
Bayesian and non-bayesian estimation of stress–strength model for Pareto type I distribution

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Abstract

This article examines statistical inference for $R = P(Y < X)$ where X and Y are independent but not identically distributed Pareto of the first kind (Pareto (I)) random variables with same scale parameter but different shape parameters. The Maximum likelihood, uniformly minimum variance unbiased and Bayes estimators with Gamma prior are used for this purpose. Simulation studies which compare the estimators are presented. Moreover, sensitivity of Bayes estimator to the prior parameters is considered.

Keywords: Bayesian estimator; Maximum likelihood estimator (MLE); Pareto of first kind; uniformly minimum variance unbiased estimator (UMVUE); stress-strength model

1. Introduction

The problem of estimating $R = P(Y < X)$ arises in the context of mechanical reliability of a system with strength X and stress Y , then reliability R is chosen as a measure of system reliability. In a stress-strength model, the system fails if and only if $X \leq Y$, at any time. This model, first considered by Birnbaum [1], is commonly used in many engineering applications such as civil, mechanical and aerospace. Recently, a number of papers have dealt with the stress-strength reliability problem. Several distributions have been used in the literature as failure models. For references see the book of Kotz et al. [2] and the articles [3-18].

The main aim of this paper is to discuss the inference of $R = P(Y < X)$, when X and Y are two independent, but not identically random variables belonging to Pareto (I) distribution with two parameters.

The Pareto distribution is a power law probability distribution that coincides with social, scientific, geophysical and many other types of observable phenomena. Recently it has been used to study ozone levels in the upper atmosphere and tensile strength of nylon carpet fibers. It has played a very important role in the investigation of city population occurrence of natural resources, insurance risk, and business failures.

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The probability density functions (pdf's) of X and Y are given, respectively, by

$$f(x) = \theta_1 b^{\theta_1} x^{-(\theta_1+1)}, \quad x \geq b, \theta_1, b > 0 \quad (1)$$

$$g(y) = \theta_2 b^{\theta_2} y^{-(\theta_2+1)}, \quad y \geq b, \theta_2, b > 0 \quad (2)$$

Then, it can be shown that

$$R = P(Y < X) = \iint_{b < y < x} f(x, y) dx dy = \frac{\theta_2}{\theta_1 + \theta_2}, \quad (3)$$

in which R is free of b .

The paper is organized as follows: In Section 2, MLE, UMVUE, and Bayesian estimators of R are obtained. In Section 3, simulation studies are done to compare the different estimates of R together with sensitivity of the Bayes estimators to the parameter of the prior distribution represented in Section 2.

2. Estimation of Reliability

2.1. Maximum Likelihood Estimation

Let X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_m be the two

independent random samples taken from the Pareto distribution with parameters (θ_1, b) and (θ_2, b) , respectively, and let b be known. Then, likelihood and log-likelihood functions based on the above samples are given by

$$L(x, y; \theta_1, \theta_2, b) = \theta_1^n \theta_2^m b^{n\theta_1 + m\theta_2} \prod_{i=1}^n x_i^{-(\theta_1+1)} \prod_{j=1}^m y_j^{-(\theta_1+1)}, \quad (4)$$

$$l(x, y; \theta_1, \theta_2, b) = n \log \theta_1 + m \log \theta_2 + (n\theta_1 + m\theta_2) \log b - (\theta_1 + 1) \sum_{i=1}^n \log x_i - (\theta_2 + 1) \sum_{j=1}^m \log y_j. \quad (5)$$

Using (5), the MLE's for θ_1 and θ_2 are, respectively,

$$\hat{\theta}_1 = n / \sum_{i=1}^n \log (x_i b^{-1}), \quad (6)$$

$$\hat{\theta}_2 = m / \sum_{j=1}^m \log (y_j b^{-1}). \quad (7)$$

Hence \hat{R}_1 , the MLE of R , is written as follows

$$\hat{R}_1 = \frac{\hat{\theta}_2}{\hat{\theta}_1 + \hat{\theta}_2}. \quad (8)$$

