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Ideal theory in Γ-semihyperrings

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Abstract

The concept of Γ-semihyperring is a generalization of semiring, a generalization of semihyperring and a generalization of Γ-semiring. Since the theory of ideals plays an important role in the theory of Γ-semihyperring, in this paper, we will make an intensive study of the notions of Noetherian, Artinian, simple and regular Γ-semihyperrings. The bulk of this paper is devoted to stating and proving analogues to several theorems in the theory of Γ-semihyperrings.

Keywords: Semihyperring; Γ-semihyperring; Γ-hyperring; simple Γ-semihyperring; regular Γ-hyperring

1. Preliminaries and basic definition

In 1964, Nobusawa [1] introduced Γ-rings as a generalization of ternary rings. Barnes [2] slightly weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. Barnes [2], Luh [3] and Kyuno [4] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory.

The hyperstructure theory was born in 1934, when the notion of a hypergroup was introduced [5]. One of the first books, dedicated especially to hypergroups, is "Prolegomena of Hypergroup Theory" written by Corsini in 1993 [6]. Another book on "Hyperstructures and Their Representations", by Vougiouklis, was published one year later [7]. We mention here another important book for the applications in Geometry and for the clearness of the exposition, written by W. Prenowitz and J. Jantosciak [8]. Another book [9] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, let H be a non-empty set. Then, the map $\circ : H \times H \to P^*(H)$ is called a hyperoperation when $P^*(H)$ is the family of non-empty subsets of H. Let H be a non-empty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called *hypergroupoid*. $(H, \circ)$ is called a semihypergroup if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$. Moreover, if for every $x \in H$, $x \circ H = H = H \circ x$, then $(H, \circ)$ is called a *hypergroup*. Also, many authors studied different aspects of semihypergroups, for instance, Bonansinga and Corsini [10, 11, 12], Davvaz [13], Davvaz and Poursalavati [14], Fasino and Freni [15], Gutan [16] and Leoreanu [17].

We say that a hypergroup $(H, \circ)$ is *canonical* if

(1) it is commutative ($x \circ y = y \circ x$, for every $x, y \in H$),

(2) it has a scalar identity (also called scalar unit), which means that
$\exists e \in H \ni \forall x \in H, e \circ x = x \circ e = x$,

(3) every element has a unique inverse, which means that for all $x \in H$, there exists a unique $x^{-1} \in H$, such that $e \in x \circ x^{-1}$,

(4) it is reversible, which means that if $x \in y \circ z$, then there exist the inverses y^{-1} of y and z^{-1} of z, such that $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$.

The notion of a *multiplicative hyperring* was introduced by Rota [18] in 1982. The multiplication is a hyperoperation, while the addition is an operation, that is why it was called a multiplicative hyperring. A triple $(R, +, \cdot)$ is called a

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multiplicative hyperring if (1) $(R,+)$ is an abelian group, (2) $(R,\cdot)$ is a semihypergroup, 3) for all $a,b,c \in R$, we have $a.(b+c) \subseteq a.b + a.c$ and $(b+c).a \subseteq b.a + c.a$. If in (3) we have equalities instead of inclusions, then we say that the multiplicative hyperring is *strongly distributive*.

The notion of Γ-semiring was introduced by Rao [18, 19] as a generalization of Γ-ring as well as of semiring. For example, let S be the additive commutative semigroup of all $m \times n$ matrices over the set of all non-negative integers and Γ be the additive commutative semigroup of all $n \times m$ matrices over the same set. Then, S is a Γ-semiring if $a\alpha b$ denotes the usual matrix product of a, α, b where $a,b \in S$ and $\alpha \in \Gamma$. Dutta and Sardar [20] gave the meaning of left and right operator semirings for a given Γ-semiring. Let $(R,+,\circ)$ be an arbitrary semiring and $\Gamma = \{\circ\}$. It is easy to see that R is a Γ-semiring. Thus a semiring can be considered as a Γ-semiring. Many classical notions of semiring have been extended to Γ-semiring.

In [21, 22], Davvaz et. al. studied the notion of a Γ-semihypergroup as a generalization of a semihypergroup. Many classical notions of semigroups and semihypergroups have been extended to Γ-semihypergroups and a lot of results on Γ-semihypergroups are obtained.

Let $(R,+)$ be a hypergroupoid and Γ be a non-empty set. Then, R is called a Γ- *hyperring* if there exists a mapping $R \times \Gamma \times R \to R$ (images denoted by $a\alpha b$ for all $a,b \in R$, and $\alpha \in \Gamma$) satisfying the following conditions:

(1) $(R,+)$ is a canonical hypergroup,

(2) there exists a zero element that a bilaterally absorbing element, i.e., $x\alpha 0 = 0\alpha x = 0, x + 0 = x$, for every $\alpha \in \Gamma$ and $x \in R$,

(1) $a\alpha(b+c) = a\alpha b + a\alpha c$,

(2) $(a+b)\alpha c = a\alpha c + b\alpha c$,

(3) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

Let R be a commutative semihypergroup and Γ be a commutative group. Then, R is called a Γ– *semihyperring* if there exists a map $R \times \Gamma \times R \to P^*(R)$ (image to be denoted by $a\alpha b$ for $a,b \in R$ and $\alpha \in \Gamma$) and $P^*(R)$ is the set of all non-empty subsets of R satisfying the conditions 3, 4, 5 and $a(\alpha+\beta)b = a\alpha b + a\beta b$ for every $a,b \in R$ and $\alpha, \beta \in \Gamma$. Let R be Γ-semihyperring. Then, $(R_\alpha, \circ)$ is a semihypergroup for every $\alpha \in \Gamma$. (Let α be a fixed element in Γ. We define $a \circ b = a\alpha b$ for all $a,b \in S$).

In the above definition, if R is a semigroup, then R is called a *multiplicative* Γ*-semihyperring*. A Γ-semihyperring R is called *commutative* if $x\alpha y = y\alpha x$ for every $x,y \in R$ and $\alpha \in \Gamma$. We say that Γ-semihyperring R with zero, if there exists $0 \in R$ such that $a \in a+0$ and $0 \in 0\alpha a, 0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$. Let A and B be two non-empty subsets of Γ-semihyperring R. We define

$A + B = \{t \in R \mid t \in a+b \ \ a \in A, b \in B\}$,

$A\Gamma B = \{t \in R \mid t \in a\alpha b \ a \in A, b \in B, \alpha \in \Gamma\}$,

$A\Gamma^{\sum} B = \{t \in R \mid t \in \sum_{i=1}^{n} a_i\alpha_i b_i, \ a_i \in A, b_i \in B, \alpha_i \in \Gamma, n \in \mathbb{N}\}$,

$\mathbb{N}X = \{t \in R \mid t \in \sum_{i=1}^{n} n_i x_i \ \ x_i \in X, n, n_i \in \mathbb{N}\}$.

A non-empty subset R_1 of Γ-semihyperring R is called a Γ *sub-semihyperring* if it is closed with respect to the multiplication and addition. In other words, a non-empty subset R_1 of Γ-semihyperring R is a sub Γ-semihyperring if $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$. A right (left) ideal I of a Γ-semihyperring R is an additive sub semihypergroup $(R,+)$ such that $I\Gamma R \subseteq R$ $(R\Gamma I \subseteq I)$. If I is both right and left ideal of R, then we say that I is a *two-sided ideal* or simply an *ideal* of R. Let X be a non-empty subset of Γ-semihyperring R. By the term left ideal $< X >_l$ (respectively, right ideal $< X >_r$) of R generated by X, that is, the intersection of all left ideals (respectively, right ideals) of R contains X. Hence,

(1) $< X >_l = \mathbb{N}X + R\Gamma^{\sum} X$,

(2) $< X >_r = \mathbb{N}X + X\Gamma^{\sum} R$,

(3) $< X > = \mathbb{N}X + R\Gamma^{\sum} X + X\Gamma^{\sum} R + R\Gamma^{\sum} X \Gamma^{\sum} R$.

A non-empty subset I of a Γ-hyperring R is a left (right) ideal if and only if

(1) $a,b \in I$ implies $a - b \subseteq I$,

(2) $a \in I$, $r \in R$ and $\alpha \in \Gamma$ imply $r\alpha a \in I (a\alpha r \in I)$.

Let I be an ideal of a Γ-hyperring R such that $x + I - x \subseteq I$ for all $x \in R$. Then, I is called a *normal ideal* of R. If I is a normal ideal of a Γ-hyperring R, then we define the relation $x \equiv y (mod I)$ if and only if $(x-y) \cap I \neq \emptyset$. This relation is denoted by xI^*y. We define the following operation and hyperoperation on the set of all classes $[R : I^*] = \{I^*(x) \mid x \in R\}$, as follows:

$I^*(x) \oplus I^*(y) = \{I^*(z) \mid z \in I^*(x) + I^*(y)\}$,

$I^*(x)\overline{\alpha} I^*(y) = I^*(x\alpha y)$.

