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Approximation of stochastic advection-diffusion equation using compact finite difference technique

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Abstract

In this paper, we propose a new method for solving the stochastic advection-diffusion equation of Ito type. In this work, we use a compact finite difference approximation for discretizing spatial derivatives of the mentioned equation and semi-implicit Milstein scheme for the resulting linear stochastic system of differential equation. The main purpose of this paper is the stability investigation of the applied method. Finally, some numerical examples are provided to show the accuracy and efficiency of the proposed technique.

Keywords: Stochastic partial differential equation; compact finite difference scheme; stability; semi-implicit Milstein method

1. Introduction

In recent years, there has been interest regarding the study of stochastic partial differential equations (SPDEs). SPDEs can describe the dynamics of stochastic processes defined on space-time continuum. These equations have been widely used to model many applications in engineering and mathematical sciences.

Analytical solution can be obtained for very few SPDEs and some authors have studied them theoretically [1-4]. One hope is that using numerical methods to generate solutions to such equations will lead to better understanding of the equations. For numerical simulation of solution of SPDEs, some authors have used the finite element approximation [5-7] and others have used finite difference scheme for approximation solution of SPDE's. Roth used an explicit finite difference method to approximate the solution of some stochastic hyperbolic equations [8]. Soheili et al. presented two methods for solving linear parabolic SPDE's based on the Saul'yev method and a high order finite difference scheme [9].

Kamrani and Hosseini reported explicit and implicit finite difference method for general SPDE [10]. Some authors used spectral method for spatial variable discertization and solved the resulting system of SODE via the Crank-Nicolson scheme or stochastic Runge-Kuttamethod [11, 12]. The Wiener Chaos expansion is another method that we can use for the solution of SPDEs [13].

In this paper, we consider the one dimensional stochastic advection-diffusion equation:

$$\begin{cases} u_t(x,t) = \alpha u_{xx}(x,t) + \beta u_x(x,t) \\ \qquad\qquad + \sigma u(x,t)\dot{W}(t) \\ u(x,t_0) = u_0(x) \;, \\ u(0,t) = f_1(t),\; u(X,t) = f_2(t), \end{cases} \tag{1}$$

where $t \in [t_0, T]$, $x \in [0, X]$. In Eq(1) $\alpha > 0, \beta, \sigma$ are constants and $\dot{W}(t)$ is a random noise which is related to the Brownian motion $W(t)$.

Equation(1) can be rewritten as:

$$u(x,t) = u(x,0) + \int_0^t (\alpha\, u_{xx}(x,s) + \beta\, u_x(x,s))\, ds \\ + \int_0^t \sigma\, u(x,s)\, dW(s)\,.$$

The stochastic integral is the Ito-integral with respect to $\mathbb{R}^1$-valued Wiener process $(W(t), F_t)_{t\in[0,T]}$ defined on a complete probability space (Ω, F, P), adapted to the standard filtration

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$(F_t)_{t\in[0,T]}$.

The outline of this paper is as follows: in section 2, we state a compact finite difference scheme for discretizing spatial derivatives of stochastic partial differential equation and a stochastic differential system is obtained. In section 3, the semi-implicit Milstein method is applied for this system. In this section we investigate stability of this method under an important theorem. Finally, in the last section, numerical examples are presented.

2. Compact finite difference for stochastic advection-diffusion equation

In this section, we introduce the standard compact approximations for the spatial derivatives of Eq(1). Consider the following differential equation:

$$\alpha u_{xx} + \beta u_x + \sigma u\dot{W}(t) = f(x) \ , x\in[0, X]. \quad (2)$$

If we denote the central difference scheme of order two for second and first order derivatives of u as

$$\delta_x^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}, \ \delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

respectively, then we obtain the following approximation for Eq(2) at point x_i:

$$\alpha\delta_x^2 u_i + \beta\delta_x u_i + \sigma u_i\dot{W}(t) - \tau_i = f_i \quad (3)$$

in which $\tau_i = \frac{\Delta x^2}{12}(\alpha\frac{\partial^4 u}{\partial x^4} + 2\beta\frac{\partial^3 u}{\partial x^3})_i$.