2.2. Some properties of \hat{R}_1

Since b is known, \hat{R}_1 can be expressed as

$$\hat{R}_1 = 1 / \left(1 + \frac{n \sum_{j=1}^m \log(y_j b)}{m \sum_{i=1}^n \log(x_i b)} \right). \quad (9)$$

Since

$$2\theta_1 \sum_{i=1}^n \log(x_i b^{-1}) \sim x_{2n}^2,$$

and

$$2\theta_2 \sum_{j=1}^m \log(y_j b^{-1}) \sim x_{2m}^2.$$

So

$$\frac{\theta_2}{\theta_1} \left(\frac{1}{\hat{R}_1} - 1 \right) = \frac{n\theta_2 \sum_{j=1}^m \log(y_j b^{-1})}{m\theta_1 \sum_{i=1}^n \log(x_i b^{-1})} \sim F_{(2m, 2n)}.$$

From this fact we shall study some properties of \hat{R}_1 (Lindley [19]) and it could be shown that

$$E(\hat{R}_1) = \frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n}.$$

$$\left[1 + \frac{n+m-1}{m(n-2)} \left(1 - \frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n} \right)^2 \right]. \quad (10)$$

Note that if m is fixed,

$$\lim_{n \rightarrow \infty} E(\hat{R}_1) = R \left[1 + \frac{1}{m} (1-R)^2 \right], \quad (11)$$

and

$$\lim_{m, n \rightarrow \infty} E(\hat{R}_1) = R, \quad (12)$$

\hat{R}_1 is asymptotically unbiased,

$$V(\hat{R}_1) = \frac{n+m-1}{m(n-2)} \left(1 - \frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n} \right)^2 \left(\frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n} \right)^2, \quad (13)$$

then

$$\lim_{n \rightarrow \infty} V(\hat{R}_1) = \frac{1}{m} \left(1 - \frac{\theta_2}{\theta_2 + \theta_1} \right)^2 \left(\frac{\theta_2}{\theta_2 + \theta_1} \right)^2, \quad (14)$$

and

$$\lim_{m, n \rightarrow \infty} V(\hat{R}_1) = 0, \quad (15)$$

i.e. \hat{R}_1 is consistent estimator for R .

The mean square error (MSE) of \hat{R}_1 is given as

$$\begin{aligned} MSE(\hat{R}_1) &= V(\hat{R}_1) + [R - E(\hat{R}_1)]^2 \\ &= \frac{n+m-1}{m(n-2)} \left(1 - \frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n} \right)^2 \left(\frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n} \right)^2 \\ &\quad + \left\{ \frac{\theta_2}{\theta_2 + \theta_1} - \frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n} \right\}^2. \end{aligned}$$

$$\left[1 + \frac{n+m-1}{m(n-2)} \left(1 - \frac{\theta_2(n-1)}{\theta_2(n-1) + \theta_1 n} \right)^2 \right]^2, \quad (16)$$

this tends to zero as $m, n \rightarrow \infty$.

2.3. Uniformly Minimum Variance Unbiased Estimator (UMVUE) of R

Let X_1, X_2, \dots, X_n be a random sample from Pareto (θ_1, b) where b is known. Set $W_i = \log(X_i b^{-1})$, $i = 1, 2, \dots, n$, then W_i is Expo (θ_1) , and hence $\sum_{i=1}^n W_i$ is sufficient statistic for θ_1 . Similarly, if Y_1, Y_2, \dots, Y_m is a random sample from Pareto (θ_2, b) where b is known and

$$V_j = \log(Y_j b^{-1}), \quad j = 1, 2, \dots, m, \quad \text{then}$$

$\sum_{j=1}^m V_j$ is sufficient statistic for θ_2 . Moreover,

$\left(\sum_{i=1}^n W_i, \sum_{j=1}^m V_j \right)$ is jointly sufficient statistic for (θ_1, θ_2) .

