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Some kinds of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of ternary semigroups

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Abstract

Generalizing the concepts of $(\in, \in \vee q)$-fuzzy (left, right, lateral) ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals and $(\in, \in \vee q)$-fuzzy bi (generalized bi-) ideals in ternary semigroups, the notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (left, right, lateral) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi (generalized bi-) in ternary semigroups are introduced and several related properties are investigated. Some new results are obtained.

Keywords: Ternary semigroups; $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals; $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals; $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals bi-ideals

1. Introduction

Lehmer in 1932 introduced the notion of ternary semigroup in his pioneering paper [1]. To develop the theory of ternary semigroups, ideal theory plays an important role. Sioson in [2], studied ternary semigroups with a special reference of ideals and radicals and characterized regular ternary semigroups by the properties of quasi-ideals. Dixit and Dewan studied quasi-ideals and bi-ideals of ternary semigroups [3]. Zadeh introduced the concept of fuzzy subset of a set in his seminal paper [4]. There has been rapid growth worldwide in the interest of fuzzy set theory and its applications from the past two decades. An extensive account of applications of fuzzy set theory have been found in different fields such as expert system, coding theory, computer science, artificial intelligence, control engineering, information sciences, operation research, robotics etc. After the introduction of fuzzy set by Zadeh, reconsideration of the concepts of classical mathematics began. On the other hand, because of the importance of group theory in mathematics as well as its applications in many areas, the notion of fuzzy subgroup was introduced by Rosenfeld [5]. Kuroki is responsible for much of the fuzzy ideal theory of semigroups (see [6-10]). The monograph by Mordeson et al. concentrates on the theory of fuzzy semigroups and its applications in fuzzy coding theory, fuzzy finite state machines and fuzzy languages (see [11]). Pu and Liu in [12], gave the idea of quasi-coincidence of a fuzzy point with a fuzzy set. Murali [13], introduced the notion of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. Bhakat in [14, 15], and Bhakat and Das [16, 17], introduced the concept of (α, β)-fuzzy subgroup by using the combined notions of `belongingness' and `quasi-coincidence' between a fuzzy point and a fuzzy subset and introduced the concept of $(\in \vee q)$-level subset, $(\in, \in \vee q)$-fuzzy normal, quasinormal, maximal subgroup and $(\in, \in \vee q)$-fuzzy subgroup. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems with other algebraic structures. Davvaz defined $(\in, \in \vee q)$-fuzzy subnear rings and ideals of a near ring in his paper [18]. Kazanci and Yamak in [19], discussed $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup. In [20], Shabir et al. characterized

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regular semigroups by $(\in, \in \vee q)$-fuzzy ideals. Dudek et al. in [21] and Ma and Zhan in [22] defined (α, β)-fuzzy ideals in hemirings, and investigated some related properties of hemirings. Akram in [23], studied $(\in, \in \vee q)$-fuzzy K-algebra. Zhan and Yin considered $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subnear-rings and ideals of a near-ring in [24]. Ma et al. in [25], introduced the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of BCI-algebras. Recently Shabir and Ali characterized semigroups by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals [26].

In the current paper we initiate the study of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi (generalized bi-) ideals of ternary semigroups and investigated several related properties.

2. Preliminaries

A ternary semigroup is an algebraic structure $(T, [\ ])$ such that T is a non-empty set and $[\ ]: T^3 \to T$ a ternary operation satisfying the associative law: $[[xyz]uv] = [x[yzu]v] = [xy[zuv]]$ for all $x, y, z, u, v \in T$. For the sake of simplicity we write $[xyz]$ as `xyz' and consider the ternary operation as multiplication. A non-empty subset A of a ternary semigroup T is called a ternary subsemigroup of T if $AAA \subseteq A$. By a left (right, lateral) ideal of a ternary semigroup T we mean a non-empty subset A of T such that $TTA \subseteq A$ $(ATT \subseteq A, TAT \subseteq A)$. If a non-empty subset A of T is a left and right ideal of T, then it is called a two sided ideal of T. If a non-empty subset A of a ternary semigroup T is a left, right and lateral ideal of T, then it is called an ideal of T. A non-empty subset A of a ternary semigroup T is called a quasi-ideal of T if $ATT \cap TAT \cap TTA \subseteq A$ and $ATT \cap TTATT \cap TTA \subseteq A$. A non-empty subset A of a ternary semigroup T is called a generalized bi-ideal of T if $ATATA \subseteq A$. A ternary subsemigroup of a ternary semigorup T is called bi-ideal if A is a generalized bi-ideal of T. It is clear that every left (right, lateral) ideal of T is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal of T.

A fuzzy subset f of a universe X is a function from X into the unit closed interval $[0,1]$, that is, $f : X \to [0,1]$.