Then, $[R : I^*]$ is a $\overline{\Gamma}$-hyperring of which

$\overline{\Gamma} = \{\overline{\alpha} \mid \alpha \in \Gamma\}$.

An ideal I of a Γ-hyperring R is called *null* if $I\Gamma I = \{0\}$ and is called *idempotent* if $I\Gamma I = I$.

Let (R_1, Γ_1) and (R_2, Γ_2) be two Γ_1- and Γ_2-semihyperrings, respectively and $f : \Gamma_1 \to \Gamma_2$ be a map. Then, $\psi : R_1 \to R_2$ is called a (Γ_1, Γ_2)-*homomorphism* or (shortly, homomorphism), if for every $x, y \in R$ and $\alpha \in \Gamma$,

(1) $\psi(x + y) = \{\psi(t) \mid t \in x + y\} \subseteq \psi(x) + \psi(y)$,
(2) $\psi(x\alpha y) = \{\psi(t) \mid t \in x\alpha y\} \subseteq \psi(x) f(\alpha)\psi(y)$,
(3) $f(x + y) = f(x) + f(y)$.

In the above definition, if $\psi(x + y) = \psi(x) + \psi(y)$ and $\psi(x\alpha y) = \psi(x) f(\alpha)\psi(y)$, then ψ is called a *strong homomorphism*. The set $ker\psi = \{(a,b) \in R_1 \times R_2 \mid \psi(a) = \psi(b)\}$ is called the *kernel* of ψ. An ordered set (ψ, f) is called an *epimorphism* if $\psi : R_1 \to R_2$ and $f : \Gamma_1 \to \Gamma_2$ be surjective and is called an *isomorphism* if $\psi : R_1 \to R_2$ and $f : \Gamma_1 \to \Gamma_2$ are bijective.

Let ρ be an equivalence relation, A and B be two non-empty subsets of R. We define $(A,B) \in \overline{\rho}$ if for every $a \in A$ there exists $b \in B$ such that $(a,b) \in \rho$ and for every $c \in B$ there exists $d \in A$ such that $(d,c) \in \rho$ and $(A,B) \in \overline{\overline{\rho}}$ if for every $a \in A$ and $b \in B$ $(a,b) \in \rho$. Let R be a Γ-semihyperring and $\alpha \in \Gamma$. An equivalence relation ρ on R is called *regular* if for every $x \in R$ and $\alpha \in \Gamma$,

$$(a,b) \in \rho \ \ imply \ \ (a + x)\overline{\rho}(b + x), (a\alpha x)\overline{\rho}(b\alpha x) \ \ and \ \ (x\alpha a)\overline{\rho}(x\alpha b).$$

Let I be a non-empty subset of Γ-semihyperring R. We say that I is a 2-ideal of R if I satisfies the following condition: $I + R \subseteq I$, $I\alpha R \subseteq I$, $R\alpha I \subseteq I$. for every $\alpha \in \Gamma$.

A 2-ideal I of Γ-semihyperring R generate the following regular relation on R:
$x\rho_I y \Leftrightarrow x = y$ or $x, y \in I$.

It is easy to see that ρ_I is reflexive, symmetric, transitive and regular. We shall call a regular relation of this type a *Rees relation*.

Example 1. Let R be a Γ-hyperring and I be a normal ideal of R. Then, the relation $x \equiv y\ (modI)$ is a regular relation on R.

Proposition 1.1. Let R be a Γ-semihyperring and ρ be a regular relation on R. Then, R/ρ is a $\widehat{\Gamma}$-semihyperring with respect to the following hyperoperation:
$\rho(a) \oplus \rho(b) = \{\rho(c) \mid c \in \rho(a) + \rho(b)\}$,
$\rho(a)\ \hat{\alpha}\rho(b) = \{\rho(d) \mid d \in \rho(a)\alpha\rho(b)\}$,

where $\widehat{\Gamma} = \{\hat{\alpha} \mid \alpha \in \Gamma\}$.

Example 2. Let $(R, +, \circ)$ be a semihyperring such that $x \circ y = x \circ y + x \circ y$, Γ be a commutative group. We define $x\alpha y = x \circ y$ for every $x, y \in R$ and $\alpha \in \Gamma$. Then, R is a Γ-semihyperring.

Example 3. Let $(R, +, \circ)$ be a semiring and $(\Gamma, +)$ be a subgroup of $(R, +)$ and I be an ideal of R such that $I \circ \Gamma = \Gamma \circ I = \{0\}$. We define $x\alpha y = x \circ \alpha \circ y + I$ for every $x, y \in R$ and $\alpha \in \Gamma$. Then, R is a multiplicative Γ-semihyperring.

Example 4. Let $R = Z_4$ and $I = \Gamma = \{\overline{0}, \overline{2}\} \subseteq \mathbb{Z}_4$.

Then, R is a multiplicative $\widehat{\Gamma}$-semihyperring with the following hyperoperation: $x\hat{\alpha}y = \{\overline{0}, \overline{2}\}$, where

$x, y \in R$, $\hat{\alpha} \in \widehat{\Gamma}$ and $\widehat{\Gamma} = \{\hat{\alpha} \mid \alpha \in \Gamma\}$.

2. Simple Γ – semihyperrings

In this section R is a Γ-semihyperring such that it has an element 0 with the following property:
$x \in x + 0$, $0 + 0 = \{0\}$ *and* $x\alpha 0 = 0\alpha x = \{0\}$,
for every $x, y \in R$ and $\alpha \in \Gamma$. Hence $\{0\}$ is an ideal of R.

Definition 2.1. A Γ-semihyperring R is called simple (right simple)if
(1) $\{0\}$ and R are the only ideals (right ideals),
(2) $R\Gamma R \neq \{0\}$.

In the same way, we can define a simple Γ-hyperring. An ideal I of Γ-semihyperring R is called *simple*, if I is a simple Γ-semihyperring. This means that $\{0\}$ and I are only ideals of I and $I\Gamma I \neq \{0\}$.

Example 5. Let $(F, +, \circ)$ be a field, $(\Gamma, +)$ be a subgroup of $(F, +)$ such that $1_F \in \Gamma$ and $\{A_t\}_{t \in F}$ be a family of non-empty disjoint sets in which $|A_o| = 1$. Then, $S = \bigcup_{t \in F} A_t$ is a simple Γ-semihyperring with the following hyperoperations: $x \oplus y = A_{g_1 + g_2}$, $x\alpha y = A_{g_1 \alpha g_2}$, where $x \in A_{g_1}, y \in A_{g_2}$, $g_1, g_2 \in \Gamma$ and $\alpha \in \Gamma$.

Example 6. Let $\{A_n \mid n \in R\}$ be a family of disjoint set such that

$$A_n = \begin{cases} \{0\} & n = 0 \\ (0,1) & 0 < n < 1 \\ [m, m+1) & m \le n < m+1. \end{cases}$$

Then, for every $x \in R$ there exists $n \in R$ such that $x \in A_n$. So, R is a simple R-semihyperring with the following hyperoperation: $x \oplus y = A_{n+m}$, $x\alpha y = A_{n\alpha m}$, where $x \in A_n$, $y \in A_m$ and $\alpha \in R$.

Example 7. Let $R = \{a,b,c\}$ and $\Gamma = \{\alpha, \beta\}$. Then, R is a simple Γ-hyperring with the following operations and hyperoperation:

$\oplus$	a	b	c
a	a	b	c
b	b	{a,b}	c
c	c	c	{a,b,c}

α	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

β	a	b	c
a	a	a	a
b	a	c	b
c	a	b	c

where $\Gamma = \{\alpha, \beta\}$.

Lemma 2.2. A Γ-semihyperring R is simple if and only if $R\Gamma \sum a\Gamma \sum R = R$ for every $a \in R \setminus 0$.

Proof: Suppose that R is a simple Γ-semihyperring. Then, $R\Gamma \sum R$ is an ideal of R. Since R is a simple Γ-semihyperring, $R\Gamma \sum R$ is distinct from $\{0\}$, hence it must be coincide with R, and it follows:

$$R\Gamma \sum R\Gamma \sum R = R\Gamma \sum R = R.$$

Let $a \ne 0$ be an element of R. Then, $R\Gamma \sum a\Gamma \sum R$ is an ideal of R and so either $R\Gamma \sum a\Gamma \sum R = R$ or $R\Gamma \sum a\Gamma \sum R = \{0\}$. If $R\Gamma \sum a\Gamma \sum R = \{0\}$, then the set $I = \{x \in R \mid R\Gamma \sum x\,\Gamma \sum R = \{0\}\}$, contains a non-zero element a. Let $x, y \in I$. Then, $R\Gamma \sum x\Gamma \sum R = \{0\}$ and $R\Gamma \sum y\,\Gamma \sum R = \{0\}$. Since

$$R\Gamma \sum (x+y)\Gamma \sum R \subseteq R\Gamma \sum x\Gamma \sum R + R\Gamma \sum y\Gamma \sum R,$$

I is an ideal of R which implies that $I = \{0\}$ or $I = R$. If $I = R$, then $R\Gamma \sum x\Gamma \sum R = \{0\}$ for every $x \in R$. Since R is simple and $\{0\} \ne R\Gamma R \subseteq R\Gamma \sum R$, we have $R\Gamma \sum R = R$. But this implies that $R = R\Gamma \sum R\Gamma \sum R = \{0\}$ which is a contradiction. Hence, $R\Gamma \sum a\Gamma \sum R = R$.

