3.1.** Helix in Euclidean space is a helix in  $\frac{\lambda}{2}$ -Sasakian space, too. Also, it is a  $\frac{\lambda}{2}$ -Legendre curve.

**Corollary 3.2.**  $\gamma: I \rightarrow \mathbb{N}^3$  be a  $\frac{\lambda}{2}$ -non-Legendre curve. Then,

**i)** If  $\gamma$  is a geodesic, it satisfies the following equation:

$$\tilde{D}_{V_1}^{V_1} = g_\lambda([e_1, e_2], V_1) \varphi V_1.$$

**ii)** If  $\gamma$  is a circle,

$$\lambda = \frac{2\alpha\sigma}{\sqrt{1-\sigma^2}}$$

or

$$\lambda = -\frac{2\alpha\sigma}{\sqrt{1-\sigma^2}} + \dot{\theta}(t)r^2.$$

where  $\alpha = r \cos \theta(t)$  and  $\beta = r \sin \theta(t)$ .

**iii)** If  $\gamma$  is a circular helix,

$$\tau = -\frac{\lambda}{2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}}.$$

**iv)** If  $\gamma$  is a helix,

$$\alpha^2 + \beta^2 = c^2 \left( \frac{\lambda}{2} + \frac{\alpha\beta - \alpha\beta}{\alpha^2 + \beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}} \right)^2.$$

**Proof: i)** If  $\gamma$  is a geodesic,  $\kappa = \tau = 0$ . So, from (7) we say that  $\alpha = \beta = 0$  and  $\tau$  is indefinite.

On the other hand, if  $\gamma$  is a geodesic,  $D_{V_1} V_1 = 0$ . So, from (5) we get

$$\tilde{D}_{V_1}^{V_1} = g_\lambda([e_1, e_2], V_1) \varphi V_1.$$

**ii)** If  $\gamma$  is a circle,  $\kappa$  is a non-zero constant. In which case there are two situations:

**a)** We assume that  $\alpha$  and  $\beta$  are constants. Thus,

$$\tau = -\frac{\lambda}{2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}} = 0$$

or

$$\lambda = \frac{2\alpha\sigma}{\sqrt{1-\sigma^2}}.$$

**b)** We assume that  $\kappa$  is a non-zero constant and  $\alpha$  and  $\beta$  are not constants. Hence, if  $\alpha$  and  $\beta$  are chosen as  $r \cos \theta(t)$  and  $r \sin \theta(t)$ , respectively, it is found that

$\alpha^2 + \beta^2 = r^2$   
and

$$\alpha\dot{\beta} - \dot{\alpha}\beta = \dot{\theta}(t)r^2.$$

Since  $\tau = 0$ , from (12) we get

$$\lambda = -\frac{2\alpha\sigma}{\sqrt{1-\sigma^2}} + \dot{\theta}(t)r^2.$$

**iv)** If  $\gamma$  is a helix,  $\frac{\kappa}{\tau} = c, c \neq 0 = \text{const}$  and from (7) and (8)

$$\alpha^2 + \beta^2 = c^2 \left( \frac{\lambda}{2} + \frac{\alpha\beta - \alpha\beta}{\alpha^2 + \beta^2} + \frac{\alpha\sigma}{\sqrt{1-\sigma^2}} \right)^2.$$

**Example 3.2.**

$\gamma: I \rightarrow \mathbb{N}^3, \gamma(t) = (r \cos t, r \sin t, c)$  is a curve in  $\mathbb{N}^3$ . If we assume that,

$$x = r \cos t$$

$$y = r \sin t$$

$$z = c$$

we get

$$\dot{\gamma}(t) = \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right)_{\gamma(t)}.$$

Thus using (3), we obtain

$$\begin{cases} \theta^1(\dot{\gamma}(t)) = -y \\ \theta^2(\dot{\gamma}(t)) = x \\ \theta^3(\dot{\gamma}(t)) = -r^2. \end{cases}$$

So, we can say that  $\gamma$  is not a  $\frac{\lambda}{2}$ -Legendre curve. On the other hand, we have

$$\dot{\gamma}(t) = (-ye_1 + xe_2 - r^2e_3)_{\gamma(t)}$$

and

$$\|\dot{\gamma}(t)\| = \sqrt{[\theta^1(\dot{\gamma}(t))]^2 + [\theta^2(\dot{\gamma}(t))]^2 + [\theta^3(\dot{\gamma}(t))]^2}.$$

Thus, we get

$$V_1 = -\frac{y}{r\sqrt{r^2+1}}e_1 + \frac{x}{r\sqrt{r^2+1}}e_2 - \frac{r}{\sqrt{r^2+1}}e_3$$

and

$$\varphi V_1 = -\frac{x}{r\sqrt{r^2+1}}e_1 - \frac{y}{r\sqrt{r^2+1}}e_2.$$

Moreover, from (5) we have

$$\begin{aligned} D_{V_1}V_1 &= \frac{\lambda}{2}V_1 \wedge V_1 - g_\lambda([e_1, e_2], V_1)\varphi V_1 + \tilde{D}_{V_1}^{V_1} \\ &= -g_\lambda([e_1, e_2], V_1)\varphi V_1 + \widehat{D}_{V_1}V_1 \\ &= \left(\frac{1}{r\sqrt{r^2+1}} + \frac{\lambda}{2\sqrt{r^2+1}}\right)\varphi V_1. \end{aligned}$$

Since,

$$D_{V_1}V_1 = \alpha \frac{\varphi V_1}{\sqrt{1-\sigma^2}} + \beta \frac{\xi - \sigma V_1}{\sqrt{1-\sigma^2}}, \alpha, \beta \in \mathbb{R}$$

we obtain

$$\alpha = \frac{1}{r^3+r} + \frac{\lambda r}{2r^2+2}$$

and  $\beta = 0$ . On the other hand, we get

$$\kappa = \left| \frac{1}{r^3+r} + \frac{\lambda r}{2r^2+2} \right|$$

And

$$\tau = -\frac{\lambda}{2} \left( \frac{1}{r^2+1} \right) - \left( \frac{1}{r^2+1} \right)$$

where  $\kappa$  and  $\tau$  are the curvature and the torsion of  $\gamma$ , respectively. As a result, we say that  $\kappa$  and  $\tau$  are non-zero constants. Namely,  $\gamma$  is a circular helix.

**Result 3.2.** Circle in Euclidean space  $IE^3$  is a circular helix in  $\frac{\lambda}{2}$ -Sasakian space.

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