Let

$$T = \begin{cases} 1 & \text{if } v_1 < w_1 \\ 0 & \text{if } v_1 \geq w_1 \end{cases}$$

Then

$$\begin{aligned} E(T) &= P(V_1 < W_1) = P(\log(y_1 b^{-1}) < \log(x_1 b^{-1})) \\ &= P(y_1 < x_1) = P(y < x) = R. \end{aligned} \quad (17)$$

Therefore, T is an unbiased estimator for R . By using either Rao-Blackwell and Lehmann-Scheffe's Theorems, or Tong's results [20, 21], we obtain UMVUE for R as

$$\begin{aligned} \hat{R}_2 &= E\left(T \mid \sum_{i=1}^n W_i, \sum_{j=1}^m V_j\right) \\ &= \begin{cases} \sum_{i=0}^{n-1} a_i Q^{-i} & \text{if } Q \geq 1 \\ 1 - \sum_{j=0}^{m-1} c_j Q^j & \text{if } Q < 1, \end{cases} \end{aligned} \quad (18)$$

where

$$Q = \frac{\sum_{i=1}^n W_i}{\sum_{j=1}^m V_j}$$

and

$$a_i = (-1)^i \frac{(n-1)! (m-1)!}{(n-i-1)! (m+i-1)!},$$

$$c_j = (-1)^j \frac{(n-1)! (m-1)!}{(n+j-1)! (m-j-1)!}.$$

To find the MSE of \hat{R}_2 which is given by

$$MSE(\hat{R}_2) = E(\hat{R}_2 - R)^2 = E(\hat{R}_2)^2 - R^2, \quad (19)$$

we proceed to get

$$\begin{aligned} E[(\hat{R}_2)^2] &= \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} a_i a_k E(Q^{-i-k} \mid Q \geq 1) P(Q \geq 1) \\ &+ P(Q < 1) - 2 \sum_{j=0}^{m-1} c_j E(Q^j \mid Q < 1) P(Q < 1) \\ &+ \sum_{j=0}^{m-1} \sum_{s=0}^{m-1} c_j c_s E(Q^{j+s} \mid Q < 1) P(Q < 1), \end{aligned} \quad (20)$$

Equation (20) depends on the evaluation of $E(Q^l \mid Q < 1) P(Q < 1)$ and $E(Q^{-l} \mid Q \geq 1) P(Q \geq 1)$. For this purpose, we first obtain the *pdf* of Q . Since $\sum_{i=1}^n W_i, \sum_{j=1}^m V_j$ have respectively the independent distributions

$\Gamma(n, \theta_1)$ and $\Gamma(m, \theta_2)$, then $\frac{n\rho}{mQ} \sim F_{(2m, 2n)}$

where,

$$\rho = \frac{\theta_2}{\theta_1}, \quad Q = \frac{\sum_{i=1}^n W_i}{\sum_{j=1}^m V_j},$$

thus the *pdf* of Q is

$$f(q) = \frac{\rho^{-n}}{\beta(n, m)} q^{n-1} \left(1 + \frac{q}{\rho} \right)^{-(n+m)}, \quad q > 0. \quad (21)$$

Then, for $l > 0$

$$\begin{aligned} E(Q^l | Q < 1) P(Q < 1) \\ = \frac{\rho^{-n}}{\beta(n, m)} \int_0^1 q^{l+n-1} (1+q/\rho)^{-(n+m)} dq \\ = \frac{\rho^{-n}}{(l+n)\beta(n, m)} F_{2:1} \left(l+n, n+m, l+n+1, -\frac{1}{\rho} \right). \end{aligned} \quad (22)$$

Interchanging n with m and replacing ρ by $\frac{1}{\rho}$ in (22), we have

$$E(Q^{-l} | Q \geq 1) P(Q \geq 1) = \frac{\rho^m}{(l+m)\beta(n, m)} F_{2:1}(l+m, n+m, l+m+1, -\rho). \quad (23)$$

We let $l = 0$ in (22), $P(Q < 1)$ has the form

$$P(Q < 1) = \frac{\rho^{-n}}{n\beta(n, m)} F_{2:1} \left(n, n+m, n+1, -\frac{1}{\rho} \right), \quad (24)$$

where $F_{2:1}(\dots)$ is the generalized hypergeometric function, [22]. Substituting (22)-(24) in (20), we then show that, using (19), the MSE of \hat{R}_2 is represented by hypergeometric function.