In order to obtain a higher order scheme, the fourth and third derivatives of u can be approximated [14, 15]:

$$\frac{\partial^2 u}{\partial x^2}|_i = \frac{1}{\alpha}(f - \beta u_x - \sigma u\dot{W})_i$$

$$\approx \frac{1}{\alpha}(f_i - \beta\delta_x u_i - \sigma u_i\dot{W})$$

$$\frac{\partial^3 u}{\partial x^3}|_i = \frac{1}{\alpha}(f_x - \beta u_{xx} - \sigma u_x\dot{W})_i$$

$$\approx \frac{1}{\alpha}(\delta_x f_i - \beta\delta_x^2 u_i - \sigma\delta_x u_i\dot{W})$$

$$\frac{\partial^4 u}{\partial x^4}|_i = \frac{1}{\alpha}(f_{xx} - \beta u_{xxx} - \sigma u_{xx}\dot{W})_i$$

$$\approx \frac{1}{\alpha}\Big(\delta_x^2 f_i - \frac{\beta}{\alpha}(\delta_x f_i - \beta\delta_x^2 u_i - \sigma\delta_x u_i\dot{W}) - \sigma\delta_x^2 u_i\dot{W} \Big).$$

We substitute the above implications in Eq(3):

$$\alpha\delta_x^2 u_i + \beta\delta_x u_i + \sigma u_i\dot{W}(t) \quad (4)$$
$$\frac{\Delta x^2}{12}[\{\delta_x^2 f_i - \frac{\beta}{\alpha}(\delta_x f_i - \beta\delta_x^2 u_i - \sigma\delta_x u_i\dot{W})$$
$$- \sigma\delta_x^2 u_i\dot{W}\} + 2\frac{\alpha}{\beta}\{\delta_x f_i - \beta\delta_x^2 u_i - \sigma\delta_x u_i\dot{W}\}] = f_i$$

Eq(4) can be rewritten as:

$$\alpha\delta_x^2 u_i + \beta\delta_x u_i + \sigma u_i\dot{W}(t) \quad (5)$$
$$- \frac{\Delta x^2}{12}[\frac{\beta^2}{\alpha}\delta_x^2 u_i + \frac{\beta}{\alpha}\sigma\delta_x u_i\dot{W} - \sigma\delta_x^2 u_i\dot{W}$$
$$- 2\frac{\beta^2}{\alpha}\delta_x^2 u_i - 2\frac{\beta}{\alpha}\sigma\delta_x u_i\dot{W}] = f_i + \frac{\Delta x^2}{12}(\delta_x^2 f_i$$
$$- \frac{\beta}{\alpha}\delta_x f_i + 2\frac{\beta}{\alpha}\delta_x f_i).$$

For an integer positive M, if $\Delta x = \frac{X}{M}$ and Δt denote the spatial step size and time step size, respectively, so we define:

$$x_j = j\,\Delta x \qquad j = 0,1,\ldots,M$$
$$t_j = j\,\Delta t \qquad j = 0,1,2,\ldots .$$

In order derivative of high-order difference algorithm, we must discrete Eq(1) in space at point x_i according to Eq(5) to obtain a system of stochastic differential equation as follows:

$$[\frac{\alpha}{\Delta x^2} + \frac{\beta^2}{12\alpha} - \frac{\beta}{2\Delta x} - \frac{\beta\sigma\Delta x}{24\alpha}\dot{W} + \frac{\sigma}{12}\dot{W}]u_{i-1} +$$
$$[\frac{5}{6}\sigma\dot{W} - \frac{2\alpha}{\Delta x^2} - \frac{\beta^2}{6\alpha}]u_i \quad (6)$$
$$+ [\frac{\alpha}{\Delta x^2} + \frac{\beta^2}{12\alpha} - \frac{\beta}{2\Delta x} - \frac{\beta\sigma\Delta x}{24\alpha}\dot{W} + \frac{\sigma}{12}\dot{W}]u_{i+1}$$
$$= [\frac{1}{12} - \frac{\beta\Delta x}{24\alpha}]u'_{i-1} + \frac{5}{6}u'_i + [\frac{1}{12} + \frac{\beta\Delta x}{24\alpha}]u'_{i+1}$$