Conversely, suppose that $R\Gamma \sum a\Gamma \sum R = R$ for all $a \in R \setminus 0$. Then, $R\Gamma \sum R \ne \{0\}$. If I is an ideal of R containing a non-zero element a, then $R = R\Gamma \sum a\Gamma \sum R \subseteq R\Gamma \sum I\Gamma \sum R \subseteq I$, and so $I = R$. Therefore, R is a simple Γ-semihyperring.

Lemma 2.3. If I is a non-zero minimal ideal of Γ-semihyperring R, then either $I\Gamma \sum I = \{0\}$ or I is a simple ideal of R.

Proof: Since $I\Gamma \sum I$ is an ideal of R contained in I, we must have either $I\Gamma \sum I = \{0\}$ or $I\Gamma \sum I = I$. Suppose that $I\Gamma \sum I = I$. Then, $(I\Gamma \sum I)\Gamma \sum I = I\Gamma \sum I$. Therefore, $I\Gamma \sum I\Gamma \sum I = I\Gamma \sum I$. If a is a non-zero element of I, then $< a >$ is a non-zero ideal of R contained in I. Since I is a minimal ideal of R, we have $I = < a >$. Since $< a > = \mathbb{N}a + R\Gamma \sum a + a\Gamma \sum R + R\Gamma \sum a\Gamma \sum R$, we obtain

$$\begin{aligned} I\Gamma \sum a\Gamma \sum I &\subseteq I\Gamma \sum (\mathbb{N}a + R\Gamma \sum a + a\Gamma \sum R + R\Gamma \sum a\Gamma \sum R)\Gamma \sum I \\ &\subseteq I\Gamma \sum a\Gamma \sum I. \end{aligned}$$

Hence, $I\Gamma \sum a\Gamma \sum I = I$ and so I is a simple ideal of R.

Definition 2.4. Let R be a Γ-semihyperring and for every $\alpha \in \Gamma \setminus 0$ there exists $1_\alpha \in \Gamma$ such that for every $x \in R$, $x \in x\alpha 1_\alpha$ and $x \in 1_\alpha \alpha x$. Then, R is called a Γ-semihyperring with Γ-identity.

Lemma 2.5. Let R be a right simple Γ-semihyperring with Γ-identity. Then, $R \setminus 0$ is a multiplicative close subset of R.

Proof: We show that for every $a, b \in R$ and $\alpha \in \Gamma$, $a\alpha b \subseteq R \setminus 0$. Suppose that $a, b \in R \setminus 0$, $\alpha \in \Gamma$ but $a\alpha b = \{0\}$. Let $\theta = \{x \in R \mid a\alpha x = \{0\}\}$. Then, θ is a right ideal of R. Since $\{0, b\} \subseteq \theta$, we obtain $\theta = R$ which implies that $a\alpha R = \{0\}$. Hence,

$a \in a\alpha 1_\alpha = \{0\}$. This is a contradiction. Therefore, $a\alpha b \subseteq R \setminus 0$.

Proposition 2.6. Let I be a simple left ideal of Γ-semihyperring R and for every left ideal J of R, $y\alpha r \cap J \neq \emptyset$ implies that $y\alpha r \subseteq J$ where $y \in I$. Then, for every $\alpha \in \Gamma$ and $r \in R$, $I\alpha r$ is either $\{0\}$ or a minimal left ideal of R.

Proof: Suppose that $I\alpha r \neq \{0\}$, where $\alpha \in \Gamma$ and $r \in R$. Obviously, $I\alpha r$ is a left ideal of R. In order to show that it is a minimal left ideal, let A be a left ideal of R contained in $I\alpha r$. Assume that $\theta = \{x \in I \mid x\alpha r \subseteq A\}$. Then, $\theta\alpha r \subseteq A$. Let $x \in A$. Then, there exists $y \in I$ such that $x \in y\alpha r$. Hence $A \cap y\alpha r \neq \emptyset$ which implies that $y\alpha r \subseteq A$. So, $\theta\alpha r = A \subseteq I\alpha r$. Since I is a simple left ideal and θ is a left ideal of R, then $\theta = \{0\}$ or $\theta = I$. Hence $A = \{0\}$ or $A = I\alpha r$.

Proposition 2.7. Let I be a minimal simple ideal of R such that $L \neq \{0\}$ is a left ideal contained in I. So, $L\Gamma\sum L \neq \{0\}$.

Proof: Since $L\Gamma\sum R$ is an ideal of R contained in I, $L\Gamma\sum R = \{0\}$ or $L\Gamma\sum R = I$. If $L\Gamma\sum R = \{0\}$, then L is an ideal of R. Hence, $L = I$ and $I\Gamma\sum I = L\Gamma\sum I \subseteq L\Gamma\sum R = \{0\}$. That is a contradiction. Hence, $L\Gamma\sum R = I$. Since $I = I\Gamma\sum I = (L\Gamma\sum R)\Gamma\sum(L\Gamma\sum R) \subseteq (L\Gamma\sum L)\Gamma\sum R$. We conclude that $L\Gamma\sum L \neq \{0\}$.

Lemma 2.8. Let R be a multiplicative Γ-semihyperring with Γ-identity and I be a simple left ideal of R. Then, $I = R\Gamma\sum a$ for every $a \in I \setminus 0$.

Proof: Suppose that $a \neq 0$ is an element of I. Then, $R\Gamma\sum a$ is a left ideal of R contained in I. Therefore, $R\Gamma\sum a = \{0\}$ or $R\Gamma\sum a = I$. If $R\Gamma\sum a = \{0\}$, then $\{0, a\}$ is a left ideal of I. Therefore, $I\Gamma\sum I = \{0\}$ which is a contradiction. Hence, $R\Gamma\sum a = I$.

Theorem 2.9. Let R be a simple Γ-hyperring containing a non-zero minimal left ideal. Then, R is the union of its minimal left ideals.

Proof: Suppose that R is a simple Γ-hyperring and I is a non-zero minimal left ideal. Then, $I\Gamma\sum R$ is an ideal of R and so either $I\Gamma\sum R = \{0\}$ or $I\Gamma\sum R = R$. Suppose that $I\Gamma\sum R = \{0\}$. Then, I is an ideal of R. Since $I \neq \{0\}$, it follows that $I = R$ and so $R\Gamma R \subseteq R\Gamma\sum R = \{0\}$. It is a contradiction. We conclude that $I\Gamma\sum R = R$ and so there exists $a \in R$ such that $I\alpha a \neq \{0\}$. Let J be a non-zero left ideal of R contained in $I\alpha a$. Then, $\Psi = \{b \in I \mid b\Gamma a \subseteq J\}$, being a non-zero left ideal of R contained in I and so $J \subseteq I\alpha a = \Psi\alpha a \subseteq J$. Now, let $H = \cup\{I\alpha a \mid a \in R, \alpha \in \Gamma\}$. Then, certainly H is a non-zero left ideal. Let $x \in H$, $y \in R$ and $\beta \in \Gamma$. Then, there exist $\alpha \in \Gamma$, $a \in R$ such that $x \in I\alpha a$. Hence, $x\beta y \in (I\alpha a)\beta y = I\alpha(a\beta y) \subseteq H$. Since R is simple, $H = R$, and R is the union of minimal left ideals.

The dual of the previous theorem is true. Therefore, if R is a simple Γ-hyperring and contains a minimal right ideal, then R is the union of its minimal right ideal.

Proposition 2.10. Suppose that R is a simple Γ-hyperring containing at least one minimal left ideal and one minimal right ideal. Then, for every minimal left ideal L of R there exists a minimal right ideal R_1 such that $L\Gamma R_1 = R$.

Proof: Let L be a minimal left ideal of R. Since R is a simple Γ-hyperring, $L\Gamma\sum R = \{0\}$ or $L\Gamma\sum R = R$. By Theorem 2.9 there exist $a \in R$ and $\alpha \in \Gamma$ such that $L\alpha a \neq \{0\}$. Since $\{0\} \neq L\alpha a \subseteq L\Gamma R_1$, by the dual of the previous theorem there is a minimal right ideal R_1 such that $a \in R_1$. Since R is a simple Γ-semihyperring, $L\Gamma R_1$ must coincide with R.