A fuzzy subset f of a universe X of the form

$$f(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set f in a set X, Pu and Liu [12] gave meaning to the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set f written $x_t \in f$ (resp. $x_t q f$) if $f(x) \geq t$ (resp. $f(x) + t > 1$), and in this case, $x_t \in \vee q f$ (resp. $x_t \in \wedge q f$) means that $x_t \in f$ or $x_t q f$ (resp. $x_t \in f$ and $x_t q f$). By $x_t \overline{\alpha} f$ we mean that $x_t \alpha f$ does not hold.

For any two fuzzy subsets f and g of T, $f \leq g$ means that, for all $x \in T$, $f(x) \leq g(x)$. The symbols $f \wedge g$ and $f \vee g$ will mean the following fuzzy subsets of T.

$(f \wedge g)(x) = f(x) \wedge g(x)$

$(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in T$.

Let f, g and h be three fuzzy subsets of a ternary semigroup T. The product $f \circ g \circ h$ is a fuzzy subset of T defined by:

$$(f \circ g \circ h)(a) = \begin{cases} \bigvee_{a = xyz} \{f(x) \wedge g(y) \wedge h(z)\} & \text{if there exist } x, y, z \in T \text{ such that } a = xyz \\ 0 & \text{otherwise.} \end{cases}$$

Let f be a fuzzy subset of a ternary semigroup T. Then the set

$$U(f; t) = \{x \in T : f(x) \geq t\},$$

where $t \in [0,1]$, is called a level subset of f.

Definition 1. A fuzzy subset f of a ternary semigroup T is a fuzzy ternary subsemigroup of T if $f(xyz) \geq f(x) \wedge f(y) \wedge f(z)$ for all $x, y, z \in T$.

Definition 2. A fuzzy subset f of a ternary semigroup T is a fuzzy left (right, lateral) ideal of T if $f(xyz) \geq f(z)$ $(f(xyz) \geq f(x), f(xyz) \geq f(y))$ for all $x, y, z \in T$.

Definition 3. A fuzzy subset f of a ternary semigroup T is called a fuzzy generalized bi-ideal of T if $f(xuyvz) \geq f(x) \wedge f(y) \wedge f(z)$ for all $x, y, z, u, v \in T$, and is called a fuzzy bi-ideal of T if it is both a fuzzy ternary subsemigroup and a fuzzy generalized bi-ideal of T.

If $A \subseteq T$, then the characteristic function of A is a function C_A of T onto $\{0,1\}$ defined by:

$$C_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Let γ, $\delta \in [0,1]$ be such that $\gamma < \delta$. For a fuzzy point x_t and a fuzzy subset f of a ternary semigroup T, we say that:

(1) $x_t \in_\gamma f$ if $f(x) \geq t > \gamma$

(2) $x_t q_\delta f$ if $f(x) + t > 2\delta$

(3) $x_t \in_\gamma \vee q_\delta f$ if $x_t \in_\gamma f$ or $x_t q_\delta f$

(4) $x_t \in_\gamma \wedge q_\delta f$ if $x_t \in_\gamma f$ and $x_t q_\delta f$

(5) $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold.

3. (α, β)-fuzzy ideals

Throughout this paper $\gamma, \delta \in [0,1]$, where $\gamma < \delta$, α, $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ and $\alpha \neq \in_\gamma \wedge q_\delta$.

We omit the case of $\alpha = \in_\gamma \wedge q_\delta$, because if f is a fuzzy subset of a ternary semigroup T such that $f(x) \leq \delta$ for all $x \in T$ and $t \in [0,1]$ be such that $x_t \in_\gamma \wedge q_\delta f$, then $f(x) \geq t > \gamma$ and $f(x) + t > 2\delta$. This implies that $2\delta < f(x) + t \leq f(x) + f(x) = 2f(x)$.

$2\delta < f(x) + t \leq f(x) + f(x) = 2f(x)$ which implies $f(x) > \delta$. Thus $\{x_t : x_t \in_\gamma \wedge q_\delta f\} = \phi$.

Definition 4. A fuzzy subset f of a ternary semigroup T is called an (α, β)-fuzzy ternary subsemigroup of T if

$(T1)$ $x_t \alpha f$, $y_r \alpha f$ and $z_s \alpha f$ implies $(xyz)_{\min\{t,r,s\}} \beta f$ for all $x, y, z \in T$ and $t, r, s \in (\gamma, 1]$.

Definition 5. A fuzzy subset f of a ternary semigroup T is called an (α, β)-fuzzy left (right, lateral) ideal of T, if it satisfies:

$(T2)$ $z_t \alpha f$ implies $(xyz)_t \beta f$ $((zxy)_t \beta f, (xzy)_t \beta f)$ for all $x, y, z \in T$ and $t \in (\gamma, 1]$.