Let the boundary conditions be homogeneous, then our system can be written as:

$$AU' = (B + \sigma\dot{W}A)U, \quad (7)$$

in which A and B are tridiagonal matrices as follows:

$$A = Tri[\frac{1}{12} - \frac{\beta\Delta x}{24\alpha}, \frac{5}{6}, \frac{1}{12} + \frac{\beta\Delta x}{24\alpha}]$$
$$B = Tri[\Sigma_1, \Sigma_2, \Sigma_3]$$

where

$$\Sigma_1 = \frac{\alpha}{\Delta x^2} + \frac{\beta^2}{12\alpha} - \frac{\beta}{2\Delta x} \quad ,$$
$$\Sigma_2 = -\frac{2\alpha}{\Delta x^2} - \frac{\beta^2}{6\alpha}$$
$$\Sigma_3 = \frac{\alpha}{\Delta x^2} + \frac{\beta^2}{12\alpha} + \frac{\beta}{2\Delta x},$$

and

$$U = [u_1(t), \ldots, u_{M-1}(t)]^T$$
$$U' = [u_1'(t), \ldots, u_{M-1}'(t)]^T.$$

Theorem 1. The matrix A is invertible.

proof: see [14].

Therefore we can have the following stochastic system:

$$U' = (A^{-1}B)U + \sigma\dot{W}(t)U. \qquad (8)$$

Before proving the main theorem, we state the following lemma:

Lemma 1. If $\frac{b^2}{4ac} = \cos^2(\frac{\theta}{2}), 0 \le \theta \le \pi$, then

$$\frac{2b^2 - 4ac + 2b\sqrt{b^2 - 4ac}}{4ac} = \cos(\theta) + i\sin(\theta).$$

proof: see [14].

Theorem 2. All eigenvalues of $C = A^{-1}B$ have negative real-parts.

proof: Let λ and υ be eigenvalue and correspondingeigenvector of C, respectively. So we have $A^{-1}B\upsilon = \lambda\upsilon$ or $B\upsilon = A\lambda\upsilon$. We rewrite this implication as follows:

$$(B_{i,i-1} - \lambda A_{i,i-1})\upsilon_{i-1} + (B_{i,i} - \lambda A_{i,i})\upsilon_i + (B_{i,i+1} - \lambda A_{i,i+1})\upsilon_{i+1} = 0. \qquad (9)$$

Let

$$a = B_{i,i+1} - \lambda A_{i,i+1},$$
$$b = B_{i,i} - \lambda A_{i,i},$$
$$c = B_{i,i-1} - \lambda A_{i,i-1}.$$

If $b^2 - 4ac = 0$, then (via the Mathematica software) we have:

$$\lambda_1 = \frac{12(-24\alpha^3 - \alpha\beta^2\Delta x^2 + 4\sqrt{3}\sqrt{12\alpha^6 - \alpha^4\beta^2\Delta x^2})}{96\alpha^2\Delta x^2 + \beta^2\Delta x^4}$$
$$\lambda_2 = \frac{12(-24\alpha^3 - \alpha\beta^2\Delta x^2 - 4\sqrt{3}\sqrt{12\alpha^6 - \alpha^4\beta^2\Delta x^2})}{96\alpha^2\Delta x^2 + \beta^2\Delta x^4}.$$

λ_2 has negative real part for each possible value of $\alpha, \beta, \Delta x$. If $12\alpha^6 - \alpha^4\beta^2\Delta x^2 \le 0$ then λ_1 has negative real part. For $12\alpha^6 - \alpha^4\beta^2\Delta x^2 \ge 0$, we have:

$$12\alpha^6 - \alpha^4\beta^2\Delta x^2 \le 12\alpha^6 \Rightarrow$$
$$4\sqrt{3}\sqrt{12\alpha^6 - \alpha^4\beta^2\Delta x^2} \le 24\alpha^3$$
$$\Rightarrow -24\alpha^3 - \alpha\beta^2\Delta x^2 + 4\sqrt{3}\sqrt{12\alpha^6 - \alpha^4\beta^2\Delta x^2} \le -\alpha\beta^2\Delta x^2 \le 0 \quad ,$$

so, λ_1 is negative.