Proposition 2.11. Let I be a proper normal ideal of Γ-hyperring R, A be the set of ideals of R containing I and B be the set of ideals of $[R : I^*]$. Then, the map $\phi : J \mapsto [J : I^*]$ is inclusion-preserving bijection from A onto B.

Proof: Since J is an ideal of R, $[J : I^*]$ is an ideal of $[R:I^*]$. Hence ϕ is well-defined. Let J_1 and J_2 be two ideals of R such that $\phi(J_1)=\phi(J_2)$. Then, $[J_1:I^*]=[J_2:I^*]$. For every element $x\in J_1$, there is $y\in J_2$ such that $x\equiv y\ (modI)$ and so $x\in a+y$ for some $a\in I$. This implies that $J_1\subseteq J_2$. In the same way $J_2\subseteq J_1$. Hence, $J_1=J_2$. Let T be an ideal of $[R:I^*]$. Then, $\omega=\{x\in R \mid I^*(x)\in T\}$ is an ideal of R and $\phi(\omega)=T$. Therefore, ϕ is bijective.

Proposition 2.12. If I, J are ideals of a Γ-hyperring R such that $I\subset J$, I is a normal ideal and there is no ideal K of R such that $I\subset K\subset J$. Then, $[J:I^*]$ is either simple or null ideal.

Proof: By Proposition 2.11, $[J:I^*]$ is a minimal ideal of $[R:I^*]$. Thus $[J:I^*]$ is null or simple, by Lemma 2.3.

3. Noetherian and Artinian Γ-semihyperrings

A collection A of subsets of a Γ-semihyperring R satisfies the ascending chain condition (or Acc) if there does not exist a properly ascending infinite chain $A_1\subset A_2\subset\ldots$ of subsets from A. Recall that a subset $B\in A$ is a maximal element of A if there does not exist a subset in A that properly contains B.

Proposition 3.1. Let R be a Γ-semihyperring. Then, the following conditions are equivalent:
(1) R satisfying the Acc condition on right (left) ideals.
(2) Every non-empty family of right (left) ideals has a maximal element.
(3) Every right (left) ideal is finitely generated.

Definition 3.2. A $\Gamma-$semihyperring R is right (left) Noetherian if the equivalent conditions of the above propositions are satisfied.

In the same way, we can define an Artinian Γ-semihyperring. Let I be an ideal of a $\Gamma-$semihyperring R and I be a Noetherian $\Gamma-$semihyperring. Then, I is called a Noetherian ideal of R.

Example 8. Let $(R,+)$ be a group and $(\Gamma,+)$ be a subgroup of R. Then, R is a multiplicative Noetherian (Artinian) $\Gamma-$semihyperring with respect the following hyperoperation: $x\alpha y=R$.

Example 9. Let $A_n=[n,n+1)$ for every $n\in Z$, $S=\cup_{n\in Z}A_n$ and $\Gamma=Z$. Then, S is a Noetherian Γ-semihyperring but not an Artinian Γ-semihyperring with respect to the following hyperoperation: $x\oplus y=A_{n+m}, x\alpha y=A_{n\alpha m}$, where $x\in A_n$ and $y\in A_m$.

Proposition 3.3. Let $(R,+,\circ)$ be a commutative ring, $(\Gamma,+)$ be a subgroup of $(R,+)$ such that $1_R\in\Gamma$ and $\{A_g\}_{g\in R}$ be a collection of non-empty disjoint sets. Then, $S=\bigcup_{g\in R}A_g$ is a Γ-semihyperring with the following hyperoperation: $x\oplus y=A_{g_1+g_2}, x\alpha y=A_{g_1\alpha g_2}$, where $x\in A_{g_1}$, $y\in A_{g_2}$ and $\alpha\in\Gamma$. Therefore, R is a Noetherian (Artinian) ring if and only if S is a Noetherian (Artinian) Γ-semihyperring.

Proof: One can see that S is a Γ-semihyperring with the above hyperoperations. Let I be an ideal of R. Then, $S_I=\cup_{g\in I}A_g$ is an ideal of R.
Conversely, suppose that T is an ideal of Γ-semihyperring S. Then, $T=S_I$, where $I=\langle X\rangle$ and $X=\{g\in R \mid A_g\cap T\neq\varnothing\}$. Therefore, the commutative ring R is Noetherian (Artinian) if and only if S is Noetherion (Artinian).

Proposition 3.4. Let I be a 2-ideal of Γ-semihyperring R. Let A be the set of ideals of R containing I and B the set of ideal of R/ρ_I. Then, the map $\psi: J\to J/\rho_I$ is inclusion-preserving bijection of A onto B.

Proof: The proof is straightforward.

Proposition 3.5. Let R be a Noetherian Γ-semihyperring and I be a 2-ideal of R. Then, R/ρ_I is a Noetherian $\overline{\Gamma}$-semihyperring.

Proof: The proof is straightforward.

Theorem 3.6. Let I be a Noetherian 2-ideal of Γ-semihyperring R. If R/ρ_I is a Noetherian $\overline{\Gamma}$-semihyperring, then R is a Noetherian Γ-semihyperring.

Proof: Assume that I and R/ρ_I are Noetherian and $A_1\subseteq A_2\subseteq A_3\ldots$ be an ascending chain of ideals of R. There exist ascending chain of ideals $A_1\cap I\subseteq A_2\cap I\subseteq\ldots$, $(A_1\cup I)/\rho_I\subseteq(A_2\cup I)/\rho_I\subseteq\ldots$, in I and R/ρ_I, respectively. Then, there exists $n\in N$ such that $A_i\cap I=A_n\cap I$ and $(A_i\cup I)/\rho_I=(A_n\cup I)/\rho_I$

for all $i \geq n$. Hence, $A_i \cup I = A_n \cup I$ for all $i \geq n$. Suppose that $x \in A_i \cup I$. If $x \in I$, then $x \in A_n \cup I$. Assume that $x \in A_i$ for some $i \geq n$. Then, there exists $x_1 \in A_n \cup I$ such that $\rho_I(x) = \rho_I(x_1)$ which implies that $x = x_1$ or $x, x_1 \in I$. Therefore, $x \in A_n \cup I$ which implies that $A_i \cup I = A_n \cup I$ for all $i \geq n$. Hence, for $i \geq n$ $A_i = A_i \cap (A_i \cup I) = A_i \cap (A_n \cup I) = A_n \cup (A_i \cap I) = A_n \cup (A_n \cap I) = A_n$. So, R is Noetherian.

Proposition 3.7. Let R be a Γ-semihyperring and A, B be two Noetherian 2-ideals of R. Then, $A \cup B$ is a Noetherian sub Γ-semihyperring of R.

Proof: Since A and B are 2-ideals, then $A \cap B$ is a 2-ideal of A and B is a 2-ideal of $A \cup B$. Indeed,
$(A \cap B) + A \subseteq (A + A) \cap (B + A) \subseteq A \cap B$,
$(A \cap B)\alpha A \subseteq (A\alpha A) \cap (B\alpha A) \subseteq A \cap B.$

In the same way we can see that B is a 2-ideal of $A \cap B$. We define $\psi : A / \rho_{A\cap B} \to (A \cup B) / \rho_B$, by $\psi(\rho_{A\cap B}(x)) = \rho_B(x)$, for all $x \in A$. Let $\rho_{A\cap B}(x) = \rho_{A\cap B}(y)$ for some $x, y \in A$. Then, $\rho_B(x) = \rho_B(y)$. Hence, ψ is well-defined. Since

$$\begin{aligned}\psi(\rho_{A\cap B}(x) \oplus \rho_{A\cap B}(y)) &= \psi(\{\rho_{A\cap B}(t) \mid t \in x + y\}) \\ &= \{\rho_B(t) \mid t \in x + y\} \\ &= \rho_B(x) \oplus \rho_B(y) \\ &= \psi(\rho_{A\cap B}(x)) \oplus \psi(\rho_{A\cap B}(y)).\end{aligned}$$

In the same way, it is easy to see that $\psi(\rho_{A\cap B}(x)\overline{\alpha}\rho_{A\cap B}(y)) = \psi(\rho_{A\cap B}(x))\overline{\alpha}\psi(\rho_{A\cap B}(y))$. Hence, $A / \rho_{A\cap B} \cong (A \cup B) / \rho_B$. By the previous proposition, $A \cup B$ is Noetherian.

Lemma 3.8. Let R be an ordered Γ-semihyperring with zero. The principle ideals of R forms a chain with respect to inclusion if and only if ideals of R do so.