2.4. Bayes Estimation for R

It is remarkable that most of the Bayesian inference procedures have been developed under the usual squared error (SE) loss function (quadratic loss), which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. However, such a restriction may be impractical. For example, in the estimation of reliability and failure rate functions, an overestimate is usually much more serious than an underestimate; in this case the use of a symmetrical loss function might be inappropriate, as has been recognized by Basu and Ebrahimi [23] and Canfield [24]. Feynman [25] states that in the disaster of a space shuttle, the management overestimated the average life or reliability of the solid fuel rocket booster. This is an example of an asymmetrical loss function.

A useful asymmetric loss known as the LINEX (linear-exponential) loss function was introduced in [26], and was widely used in several papers [23, 27-29]. This function rises approximately exponentially on one side of zero, and

approximately linearly on the other side. Under the assumption that the minimal loss occurs at $\tilde{u} = u$, the LINEX loss function can be expressed as

$$L(\Delta) \propto e^{c\Delta} - c\Delta - 1; \quad c \neq 0, \quad (25)$$

where $\Delta = (\tilde{u} - u)$, \tilde{u} is an estimate of u .

The sign and magnitude of 'c' represent the direction and degree of symmetry, respectively. ($c > 0$ means overestimation is more serious than underestimation, and $c < 0$ means the opposite). For 'c' closed to zero, the LINEX loss function is approximately the Squared Error (SE) loss, and therefore almost symmetric. The posterior s-expectation of the LINEX loss function of (25) is

$$E_u((L(\hat{u} - u)) \propto \exp(cu) \cdot E_u(\exp(-cu)) - c \cdot (\hat{u} - E_u(u)) - 1. \quad (26)$$

E_u is equivalent to the posterior s-expectation with respect to the posterior pdf. The Bayes estimator \hat{u}_{BL} of u under the LINEX loss function is the value \hat{u} , which minimizes (26).

$$\hat{u}_{BL} = -\frac{1}{c} \log(E_u[\exp(-cu)]), \quad (27)$$

provided that $E_u[\exp(-cu)]$ exists, and is finite. Another useful asymmetric loss function is the General Entropy (GE) loss

$$L_2(\hat{u}, u) \propto \left(\frac{\hat{u}}{u}\right)^q - q \log\left(\frac{\hat{u}}{u}\right) - 1, \quad (28)$$

whose minimum occurs at $\hat{u} = u$. This loss function is a generalization of the Entropy-loss used in several papers where $q = 1$, see [30] and [31]. When $q > 0$, a positive error ($\hat{u} > u$) causes more serious consequences than a negative error. The Bayes estimate \hat{u}_{BG} of u under GE loss (28) is

$$\hat{u}_{BG} = (E_u[u^{-q}])^{-\frac{1}{q}}, \quad (29)$$

provided that $E_u[u^{-q}]$ exists, and is finite.

Bayes estimation of R can be obtained if θ_1 and θ_2 are assumed to be random variables which have independent gamma prior distributions with probability density functions

$$h_i(\theta_i) = \frac{\beta_i^{\varphi_i}}{\Gamma(\varphi_i)} \theta_i^{\varphi_i-1} \exp(-\beta_i \theta_i), \quad \theta_i, \varphi_i, \beta_i > 0, i = 1, 2. \quad (30)$$

It can be shown that the posterior density of (θ_1, b) given \underline{X} is, from (4) and (30),

$$h(\theta_1, b | \underline{x}) \propto \theta_1^{\delta_1-1} \exp(-\theta_1 \lambda_1), \quad (31)$$

where $\delta_1 = n + \varphi_1$ and

$$\lambda_1 = \beta_1 - n \log b + \sum_{i=1}^n \log x_i.$$

Similarly, the posterior density of (θ_2, b) can be written as

$$h(\theta_2, b | \underline{y}) \propto \theta_2^{\delta_2-1} \exp(-\theta_2 \lambda_2), \quad (32)$$

where $\delta_2 = n + \varphi_2$ and

$$\lambda_2 = \beta_2 - m \log b + \sum_{j=1}^m \log y_j.$$

The joint posterior density function of (θ_1, θ_2) is then

$$h(\theta_1, \theta_2 | \underline{x}, \underline{y}) = \frac{\lambda_1^{\delta_1} \lambda_2^{\delta_2}}{\Gamma(\delta_1) \Gamma(\delta_2)} \theta_1^{\delta_1-1} \theta_2^{\delta_2-1} \exp(-\theta_1 \lambda_1 - \theta_2 \lambda_2). \quad (33)$$