Now consider that $b^2 - 4ac \ne 0$. In this situation let

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

be roots of equation $ar^2 + br + c = 0$, then $\upsilon_i = c_1 r_1^i + c_2 r_2^i$ is solution of difference equation Eq(9). We need to have $\upsilon_0 = \upsilon_M = 0$, since $\upsilon_0 = 0$ then $c_1 = -c_2$, but we need $\upsilon_M = 0$, then $(\frac{r_1}{r_2})^M = 1$.

Let $R = \frac{r_1}{r_2}$, so

$$R = \cos(\frac{2k\pi}{M}) + i\sin(\frac{2k\pi}{M}), k = 0, 1, \cdots, M-1.$$

Since

$$R = \frac{r_1}{r_2} = \frac{2b^2 - 4ac + 2b\sqrt{b^2 - 4ac}}{4ac}$$
$$= \cos(\frac{2k\pi}{M}) + i\sin(\frac{2k\pi}{M}),$$

from the previous lemma, it is sufficient to have $\frac{b^2}{4ac} = \cos^2(\frac{\theta}{2})$. If we assume $t = \cos^2(\frac{\theta}{2})$, then by solving the $b^2 - 4act = 0$, $t \in [0,1]$, we will have:

$$\lambda_1 = \frac{2(\Lambda_1 + \sqrt{\Lambda_2})}{\Lambda_3} \quad , \quad \lambda_2 = \frac{2(\Lambda_1 - \sqrt{\Lambda_2})}{\Lambda_3},$$

where

$$\Lambda_1 = -120\alpha^3\Delta x^2 - 10\alpha\beta^2\Delta x^4 - 24\alpha^3\Delta x^2 t + 4\alpha\beta^2 t\Delta x^4 \le 0,$$
$$\Lambda_2 = 12^4\alpha^6 t\Delta x^4 - 1728\alpha^4\beta^4 t\Delta x^6 - \beta^6 t\Delta x^{10} + \beta^6 t^2\Delta x$$
$$\Lambda_3 = 100\alpha^2\Delta x^4 - 4\alpha^2 t\Delta x^4 + \beta^2 t\Delta x^6 \ge 0$$

Obviously, λ_2 has negative real part for each possible value of α, β. If $\Lambda_2 \le 0$, then λ_1 has negative real part. But if it is positive, then we have (via the Mathematica software):

$$\Lambda_1 + \sqrt{\Lambda_2} \le 0 \Leftrightarrow \Lambda_2 \le \Lambda_1^2 \Leftrightarrow$$
$$\Delta x^4(\alpha^2(4t-100) - \beta^2\Delta x^2 t)((1-t)(-144\alpha^4 - \beta^4\Delta x^4) - 12\alpha^2\beta^2\Delta x^2(2+t)) \ge 0,$$

therefore λ_1 is negative.

According to the above theorem each eigenvalue of C is in the left-half complex plane and similar to deterministic case, it is useful for stability.

3. Semi-implicit Milstein method and its stability

Consider the $n-$dimensional SDE of Ito type given by:

$$\begin{cases} dX(t) = f(X(t))\,dt + g(X(t))\,dW(t) \\ t \ge 0 \quad X(0) = X_0 \in \mathbb{R}^n \end{cases}$$

where $f:\mathbb{R}^n \to \mathbb{R}^n$ and $g:\mathbb{R}^n \to \mathbb{R}^n$, and $W(t)$ is a scalar wiener process[16]. The semi-implicit Milstein scheme for computing approximations $X_n \approx X(t_n)$ takes the form:

$$X_{n+1} = X_n + (1-\theta)\Delta t\, f(x_n) + \theta\Delta t\, f(X_{n+1}) + \sqrt{\Delta t}g(X_n)V_n + \frac{1}{2}\Delta t\, g'(X_n)\, g(X_n)\,(V_n^2 - 1); \tag{10}$$

where V_n is an independent standard Normal $(0,1)$ random variable [17], and θ is a free parameter (usually $0 \le \theta \le 1$).