Proof: Suppose that I, J are ideals of R such that $I \bigcup J$ with zero. Then, $J \subseteq I$. Let $x \in J$. We consider an element $y \in I$ such that $y \notin J$. By hypothesis, we have $< x > \subseteq < y >$ or $< y > \subseteq < x >$. If $< y > \subseteq < x >$, then $y \in < y > \subseteq < x > \subseteq J$, which is impossible. Thus we have $< x > \subseteq < y >$ and so which $x \in < x > \subseteq I$.

Lemma 3.9. Let R_1 and R_2 be Γ_1- and Γ_2-semihyperring with zero, Γ_1- and Γ_2-identity, respectively. Then, $R = R_1 \times R_2$ is a Noetherian $\widehat{(\Gamma_1, \Gamma_2)}$-semihyperring with the following hyperoperations if and only if R_1 and R_2 are Noetherian Γ_1 and Γ_2-semihyperring, respectively.

$(a_1, b_1) \oplus (a_2, b_2) = \{(x, y) \mid x \in a_1 + a_2, y \in b_1 + b_2\}$,

$(a_1, b_1)\widehat{(\alpha, \beta)}(a_2, b_2) = \{(x, y) \mid x \in a_1\alpha a_2, b_1\beta b_2\}$.

Proof: The proof is straightforward.

Proposition 3.10. Let R be a Γ-semiring with zero and Γ-identity, ρ_1 and ρ_2 be regular relations on R such that $\rho_1 \circ \rho_2 = R \times R$ and $\rho_1 \cap \rho_2 = Id_R$. Then, R / ρ_1 and R / ρ_2 are Noetherian $\overline{\Gamma}$-semirings if and only if R is a Noetherion Γ-semiring.

Proof: Let $\psi : R \to R / \rho_1 \times R / \rho_2$ defined by $\psi(x) = (\rho_1(x), \rho_2(x))$. We show that ψ is a homomorphism. We have
$\psi(x + y) = (\rho_1(x + y), \rho_2(x + y))$
$= (\rho_1(x), \rho_2(x)) \oplus (\rho_1(y), \rho_2(y))$
$= \psi(x) \oplus \psi(y)$.

Let $f : \Gamma \to \overline{\Gamma} \times \overline{\Gamma}$ defined by $f(\alpha) = (\overline{\alpha}, \overline{\alpha})$. In the same way, $\psi(x\alpha y) = \psi(x) f(\alpha) \psi(y)$. Also, we have

$$\begin{aligned}ker\psi = \{(a,b) \mid \psi(a) = \psi(b)\} &= \{(a,b) \mid \rho_1(a) \\ &= \rho_1(b), \rho_2(a) = \rho_2(b)\} = Id_R.\end{aligned}$$

Let $(\rho_1(x), \rho_2(y)) \in R / \rho_1 \times R / \rho_2$. There exists $c \in R$ such that $(a, c) \in \rho_1$ and $(c, b) \in \rho_2$ which implies that $\psi(c) = (\rho_1(c), \rho_2(c)) = (\rho_1(x), \rho_2(y))$. Hence $R \cong R / \rho_1 \times R / \rho_2$. By Lemma 3.9, R is a Noetherian Γ-semihyperring if and only if R / ρ_1 and R / ρ_2 are Noetherian $\overline{\Gamma}$-semihyperrings.

In the previous proposition, let R be a Γ-semihyperring. Then, ψ is a homomorphism but is not a strong homomorphism. Let R be a Γ-semihyperring. An ideal P of R is called prime if $A\Gamma B \subseteq P$, implying that $A \subseteq P$ or $B \subseteq P$, where A and B are ideals of R. An ideal M of Γ-

semihyperring R is called maximal, if $M \neq R$ and there are no ideals in " between" M and R. In other words, if I is an ideal which contains M as a subset, then either $I = R$ or $I = M$. Let R be a commutative Γ-semihyperring with zero and Γ-identity. Then, every maximal ideal of R is prime.

Example 10. Let $R = \{a,b,c,d\}$, $\Gamma = \mathbb{Z}_2$ and $\alpha = \bar{0}, \beta = \bar{1}$. Then, R, is a Γ-semihyperring with the following hyperoperations:

$\oplus$	a	b	c	d
a	{a,b}	{a,b}	{c,d}	{c,d}
b	{a,b}	{a,b}	{c,d}	{c,d}
c	{c,d}	{c,d}	{a,b}	{a,b}
d	{c,d}	{c,d}	{c,d}	{a,b}

β	a	b	c	d
a	{a,b}	{a,b}	{a,b}	{a,b}
b	{a,b}	{a,b}	{a,b}	{a,b}
c	{a,b}	{a,b}	{c,d}	{c,d}
d	{a,b}	{a,b}	{c,d}	{c,d}

For every $x, y \in R$ we define $x\alpha y = \{a,b\}$. In this example $P = \{a,b\}$ is a prime ideal of R.

Proposition 3.11. Let R be a commutative Γ-semihyperring with zero and Γ-identity and $M_1, M_2, ..., M_n$ different maximal ideals in R. Then, $M_1, M_2, ..., M_n$ is a proper ideal of $M_1 \Gamma M_2 \Gamma ... \Gamma M_{n-1}$.

Proof: It is straightforward.

Proposition 3.12. Let R be a multiplicative Γ-semihyperring with zero, I and J be non-empty subsets of F/ξ and R, respectively. Then,

1) If I is an ideal of F/ξ, then I_R is an ideal of R.

2) If J is an ideal of R, then $J_{F/\xi}$ is an ideal of F/ξ.

Proof: (1) Since I is a non-empty set, there exists an element $\xi\left(\prod_{i=1}^{n}(x_i, \alpha_i)\right) \in I$. Hence $\xi(0,\alpha) \in \xi\left(\prod_{i=1}^{n}(x_i, \alpha_i)\right) \odot \xi(0,\alpha) \subseteq I$ for every $\alpha \in \Gamma$, which implies that I_R is a non-empty set. Now, let $x, y \in I_R$, then $\xi(x,\alpha), \xi(y,\alpha) \in I$ for every $\alpha \in \Gamma$. Hence $\xi(x+y,\alpha) = \xi(x,\alpha) \oplus \xi(y,\alpha) \in I$ for every $\alpha \in \Gamma$ and so $x + y \in I$. Let $x \in I_R$, $y \in R$ and $\alpha \in \Gamma$. Then, $\xi(x,\alpha) \in I$ for every $\alpha \in \Gamma$. Since I is an ideal of F/ξ, $\xi(x,\alpha) \odot \xi(y,\alpha) \subseteq I$. So $\{\xi(t,\alpha) \mid t \in x\alpha y\} \subseteq I$ which implies that $x\alpha y \subseteq I$. Hence, I_R is an ideal of R.

(2) Let J be an ideal of R. Then, $0 \in J$, which implies that $\xi(0,\alpha) \in J_{F/\xi}$. Hence, $J_{F/\rho}$ is a non-empty set. Let $\xi\left(\prod_{i=1}^{n}(x_i, \alpha_i)\right)$ *and* $\xi\left(\prod_{j=1}^{m}(y_j, \beta_j)\right)$ be elements of $J_{F/\xi}$. Then, $\sum_{i=1}^{n} x_i \alpha_i x$ *and* $\sum_{j=1}^{m} y_j \beta_j x \subseteq I$ for every $x \in R$. Hence, $\sum_{i=1}^{n} x_i \alpha_i x + \sum_{j=1}^{m} y_j \beta_j x \subseteq I$ which implies that $\xi\left(\prod_{i=1}^{n} x_i \alpha_i x\right) \oplus \xi\left(\prod_{j=1}^{m} y_j \beta_j x\right) \subseteq J_{F/\xi}$. In the same way, $J_{F/\xi}$ is closed with respect to the above hyperoperation. Hence, $J_{F/\xi}$ is an ideal of F/ξ.

Theorem 3.13. Let R be a multiplicative Γ-semihyperring with Γ-identity and zero, I and J be ideals of F/ξ and R, respectively. Then, (1) $(J_{F/\xi})_R = I$, (2) $(I_R)_{F/\xi} = J$,

Proof: (1) We have

$$\begin{aligned}(J_{F/\xi})_R &= \{x \in R \mid \xi(x,\Gamma) \subseteq J_{F/\xi}\} \\ &= \{x \in R \mid \xi(x,\alpha) \in J_{F/\xi},\ for\ every\ \alpha \in \Gamma\} \\ &= \{x \in R \mid x\alpha a \subseteq J,\ for\ every\ \alpha \in \Gamma, a \in R\}.\end{aligned}$$

Since R has Γ-identity, then $(J_{F/\xi})_R \subseteq J$. Therefore, $(J_{F/\xi})_R = J$. (2) By definition, we have

$$\begin{aligned}(I_R)_{F/\xi} &= \left\{\xi\left(\prod_{i=1}^{n}(x_i,\alpha_i)\right) \mid \sum_{i=1}^{n} x_i \alpha_i x \subseteq I_R,\ for\ all\ x \in R\right\} \\ &= \left\{\xi\left(\prod_{i=1}^{n}(x_i,\alpha_i)\right) \mid \xi(t,\Gamma) \subseteq I\ \ for\ every\ t \in \sum_{i=1}^{n} x_i \alpha_i x, x \in R\right\}.\end{aligned}$$

This implies that $(I_R)_{F/\xi} F/\xi \subseteq I$. Since R is with Γ-identity, $(I_R)_{F/\xi} \subseteq I$. Hence, $(I_R)_{F/\xi} = I$.