Put $R = \frac{\theta_2}{\theta_1 + \theta_2}$, and using standard procedure

of transformations of random variables we obtain the density of R as

$$g_R(r) = \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1) \Gamma(\delta_2)} \lambda_1^{\delta_1} \lambda_2^{\delta_2} (1-r)^{\delta_1-1} r^{\delta_2-1} \left((1-r) \lambda_1 + r \lambda_2 \right)^{-(\delta_1+\delta_2)}, \quad 0 < r < 1.$$

A. Symmetric Bayes Estimation

Under the squared error (SE) loss function, the Bayes estimator of R is

$$\hat{R}_3 = E(R | x, y, b) = \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1) \Gamma(\delta_2)} \lambda_1^{-\delta_2} \lambda_2^{\delta_2} \int_0^1 (1-r)^{\delta_1-1} r^{\delta_2} \left(1 - \frac{\lambda_1 - \lambda_2}{\lambda_1} r \right)^{-(\delta_1+\delta_2)} dr. \quad (34)$$

B. Asymmetric Bayes Estimation

LINEX Loss:

If in (27), $u = r$, then the Bayes estimator of R is

$$\hat{R}_4 = \frac{-1}{c} \log \left\{ \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1) \Gamma(\delta_2)} \lambda_1^{\delta_1} \lambda_2^{\delta_2} \int_0^1 e^{-cr} (1-r)^{\delta_1-1} r^{\delta_2-1} \left((1-r) \lambda_1 + r \lambda_2 \right)^{-(\delta_1+\delta_2)} dr \right\}. \quad (35)$$

General Entropy Loss: Set $u = r$ in (29), then the Bayes estimate \hat{R}_5 of R relative to the general Entropy loss function (28) is

$$\hat{R}_5 = \left\{ \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1) \Gamma(\delta_2)} \lambda_1^{\delta_1} \lambda_2^{\delta_2} \right.$$

$$\left. \int_0^1 (1-r)^{\delta_1-1} r^{\delta_2-1} \left((1-r) \lambda_1 + r \lambda_2 \right)^{-(\delta_1+\delta_2)} dr \right\}^{-1}. \quad (36)$$

2.5. Two-Sided Confidence Intervals

To determine the confidence interval for the reliability R , we first note that $\frac{\rho}{\hat{\rho}}$ has an F -distribution with $2m, 2n$ degrees of freedom, where R may be written as

$$R = P(V < W) = P\left(\frac{\rho}{Q} < \rho\right),$$

where $W = \sum_{i=1}^n W_i$, $V = \sum_{j=1}^m V_j$ and $\frac{\rho}{Q} \sim F_{(2,2)}$.

Then $R = F_F(\rho, 2, 2)$, where $F_F(\cdot, 2, 2)$ is the cumulative distribution function of an F random variable with $2, 2$ degrees of freedom.

The *MLE*'s of θ_1 and θ_2 , by (6), (7), are

$$\hat{\theta}_1 = \frac{n}{W} \text{ and } \hat{\theta}_2 = \frac{m}{V}, \text{ respectively.}$$

If \hat{R}_1 is the *MLE* of R , then

$$\hat{R}_1 = F_F(\hat{\rho}, 2, 2), \text{ where } \hat{\rho} = \frac{\hat{\theta}_2}{\hat{\theta}_1},$$

so

$$\frac{\rho}{\hat{\rho}} = \frac{\theta_2 \hat{\theta}_1}{\theta_1 \hat{\theta}_2} = \frac{n \theta_2 V}{m \theta_1 W} \sim F_{(2m, 2n)}$$

with $1 - \alpha = 1 - (\alpha_1 + \alpha_2)$ confidence coefficient, where F_{α_1} has $1 - \alpha_1$ confidence coefficient and F_{α_2} has $1 - \alpha_2$ confidence coefficient with $2m, 2n$ degrees of freedom. We can obtain the desired interval by