The above scheme is called trapezoidal particularly if $\theta = \frac{1}{2}$, and backward Euler scheme if $\theta = 1$. We note that in the deterministic case, $g \equiv 0$, Eq(10) is called the Theta method [17, 18]. Semi-implicit Milstein method for stochastic system Eq (8) takes the form:

$$U_{n+1} = U_n + (1-\theta)\Delta t(CU_n) + \theta\Delta t(CU_{n+1}) + \sqrt{\Delta t}(\sigma U_n)V_n + \frac{1}{2}\Delta t(\sigma^2 U_n)(V_n^2 - 1); \tag{11}$$

or:

$$\underbrace{(I - \theta\Delta t\, C)}_{=F}U_{n+1} = U_n + \underbrace{(1-\theta)\Delta t\, C}_{=G}U_n + \sqrt{\Delta t}V_n\,\sigma U_n + \frac{1}{2}\Delta t(V_n^2 - 1)\sigma^2 U_n \tag{12}$$

where $U_n \simeq [u(x_1,t_n), u(x_2,t_n), \cdots, u(x_{M-1},t_n)]^T$.

In component form, we have:

$$\sum_{j=1} F_{kj} U_{n+1}^j = U_n^k + \sum_{j=1} G_{kj} U_n^j + \sqrt{\Delta t}\,\sigma V_n U_n^k + \frac{1}{2}\Delta t\,\sigma^2\,(V_n^2 - 1)\,U_n^k, \tag{13}$$

in which $k = 1, \ldots, M-1$.

Before proving the following theorem, we define that $\sup_k \sum_j |G_{kj}| = \sum_j |G_{mj}|$ and

$$\sup_k \sum_j |F_{kj}| = \sum_j |F_{Lj}|.$$

Higham [17] and Saito [19] applied the semi-implicit Milstein method on the test equation:

$$dy(t) = \lambda\, y(t)dt + \mu\, y(t)dW(t) \ , y(0) = y_0,$$

and obtained useful properties about it's stability.

For stability, we need a norm, hence for sequence $x = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$, sup-norm is defined $\|x\|_\infty = \sqrt{\sup_k |x_k|^2}$ [8]. We refer to the paper of

Roth [8] for the following definition.

Definition 1. (Stability of a stochastic difference scheme) A stochastic difference scheme is said to be stable with respect to a norm in mean square if there exist some positive constants $\overline{\Delta x_0}$ and $\overline{\Delta t_0}$ and constants K and γ such that

$$E\,\|u^{n+1}\|^2 \le Ke^{\gamma t} E\,\|u^0\|^2, \tag{14}$$

For all

$$0 \le t = (n+1)\Delta t, 0 \le \Delta x \le \overline{\Delta x_0}, 0 \le \Delta t \le \overline{\Delta t_0}.$$

Remark 1. One interpretation of stability of a difference scheme is that for stable difference schemes small errors in the initial conditions cause small errors in the solutions. The definition allows the errors to grow, but limits them from growingless quickly than exponentially. Numerical solution can keep a similar property as n tends to infinity when it is applied to the stable SDE in mean-square.

Remark 2. For the proposed scheme the increments of Wiener process are independent of the state U_n^k.

Theorem 3. If $\frac{1+\sum_j |G_{mj}|}{\sum_j |F_{Lj}|} \le 1$ then semi-implicit Milstein scheme with $(n+1)\Delta t = t$ is stable in mean square.

Proof: Applying $E\,|.|^2$ to Eq(13) and using the independence of Wiener increments, we get

$$E[\sum_{j=1} F_{kj} U_{n+1}^j]^2 =$$

$$E[U_n^k + \sum_{j=1} G_{kj} U_n^j + \sqrt{\Delta t}\,\sigma V_n U_n^k + \frac{1}{2}\Delta t\,\sigma^2 (V_n^2 - 1) U_n^k]^2$$

$$= E\,|U_n^k|^2 + E\,|\sum_{j=1} G_{kj} U_n^j|^2 + \Delta t \sigma^2 E\,|U_n^k|^2 + \frac{1}{2}\Delta t^2 \sigma^4 E\,|U_n^k|^2 + 2E\,|U_n^k \sum_{j=1} G_{kj} U_n^j|^2 .$$

then:

$$E[\sum_{j=1} F_{kj} U_{n+1}^j]^2 \le ([1+\sum_j |G_{mj}|]^2 + \Delta t\,\sigma^2 + \frac{1}{2}\Delta t^2\,\sigma^4) \sup_k E\,|U_n^k|^2 .$$