Corollary 3.14. Let J_1 and J_2 be two ideals of R and A and B be two ideals of F/ξ such that $J_1 \subseteq J_2$ and $A \subseteq B$. Then, $(J_1)_{F/\xi} \subseteq (J_2)_{F/\xi}$ and $A_R \subseteq B_R$.

Corollary 3.15. Let R be a multiplicative Γ-semihyperring. Then, R is a Noetherian (Artinian) if and only if F/ξ is Noetherian (Artinian) multiplicative hyperring.

Corollary 3.16. Let R be an Artinian Γ-ring. Then, R is a Noetherian Γ-ring.

4. Regular Γ-hyperrings

Let R be a Γ-hyperring. An element $x \in R$ is (α,β)- regular or shortly regular if for every $x \in R$ there exist $y \in R$ and $\alpha,\beta \in \Gamma$ such that $x = x\alpha y\beta x$. Let every element of R be regular. Then, R is called a regular Γ-hyperring. An element $e \in R$ is idempotent, if there exists $\alpha \in \Gamma$ such that $e = e\alpha e$. In this case, we say that e is an α-idempotent. We denote that the set of all α-idempotent element with E_α. Hence, if E is a set of all idempotent elements of R, then $E = \bigcup_{\alpha\in\Gamma} E_\alpha$. If every element of R is an idempotent, then R is called an idempotent Γ-hyperring. For an element a in a Γ-hyperring R if there exist an element $b \in R$ and $\alpha,\beta \in \Gamma$ such that $a = a\beta b\alpha a$ and $b = b\alpha a\beta b$, then b is said to be a (α,β)- inverse of a. In this case, we write $b \in V_\alpha^\beta(a)$. Notice that an element with an inverse is necessary regular and every (α,β)-regular element has a (β,α)-inverse: if there exist $x \in R$ and $\alpha,\beta \in \Gamma$, $a = a\alpha x\beta a$, then define $b = x\beta a\alpha x$ and observe that

$$a\alpha b\beta a = a\alpha(x\beta a\alpha x)\beta a = (a\alpha x\beta a)\alpha x\beta a = a\alpha x\beta a = a.$$

$$\begin{aligned} b\beta a\alpha b &= (x\beta a\alpha x)\beta a\alpha(x\beta a\alpha x) = x\beta(a\alpha x\beta a)\alpha(x\beta a\alpha x) \\ &= x\beta(a\alpha x\beta a)\alpha x = x\beta a\alpha x = b. \end{aligned}$$

Hence, in a regular Γ-hyperring every element has inverse. Let a be a (α,β)-regular element of R. Then, $V_\beta^\alpha(a) \neq \emptyset$. An ideal I of a Γ-hyperring R is called (α,β)-regular, if $x' \in (x - x\alpha y\beta x) \cap I \neq \emptyset$ implies that there exist $x_1, x_2 \in x - x\alpha y\beta x$ such that $x' = x_1\alpha a\beta x_2$ for some $a \in I$. Let I be an ideal of a Γ-hyperring R and for every $x \in I$ there exist $\alpha,\beta \in \Gamma$ and $y \in I$ such that $x = x\alpha y\beta x$. Then, I is called a regular ideal of R.

Example 11. Let $(R,+,\cdot)$ be a regular commutative ring, $(\Gamma,+)$ be a non-empty subset of R and ρ be an equivalence relation defined as follows: $x\rho y \Leftrightarrow x = y$ or $x = -y$.

Then, the set $R/\rho = \{\rho(x) \mid x \in R\}$ becomes a regular $\overline{\Gamma}$-hyperring with respect to the hyperoperation $\rho(x) \oplus \rho(y) = \{\rho(x+y), \rho(x-y)\}$ and multiplication $\rho(x)\overline{\alpha}\rho(y) = \rho(x\alpha y)$.

Proposition 4.1. Let R be a regular Γ-hyperring with zero. Then, every principal (right) left ideal of R is generated by an idempotent element.

Proof: Suppose that x is an element of R. Then, there exist $y \in R$ and $\alpha,\beta \in \Gamma$ such that $x = x\alpha y\beta x$. Since,

$$< x >_l = \left\{ y \in R \mid y \in \sum_{i=1}^n n_i x + \sum_{j=1}^m r_j\beta_j x, i,n,m,j \in \mathbb{N} \beta_j \in \Gamma, r_j \in R, 1 \le j \le m \right\}.$$

We have $< x >_l = < y\beta x >_l$. Moreover, $(y\beta x)\alpha(y\beta x) = y\beta(x\alpha y\beta x) = y\beta x$. Hence, $y\beta x$ is an α-idempotent element of R. This completes the proof.

Proposition 4.2. Let R be a Γ-hyperring with Γ-identity and for every $x \in R$ there exists idempotent element $e \in R$ such that $x\Gamma R = e\Gamma R$. Then, R is a regular Γ-hyperring.

Proof: Since R is a Γ-hyperring with Γ-identity, there exist $x_1, x_2 \in R$ and $\alpha,\beta \in \Gamma$ such that $x = e\alpha x_1$ and $e = x\beta x_2$. Since e is an idempotent element of R, there exists $\gamma \in \Gamma$ such that $e = e\gamma e$. Hence,

$$\begin{aligned} x = e\alpha x_1 &= (e\gamma e)\alpha x_1 = (x\beta x_2)\gamma(x\beta x_2)\alpha x_1 \\ &= x\beta x_2\gamma(x\beta x_2\alpha x_1) \\ &= x\beta x_2\gamma x. \end{aligned}$$

Then, x is a regular element of R. Therefore, R is a regular Γ-hyperring.

Proposition 4.3. Let R be a regular Γ-hyperring. Then, every one-sided ideal of R is idempotent.

Proof: Suppose that I is a right ideal of R and $x \in I$. Then, $x = x\alpha y\beta x$ for some $\alpha,\beta \in \Gamma$ and $y \in R$. Consequently, $x = (x\alpha y)\beta x \in I\Gamma I$. Thus, $I\Gamma I = I$.

Proposition 4.4. Let I, J be two ideals in Γ-hyperring R such that $I \subseteq J$. If J is regular, then I is regular too.

Proof: Suppose that $x \in I$. Then, there exist $y \in J$ and $\alpha,\beta \in \Gamma$ such that $x = x\alpha y\beta x$. Hence, $z = y\beta x\alpha y$ is an element of I such that $x\alpha z\beta x = x\alpha(y\beta x\alpha y)\beta x = (x\alpha y\beta x)\alpha(y\beta x) = x\alpha y\beta x = x$.

Proposition 4.5. Let R be a Γ-hyperring and $I \subseteq J$ are ideals of R such that I is a (α,β)-regular ideal of R. Then, I and $[J : I^*]$ are both regular if and only if J is regular.

Proof: By Proposition 4.4, it is obvious that if J is regular, then $[J:I^*]$ and I are regular.

Conversely, assume that I and $[J:I^*]$ are both regular and $x \in J$. It follows from regularity $[J:I^*]$ that $(x - x\alpha y\beta x) \cap I \neq \emptyset$ for some $\alpha, \beta \in \Gamma$ and $y \in J$. Suppose that $x' \in (x - x\alpha y\beta x) \cap I$. Then, there exist $x_1, x_2 \in x - x\alpha y\beta x$ such that $x' = x_1\alpha a\beta x_2$ for some $a \in J$. Hence

$$\begin{aligned} x \in x' + x\alpha y\beta x &= x_1\alpha a\beta x_2 + x\alpha y\beta x \subseteq (x - x\alpha y\beta x)\alpha a\beta(x - x\alpha y\beta x) + x\alpha y\beta x \\ &= x\alpha a\beta x - x\alpha a\beta x\alpha y\beta x - x\alpha y\beta x\alpha a\beta x + x\alpha y\beta x\alpha a\beta x\alpha y\beta x \\ &= x\alpha(a - a\beta x\alpha y - y\beta x\alpha a + y\beta x\alpha a\beta x\alpha y + y)\beta x. \end{aligned}$$

Consequently, from which we conclude that $x = x\alpha z\beta x$ from some $z \in J$. Hence, J is a regular.

Proposition 4.6. Let R be a Γ-hyperring, I be a regular ideal and J be a normal ideal such that (α, β)-regular. Then, $I + J$ is a regular ideal of R.