$$\begin{aligned} 1 - \alpha &= P\left(F_{\alpha_1}^{-1} \leq \frac{\rho}{\hat{\rho}} \leq F_{\alpha_2}\right) \\ &= P\left(\hat{\rho} F_{\alpha_1}^{-1} \leq \rho \leq \hat{\rho} F_{\alpha_2}\right) \\ &= P\left(F(\hat{\rho} F_{\alpha_1}^{-1}, 2, 2) \leq R \leq F(\hat{\rho} F_{\alpha_2}, 2, 2)\right), \end{aligned}$$

so the two-sided $(1 - \alpha)100\%$ confidence

interval for R is given by

$$\left[F(\hat{\rho} F_{\alpha_1}^{-1}, 2, 2), F(\hat{\rho} F_{\alpha_2}, 2, 2) \right],$$

i.e.,

$$\left[\frac{1}{\frac{1}{\hat{\rho}} F_{\alpha_1}^{-1} + 1}, \frac{1}{\frac{1}{\hat{\rho}} F_{\alpha_2} + 1} \right].$$

If $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$, then the confidence interval

takes the form $\left[\frac{1}{\frac{1}{\hat{\rho}} F_{\alpha/2} + 1}, \frac{1}{\frac{1}{\hat{\rho}} F_{\alpha/2}^{-1} + 1} \right]$,

where F_{α}^{-1} denotes the inverse function of F with $1 - \alpha$ confidence coefficient and $2n, 2m$ degrees of freedom.

3. Simulation Study for the Different Estimators

Table 1. Maximum Likelihood Estimator, Uniformly Minimum Variance Unbiased and Bayes Estimator with Empirical Estimators for the Prior Parameters (The effect of changing two sample sizes over the MSE for the MLE)

$\theta_1 = 1, \theta_2 = 2, b = .2, R = .66667$							
n	m	\hat{R}_1	$MSE \hat{R}_1$	\hat{R}_2	$MSE \hat{R}_2$	\hat{R}_3	$MSE \hat{R}_3$
10	10	0.528	0.012	0.689	0.010	0.696	0.0044
	40	0.610	0.008	0.654	0.007	0.671	0.0003
	60	0.727	0.008	0.592	0.007	0.597	0.0004
20	10	0.834	0.008	0.714	0.006	0.528	0.0064
	20	0.673	0.005	0.708	0.004	0.637	0.0015
	60	0.560	0.004	0.658	0.003	0.706	0.0001
30	10	0.655	0.007	0.772	0.004	0.522	0.0070
	30	0.672	0.003	0.629	0.004	0.573	0.0008
	60	0.667	0.003	0.705	0.002	0.655	0.0001
40	10	0.650	0.006	0.746	0.005	0.588	0.0075
	40	0.597	0.003	0.610	0.003	0.592	0.0005
	60	0.710	0.002	0.635	0.002	0.653	0.0002
50	10	0.759	0.006	0.649	0.006	0.579	0.0078
	50	0.732	0.002	0.635	0.002	0.658	0.0003
	60	0.660	0.002	0.676	0.002	0.644	0.0002
60	10	0.634	0.006	0.650	0.007	0.685	0.0081
	60	0.612	0.002	0.666	0.002	0.678	0.0002

Table 2. Bayes Estimates of the Reliability Function

Prior Parameters $\varphi_1 = 1, \varphi_2 = 2, \beta_1 = 3, \beta_2 = 4$											
$\theta_1 = 1, \theta_2 = 2, b = .2, R = .66667$											
n	m	\hat{R}_1	\hat{R}_3	\hat{R}_4				\hat{R}_5			
				C=100	C=150	C=-50	C=-100	q=1	q=3	q=-.5	q=-2
20	10	0.638	0.329	0.438	0.466	0.699	0.563	0.580	0.638	0.539	0.553
60	10	0.666	0.307	0.456	0.482	0.661	0.544	0.546	0.557	0.596	0.508
60	30	0.666	0.2083	0.325	0.375	0.928	0.674	0.561	0.595	0.608	0.578
100	10	0.666	0.158	0.458	0.483	0.654	0.539	0.529	0.608	0.557	0.596
100	30	0.648	0.158	0.351	0.398	0.876	0.645	0.569	0.629	0.594	0.595