This hold for every k, so

$$[\sum_j |F_{Lj}|]^2 \sup_k E\,|U_{n+1}^k|^2 \le ([1+\sum_j |G_{mj}|]^2 + \Delta t\,\sigma^2 + \frac{1}{2}\Delta t^2\,\sigma^4) \sup_k E\,|U_n^k|^2 .$$

So,

$$\sup_k E\,|U_{n+1}^k|^2 \le ([\frac{1+\sum_j |G_{mj}|}{\sum_j |F_{Lj}|}]^2 + \frac{1}{(\sum_j |F_{Lj}|)^2}(\Delta t\,\sigma^2 + \frac{1}{2}\Delta t^2 \sigma^4)) \sup_k E\,|U_n^k|^2,$$

since $\Delta t < 1$, we have

$$\sup_k E\,|U_{n+1}^k|^2 \le (SC^2 + \frac{\Delta t}{(\sum_j |F_{Lj}|)^2}(\sigma^2 + \frac{1}{2}\sigma^4)) \sup_k E\,|U_n^k|^2$$

$$\le \cdots$$

$$\le (SC^2 + \frac{\Delta t}{(\sum_j |F_{Lj}|)^2}(\sigma^2 + \frac{1}{2}\sigma^4))^{n+1} \sup_k E\,|U_0^k|^2 .$$

where $SC = [\frac{1+\sum_j |G_{mj}|}{\sum_j |F_{Lj}|}]$.

Obviously if $[\frac{1+\sum_j |G_{mj}|}{\sum_j |F_{Lj}|}] \le 1$ then with $(n+1)\Delta t = t$, we have:

$$E\,\|U_{n+1}\|_\infty^2 \le e^{\gamma t} E\,\|U_0\|_\infty^2 ,$$

where $\gamma = \frac{\sigma^2 + \frac{1}{2}\sigma^4}{(\sum_j |F_{Lj}|)^2}$.

Therefore, according to definition1, our scheme is stable.

But there is an essential question: when is $[\frac{1+\sum_j |G_{mj}|}{\sum_j |F_{Lj}|}] \le 1$ satisfied?

For different values of α, β we have plotted $\log$ scale of $SC = [\frac{1+\sum_j |G_{mj}|}{\sum_j |F_{Lj}|}]$ against θ. It is observed that the stability condition $(SC \le 1)$ holds for $\frac{1}{2} \le \theta \le 1$ (Fig. 1).

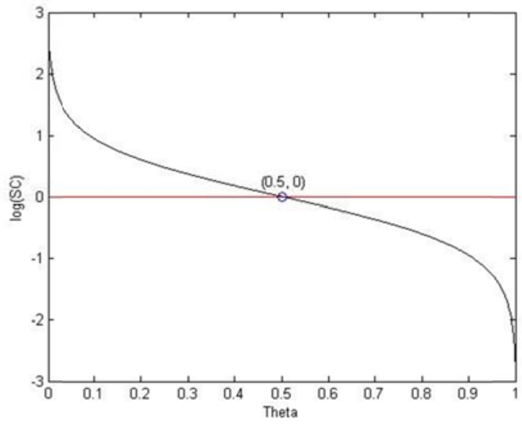

Fig. 1. Stability condition: we see $\log(SC) \le 0$ for $\frac{1}{2} \le \theta$. It means that stability condition is satisfied for this interval.

4. Numerical experiments

In this section, we consider some SPDEs and investigate the results of the previous section. For each $\{\Delta x, \Delta t\}$, 10000 runs are performed with different samples of noise(via Matlab software) and their averages are computed.