Proof: Clearly, J is a normal ideal of $I + J$ and $I \cap J$ is a normal ideal of I. Consequently, one can see $[I + J : J^*] \cong [I : (I \cap J)^*]$. Since I is regular, $[I : (I \cap J)^*]$ is regular. By Proposition 4.5, we obtain $I + J$ is regular ideal of R.

Let R be a Γ-ring such that every element of R be (α, α)-regular. (This means that for every $x \in R$ there exists $y \in R$ and $\alpha \in \Gamma$ such that $x = x\alpha y\alpha x$.) Then, R is called a (α, α)-regular Γ-ring. An ideal P of a commutative Γ-ring R is called quasi prime if $A\Gamma R\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, where A, B are subsets of R and $R\Gamma A \subseteq A$, $R\Gamma B \subseteq B$.

Theorem 4.7. Let R be a commutative Γ-ring and $\alpha \in \Gamma$ such that (1) R be a Noetherian Γ – ring, (2) R/P is $(\overline{\alpha}, \overline{\alpha})$-regular for all quasi prime ideals of R, (3) $\{0\}$ is a quasi prime ideal of R. Then, R is a (α, α)-regular Γ-ring.

Proof: Assume that R is not (α, α)-regular Γ-hyperring. Then, there is some $x \in R$ such that $x \notin x\alpha R\alpha x$. Note that $\{0\}$ is a quasi prime ideal of R such that $x \notin x\alpha R\alpha + \{0\}$. Hence,

$\theta = \{I \subseteq R \mid x \notin x\alpha R\alpha x + I, I \text{ is a quasi ideal of } R\}$

is a non-empty set. Since R is a Noetherian Γ-ring, there is a quasi ideal J in R which is maximal with resect to the property $x \notin x\alpha R\alpha x + J$. Suppose that $x + J$ be $(\overline{\alpha}, \overline{\alpha})$-regular element of R/J. Thus, there is $y \in R$ such that $x \in x\alpha y\alpha x + J \subseteq x\alpha R\alpha x + J$ which is a contradiction. Hence, R/J is not $(\overline{\alpha}, \overline{\alpha})$-regular. Therefore, J is not quasi prime ideal of R. Thus there exist ideals I_1 and I_2 such that $I_1\Gamma R\Gamma I_2 \subseteq J$ and I_1, I_2 are not subsets of J. Now, set

$A = \{r \in R \mid r\Gamma R\Gamma I_2 \subseteq J\}$ and $B = \{r \in R \mid A\Gamma R\Gamma r \subseteq J\}$.

Since J is a quasi prime ideal, we conclude that A and B are quasi prime ideals of R. Clearly, $I_1 \subseteq A$ and $I_2 \subseteq B$. Hence, A and B properly contain J. Because of maximality of J, $x \in a\alpha R\alpha x + A$ and $x \in x\alpha R\alpha x + B$. Hence, there exist $r_1, r_2 \in R$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha r_1\alpha x + a$ and $x = x\alpha r_2\alpha x + b$ where $a \in A$ and $b \in B$. We have $x - x\alpha(r_1 + r_2 - r_1\alpha x\alpha r_2)\alpha x = (x - x\alpha r_1\alpha x) - (x - x\alpha r_1\alpha x)\alpha r_2\alpha x \in A$. In the same way, we can see $x - x\alpha(r_1 + r_2 - r_1\alpha x\alpha r_2)\alpha x \in B$. Hence, $x \in x\alpha R\alpha x + A \cap B$. Note also that, since $(A \cap B)\Gamma R\Gamma(A \cap B) \subseteq A\Gamma R\Gamma B \subseteq J$, then we have $A \cap B \subseteq J$. Hence, $x \in x\alpha R\alpha x + J$, which is a contradiction. Therefore, R is a (α, α)-regular Γ-ring.

Proposition 4.8. Let R be a commutative Γ-hyperring with Γ-identity. Then, $I\Gamma J = I \cap J$ for all ideals I and J if and only if R is regular.

Proof: Suppose that R is regular and I, J are two ideals of R. By Proposition 4.3, $I \cap J$ is idempotent. Then, $I \cap J = (I \cap J)\Gamma(I \cap J) \subseteq I\Gamma J$. Hence, $I\Gamma J = I \cap J$.

Conversely, suppose that $I \cap J = I\Gamma J$, where I and J are ideals of R. Let $x \in R$. Then, $x\alpha R$ and $R\beta x$ are ideals of R. So, $(x\alpha R)\Gamma(R\beta x) = (x\alpha R) \cap (R\beta x)$. Since R has Γ-identity, $x \in x\alpha R$ and $x \in R\beta x$ and so $x \in (x\alpha R) \cap (R\beta x) = (x\alpha R)\Gamma(R\beta x) \subseteq x\alpha R\beta x$. Hence, R is regular.

Proposition 4.9. Let ρ be a regular relation on a regular Γ- hyperring R, $\rho(a)$ is a $\overline{\alpha}$-idempotent element in R/ρ and $a\alpha a$ be a (α, β)-regular element of R. Then, there exists an idempotent element $e \in R$ such that $\rho(a) = \rho(e)$, where e is an α-idempotent element of R.

Proof: It is straightforward.

Proposition 4.10. Let R be a regular Γ-hyperring and $e_1\alpha e_2$ be (α, β)-regular element of R, where

$\alpha,\beta\in\Gamma$ and $e_1\in E_\alpha,\ e_2\in E_\beta$. Then,

$$E_\alpha^\beta(e_1,e_2)=\{x\in V_\alpha^\beta(e_1\alpha e_2)\cap E_\alpha \mid x\alpha e_1 = x\beta e_1 = x, e_2\alpha x = e_2\beta x = x\},$$

is a non-empty set of R.

Proof: Since $e_1\alpha e_2$ is a (α,β)-regular, $V_\beta^\alpha(e_1\alpha e_2)$ is a non-empty set of R. Suppose that $y\in V_\beta^\alpha(e_1\alpha e_2)$ and $x=e_2\alpha y\beta e_1$. Then,

$$\begin{aligned}(e_1\alpha e_2)\beta x\alpha(e_1\alpha e_2) &= (e_1\alpha e_2)\beta(e_2\alpha y\beta e_1)\alpha(e_1\alpha e_2)\\ &= e_1\alpha(e_2\beta e_2)\alpha y\beta(e_1\alpha e_1)\alpha e_2\\ &= (e_1\alpha e_2)\alpha y\beta(e_1\alpha e_2)\\ &= e_1\alpha e_2.\end{aligned}$$

Also, we have

$$\begin{aligned}x\alpha(e_1\alpha e_2)\beta x &= (e_2\alpha y\beta e_1)\alpha(e_1\alpha e_2)\beta(e_2\alpha y\beta e_1)\\ &= e_2\alpha y\beta(e_1\alpha e_1)\alpha(e_2\beta e_2)\alpha y\beta e_1\\ &= e_2\alpha y\beta e_1\alpha e_2\alpha y\beta e_1\\ &= e_2\alpha(y\beta e_1\alpha e_2\alpha y)\beta e_1\\ &= e_2\alpha y\beta e_1\\ &= x,\end{aligned}$$

and so $x\in V_\alpha^\beta(e_1\alpha e_2)$. Also,

$$\begin{aligned}x\alpha x &= (e_2\alpha y\beta e_1)\alpha(e_2\alpha y\beta e_1)\\ &= e_2\alpha(y\beta(e_1\alpha e_2)\alpha y)\beta e_1\\ &= e_2\alpha y\beta e_1\\ &= x,\end{aligned}$$

and so $x\in E_\alpha$. Finally, it is clear that $x\alpha e_1 = x$ and $e_2\beta x = x$. Therefore, $x\in E_\alpha^\beta(e_1,e_2)$ which implies that $E_\alpha^\beta(e_1,e_2)$ is a non-empty set of R.

Proposition 4.11. Let R be a regular Γ-hyperring and $e_1\alpha e_2$ be (α,β)-regular element of R, where $\alpha,\beta\in\Gamma$ and $e_1\in E_\alpha, e_2\in E_\beta$. Then,

$$E_\alpha^\beta(e_1,e_2)=\{x\in E_\alpha \mid x\alpha e_1 = e_2\beta x = x, e_1\alpha x\alpha e_2 = e_1\alpha e_2\}.$$

Proof: Suppose that $x\in E_\alpha$, $x\alpha e_1 = e_2\beta x = x$ and $e_1\alpha x\alpha e_2 = e_1\alpha e_2$. Then,

$$\begin{aligned}(e_1\alpha e_2)\beta x\alpha(e_1\alpha e_2) &= (e_1\alpha e_2)\beta(x\alpha e_1)\alpha e_2\\ &= e_1\alpha(e_2\beta x)\alpha e_2\\ &= e_1\alpha x\alpha e_2\\ &= e_1\alpha e_2.\end{aligned}$$

$$x\alpha(e_1\alpha e_2)\beta x = (x\alpha e_1)\alpha e_2\beta x = x\alpha x = x.$$

Hence, $x\in V_\alpha^\beta(e_1\alpha e_2)$ and so $x\in E_\alpha^\beta(e_1,e_2)$.