It is clear that, the computations of $\hat{R}_i, i = 1, \dots, 5$ and their mean square errors are very complicated. Therefore, Mathematical 7 is used to evaluate \hat{R}_i and the MSE 's of \hat{R}_1, \hat{R}_2 and \hat{R}_3 , as shown in the following tables. Table (1) shows the simulation results for the MSE 's of the $MLE, UMVUE$ and Bayes estimators for R , for different values of the sample sizes. It is observed that the MSE of Bayes is smaller than the MSE of maximum likelihood. Table (2) illustrates the sensitivity of the Bayes estimators relative to asymmetric loss functions (LINEX and General Entropy) \hat{R}_4 and \hat{R}_5 for the values of the shape parameters c and q . Table (3) simplifies the sensitivity of \hat{R}_3 to the prior parameters $\varphi_1, \varphi_2, \beta_1$ and β_2 . It is clear first that, the Bayes estimator is sensitive to the prior parameters, and secondly, on keeping β_2 constant, MSE is increasing as β_1 .

Table 3. Sensitivity of Bayes Estimator to the Prior Parameters

$\theta_1 = 1, \theta_2 = 2, b = 2, R = .66667, n = 10, m = 20$						
φ_1	φ_2	β_1	β_2	\hat{R}_3	Bias	MSE \hat{R}_3
1	2	3	2	0.675807	0.009140	0.000540
			4	0.705948	0.039282	0.000792
			6	0.647106	- 0.019560	0.003069
		5	2	0.799059	0.132392	0.001352
			4	0.678427	0.011760	0.000428
			6	0.609685	- 0.056980	0.001507
	4	3	2	0.707073	0.040406	0.003145
			4	0.597962	- 0.068700	0.001105
			6	0.637956	- 0.028710	0.001108
		5	2	0.676283	0.009617	0.005345
			4	0.726487	0.059820	0.001917
			6	0.714830	0.048163	0.001052
	6	3	2	0.710516	0.043849	0.007151
			4	0.738799	0.072132	0.003222
			6	0.705436	0.038770	0.001764
		5	2	0.708526	0.041859	0.009894
			4	0.683676	-0.017010	0.005639
			6	0.755847	0.089180	0.002477
3	2	3	2	0.749162	0.082495	0.000802
			4	0.599080	- 0.067590	0.003771
			6	0.388317	- 0.278350	0.008626
		5	2	0.564465	- 0.102200	0.000318
			4	0.684025	0.017359	0.001810
			6	0.516416	- 0.150250	0.005592
	4	3	2	0.710881	0.044214	0.000932
			4	0.618519	- 0.048150	0.001211
			6	0.554267	- 0.112400	0.003576
		5	2	0.673980	0.007314	0.001707
			4	0.643080	- 0.023590	0.000811
			6	0.618309	- 0.048360	0.001815
	6	3	2	0.681611	0.014944	0.002772
			4	0.618714	- 0.047950	0.001313
			6	0.550589	- 0.116080	0.001693
		5	2	0.780480	0.113814	0.004810
			4	0.709575	0.042909	0.002034
			6	0.609194	- 0.057470	0.001402
5	2	3	2	0.662994	- 0.003670	0.003476
			4	0.635475	- 0.031190	0.008958
			6	0.490599	- 0.176070	0.016216
		5	2	0.618631	- 0.048040	0.001596
			4	0.516726	- 0.149940	0.005709
			6	0.435846	- 0.230820	0.011371
	4	3	2	0.599962	- 0.066700	0.000940
			4	0.707833	0.041166	0.003461
			6	0.595427	- 0.071240	0.008232
		5	2	0.695973	0.029306	0.000726
			4	0.579451	- 0.087220	0.001940
			6	0.533457	- 0.133210	0.005283
	6	3	2	0.567377	- 0.099290	0.001239
			4	0.633209	- 0.033460	0.001638
			6	0.624459	- 0.042210	0.003994
		5	2	0.635440	- 0.031230	0.001992
			4	0.673176	0.006510	0.001081
			6	0.626982	- 0.039680	0.002362

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