Example 1. Consider the following stochastic heat equation:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + u(x,t)\dot{W}(t), \\ u(x,0) = \sin(\pi x), \; t \in [0,0.3], \; x \in [0,1] \\ u(0,t) = u(1,t) = 0. \end{cases} \quad (15)$$

In this case, we choose $\Delta x = \frac{1}{100}, \Delta t = \frac{1}{1000}$. For different values of θ, we have Table1. From Table1 it is obvious that for $\frac{1}{2} \le \theta$ stability condition $\frac{1+\sum_j |G_{mj}|}{\sum_j |F_{Lj}|} \le 1$ holds and $E(u(0.5,0.3))$ is about 0.0519.

Example 2. Consider the following stochastic advection-diffusion equation:

$$\begin{cases} u_t(x,t) = 0.001u_{xx}(x,t) + u_x \\ \qquad -2u(x,t)\dot{W}(t), \\ u(x,0) = x^2(1-x)^2, \\ u(0,t) = u(1,t) = 0 t \in [0,1] \end{cases} \quad (16)$$

Table 1. Comparison between stability condition and different values of θ, for example 1.

θ	$\frac{1+\sum_j \lvert G_{mj} \rvert}{\sum_j \lvert F_{Lj} \rvert}$	$E(u(0.5,0.3))$
0	61	NaN
0.1	7.8571	8.994 e+246
0.2	3.7692	3.9655 e+150
0.3	2.2631	-1.749 e+83
0.4	1.48	8.3812 e+26
0.45	1.2142	3.2909
0.5	1	0.05161
0.6	0.6756	0.05175
0.7	0.4418	0.05190
0.8	0.2653	0.0520
0.9	0.1272	0.05199
1	0.0164	0.05210

For this example we use $\Delta x = \frac{1}{100}, \Delta t = \frac{1}{10}$. From Table 2 we can observe that $E(u(0.5,1))$ is about zero for $\frac{1}{2} \le \theta$.

Table 2. Representation of stability condition and $E(u(0.5,1))$ for different values of θ corresponding to example2.

θ	$\frac{1+\sum_j \lvert G_{mj} \rvert}{\sum_j \lvert F_{Lj} \rvert}$	$E(u(0.5,1))$
0	71.33	-2.8241 e+5
0.1	8.00414	0.22641
0.2	3.8008	0.10441
0.3	2.2730	-0.0405
0.4	1.4828	-0.0070
0.45	1.2154	-0.00238
0.5	1	-7.5774 e-4
0.6	0.6744	-2.4496 e-5
0.7	0.4399	9.9938 e-6
0.8	0.2630	-4.0235 e-5
0.9	0.1249	4.8299 e-5
1	0.0140	1.8300 e-4

Example 3. Consider the following stochastic advection-diffusion equation:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u_x - 2u(x,t)\dot{W}(t), \\ u(x,0) = 10x(1-x)\,, t \in [0,0.5]\,, x \in [0\,,1] \\ u(0,t) = u(1,t) = 0. \end{cases} \quad (17)$$

We choose $\Delta x = \dfrac{1}{100}, \Delta t = \dfrac{1}{1000}$ for this case. In Table3, we see the results for $E(u(0.5,0.5))$ are destroyed when $0 \le \theta < \dfrac{1}{2}$, while for $\dfrac{1}{2} \le \theta$, the stable results are obtained.

Table 3. Investigation of stability condition and $E(u(0.5,0.5))$ for different values of θ

θ	$\dfrac{1+\sum_j \lvert G_{mj} \rvert}{\sum_j \lvert F_{Lj} \rvert}$	$E(u(0.5,0.5))$
0	61	NaN
0.1	7.8571	NaN
0.2	3.7692	-4.5047 e+262
0.3	2.2631	-8.7938 e+150
0.4	1.4800	-5.8375 e+56
0.45	1.2142	-6.9331 e+12
0.5	1	0.017684
0.6	0.6756	0.01777
0.7	0.4418	0.017860
0.8	0.2653	0.01794
0.9	0.1272	0.01803
1	0.0164	0.01810

5. Conclusion

In this paper, we approximate the stochastic advection-diffusion equation using the compact finite difference technique and semi-implicit Milstein method and studied the stability condition, theoretically and numerically. Numerical experiments show that the proposed scheme is unconditionally stable for $\dfrac{1}{2} \le \theta$.

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