Conversely, let $x\in E_\alpha^\beta(e_1,e_2)$. Then, $x\in E_\alpha$ and $(e_1\alpha e_2)\beta x\alpha(e_1\alpha e_2) = e_1\alpha e_2$. Moreover,

$$\begin{aligned}(e_1\alpha e_2)\beta x\alpha(e_1\alpha e_2) &= (e_1\alpha e_2)\beta(x\alpha e_1)\alpha e_2\\ &= (e_1\alpha e_2)\beta x\alpha e_2\\ &= e_1\alpha(e_2\beta x)\alpha e_2\\ &= e_1\alpha x\alpha e_2.\end{aligned}$$

Therefore, $e_1\alpha x\alpha e_2 = e_1\alpha e_2$.

Proposition 4.12. Let R be a regular Γ-hyperring, $e_1\alpha e_2$ be (α,β)-regular element of R, where $e_1\in E_\alpha$, $e_2\in E_\beta$. Then, $E_\alpha^\beta(e_1,e_2)$ is a sub semigroup of R_α.

Proof: By Proposition 4.11, $E_\alpha^\beta(e_1,e_2)$ is a non-empty set. Let $x_1,x_2\in E_\alpha^\beta(e_1,e_2)$. Then, $x_1,x_2\in E_\alpha$ and

$$x_1\alpha e_1 = e_2\beta x_1 = x_1, e_1\alpha x_1\alpha e_2 = e_1\alpha e_2, x_2\alpha e_1 = e_2\beta x_2 = x, e_1\alpha x_2\alpha e_2 = e_1\alpha e_2.$$

Moreover,

$$\begin{aligned}x_1\alpha x_2\alpha x_1 &= (x_1\alpha e_1)\alpha x_2\alpha(e_2\beta x_1)\\ &= x_1\alpha(e_1\alpha x_2\alpha e_2)\beta x_1\\ &= x_1\alpha(e_1\alpha e_2)\beta x_1\\ &= (x_1\alpha e_1)\alpha(e_2\beta x_1)\\ &= x_1\alpha x_1 = x_1.\end{aligned}$$

It follows that $(x_1\alpha x_2)\alpha(x_1\alpha x_2) = (x_1\alpha x_2\alpha x_1)\alpha x_2 = x_1\alpha x_2$, and so $x_1\alpha x_2$ is α-idempotent element. Also,

$$(x_1\alpha x_2)\alpha e_1 = x_1\alpha(x_2\alpha e l_1) = x_1\alpha x_2,$$

$$e_2\beta(x_1\alpha x_2) = (e_2\beta x_1)\alpha x_2 = x_1\alpha x_2.$$

$$\begin{aligned}e_1\alpha(x_1\alpha x_2)\alpha e_2 &= (e_1\alpha x_1)\alpha(e_2\beta x_2)\alpha e_2\\ &= (e_1\alpha x_1\alpha e_2)\beta x_2\alpha e_2\\ &= (e_1\alpha e_2)\beta x_2\alpha e_2\\ &= e_1\alpha(e_2\beta x_2)\alpha e_2\\ &= e_1\alpha x_2\alpha e_2,\end{aligned}$$

and so $x_1\alpha x_2\in E_\alpha^\beta(e_1,e_2)$. This implies that $E_\alpha^\beta(e_1,e_2)$ is a sub semigroup of R_α.

Proposition 4.13. Let R be a regular Γ-hyperring, b is a (β,α)-regular, a is a (α,β)-regular and for every α-idempotent elements $e_1,e_2\in R$, $e_2\alpha e_1\in E_\alpha^\alpha(e_1,e_2)$. Then, $V_\alpha^\beta(b)\alpha V_\beta^\alpha(a)\subseteq V_\beta^\beta(a\alpha b)$.

Proof: Since b is a (β,α)-regular and a is a (α,β)-regular elements, $V_\alpha^\beta(b)$ and $V_\beta^\alpha(a)$ are non-empty subsets of R. Suppose that $a' \in V_\beta^\alpha(a)$ and $b' \in V_\alpha^\beta(b)$. Let $g \in E_\alpha^\alpha(a'\beta a, b\beta b')$. Then,

$$\begin{aligned}(a\alpha b)\beta(b'\alpha g\alpha a')\beta(a\alpha b) &= a\alpha(b\beta b'\alpha g)\alpha a'\beta(a\alpha b)\\ &= a\alpha(g\alpha a'\beta a)\alpha b = a\alpha g\alpha b\\ &= (a\alpha a'\beta a)\alpha g\alpha(b\beta b'\alpha b)\\ &= a\alpha(a'\beta a\alpha g\alpha b\beta b')\alpha b\\ &= a\alpha(a'\beta a\alpha b\beta b')\alpha b\\ &= (a\alpha a'\beta a)\alpha(b\beta b'\alpha b)\\ &= a\alpha b.\end{aligned}$$

$$\begin{aligned}(b'\alpha g\alpha a')\beta(a\alpha b)\beta(b'\alpha g\alpha a') &= b'\alpha(g\alpha a'\beta a)\alpha(b\beta b'\alpha g)\alpha a'\\ &= b'\alpha(g\alpha g)\alpha a' = b'\alpha g\alpha a'.\end{aligned}$$

Hence, $b'\alpha g\alpha a' \in V_\beta^\beta(a\alpha b)$. One can see that $a'\beta a$ and $b\beta b'$ are α-idempotent elements of R. Hence $(b\beta b')\alpha(a'\beta a) \in E_\alpha^\alpha(a'\beta a, b\beta b')$. This implies that $b'\alpha((b\beta b')\alpha(a'\beta a))\alpha a' \in V_\beta^\beta(a\alpha b)$.

Thus, $(b'\alpha b\beta b')\alpha(a'\beta a\alpha a') \in V_\beta^\beta(a\alpha b)$ which implies that $b'\alpha a' \in V_\beta^\beta(a\alpha b)$.

Therefore, $V_\alpha^\beta(b)\alpha V_\beta^\alpha(a) \subseteq V_\beta^\beta(a\alpha b)$.

Proposition 4.14. Let R be a regular Γ-hyperring with the set E_α of α-idempotent elements, $e_1\alpha e_2$ is a (α,α)-regular element of R where $e_1, e_2 \in E_\alpha$ and $V_\alpha^\alpha(e) \subseteq E_\alpha$ for every $e \in E_\alpha$. Then, E_α is a sub semigroup of R_α.

Proof: Suppose that $e_1, e_2 \in E_\alpha$, $x \in V_\alpha^\alpha(e_1\alpha e_2)$ and $y = e_2\alpha x\alpha e_1$. Then,

$$\begin{aligned}(e_1\alpha e_2)\alpha y\alpha(e_1\alpha e_2) &= (e_1\alpha e_2)\alpha(e_2\alpha x\alpha e_1)\alpha(e_1\alpha e_2)\\ &= e_1\alpha(e_2\alpha e_2)\alpha x\alpha(e_1\alpha e_1)\alpha e_2\\ &= (e_1\alpha e_2)\alpha x\alpha(e_1\alpha e_2)\\ &= e_1\alpha e_2.\end{aligned}$$

$$\begin{aligned}y\alpha(e_1\alpha e_2)\alpha y &= (e_2\alpha x\alpha e_1)\alpha(e_1\alpha e_2)\alpha(e_2\alpha x\alpha e_1)\\ &= e_2\alpha x\alpha(e_1\alpha e_1)\alpha(e_2\alpha e_2)\alpha x\alpha e_1\\ &= e_2\alpha(x\alpha e_1\alpha e_2\alpha x)\alpha e_1\\ &= e_2\alpha x\alpha e_1.\end{aligned}$$

Hence, $y \in V_\alpha^\alpha(e_1\alpha e_2)$. Moreover,

$$\begin{aligned}y\alpha y &= (e_2\alpha x\alpha e_1)\alpha(e_2\alpha x\alpha e_1)\\ &= e_2\alpha(x\alpha e_1\alpha e_2\alpha x)\alpha e_1 = e_2\alpha x\alpha e_1 = y,\end{aligned}$$

which implies that y is α-idempotent element. But $e_1\alpha e_2 \in V_\alpha^\alpha(y) \subseteq E_\alpha$. Hence, E_α is a sub semigroup of R_α.

5. Conclusion

In this work, we presented the concept of Γ-semihyperring which is a new kind of hyperalgebra and is a generalization of semihyperrings, hyperrings and rings and proved some results. In particular, the study of the notions of Noetherian, Artinian, simple and regular Γ-semihyperrings.

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