A fuzzy subset f of a ternary semigroup T is called an (α, β)-fuzzy two sided ideal of T if it is an (α, β)-fuzzy left ideal and an (α, β)-fuzzy right ideal of T. A fuzzy subset f of a ternary semigroup T is called an (α, β)-fuzzy ideal of T if it is an (α, β)-fuzzy left ideal, (α, β)-fuzzy right ideal and an (α, β)-fuzzy lateral ideal of T.

Definition 6. A fuzzy subset f of a ternary semigroup T is called an (α, β)-fuzzy generalized bi-ideal of T, if it satisfies:

$(T3)$ $x_t \alpha f$, $y_r \alpha f$ and $z_s \alpha f$ implies $(xuyvz)_{\min\{t,r,s\}} \beta f$ for all $x, y, z, u, v \in T$ and $t, r, s \in (\gamma, 1]$.

A fuzzy subset f of a ternary semigroup T is called an (α, β)-fuzzy bi-ideal of T, if it satisfies $(T1)$ and $(T3)$.

Theorem 1. Let $2\delta = 1+\gamma$ and f be an (α, β)-fuzzy ternary subsemigroup of a ternary semigroup T. Then the set $f_\gamma = \{x \in T : f(x) > \gamma\}$ is a ternary subsemigroup of T.

Proof: Suppose $x, y, z \in f_\gamma$. Then $f(x) > \gamma$, $f(y) > \gamma$ and $f(z) > \gamma$. Assume that $f(xyz) \leq \gamma$. If $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $x_{f(x)}\alpha f$, $y_{f(y)}\alpha f$, $z_{f(z)}\alpha f$ but $f(xyz) \leq \gamma < \min\{f(x), f(y), f(z)\}$ and $f(xyz) + \min\{f(x), f(y), f(z)\} \leq \gamma + 1 = 2\delta$. This implies that $(xyz)_{\min\{f(x), f(y), f(z)\}} \overline{\beta} f$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence $f(xyz) > \gamma$, that is, $xyz \in f_\gamma$. Also, $f(x)+1 > \gamma + 1 = 2\delta$, $f(y)+1 > \gamma+1 = 2\delta$ and $f(z)+1 > \gamma+1 = 2\delta$. This implies that $x_1 q_\delta f$, $y_1 q_\delta f$ and $z_1 q_\delta f$, but $f(xyz) \leq \gamma$ so $(xyz)_1 \overline{\in}_\gamma f$ and $f(xyz)+1 \leq \gamma+1 = 2\delta$, so $(xyz)_1 \overline{q}_\delta f$, a contradiction. Hence $f(xyz) > \gamma$, that is, $xyz \in f$. Therefore, f_γ is a ternary subsemigroup of T.

Theorem 2. Let $2\delta = 1+\gamma$ and f be an (α, β)-fuzzy left (right, lateral, two sided) ideal of a ternary semigroup T. Then the set $f_\gamma = \{x \in T : f(x) > \gamma\}$, is a left (right, lateral, two sided) ideal of T.

Proof: Let f be an (α, β)-fuzzy left ideal of T and $z \in f_\gamma$. Suppose there exist $x, y \in T$ such that $f(xyz) \leq \gamma$. Now, if $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $z_{f(z)}\alpha f$ but $f(xyz) \leq \gamma < f(z)$. This implies that $(xyz)_{f(z)} \overline{\in}_\gamma f$. Also, $f(xyz) + f(z) \leq \gamma + f(z) \leq \gamma + 1 = 2\delta$, so $(xyz)_{f(z)} \overline{q}_\delta f$. This implies that $(xyz)_{f(z)} \overline{\beta} f$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence $f(xyz) > \gamma$. This implies that $xyz \in f_\gamma$. If $\alpha = q_\delta$, then $z_1 q_\delta f$. But $f(xyz) \leq \gamma$, so $(xyz)_1 \overline{\in}_\gamma f$ and $f(xyz)+1 \leq \gamma+1 = 2\delta$. So $(xyz)_1 \overline{q}_\delta f$. Thus $(xyz)_1 \overline{\beta} f$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence $f(xyz) > \gamma$. This implies that $xyz \in f_\gamma$. Therefore, f_γ is a left ideal of T.

Theorem 3. (1) Let $2\delta = 1+\gamma$ and f be an (α, β)-fuzzy generalized bi-ideal of a ternary semigroup T. Then the set $f_\gamma = \{x \in T : f(x) > \gamma\}$ is a generalized bi-ideal of T.
(2) Let $2\delta = 1+\gamma$ and f be an (α, β)-fuzzy bi-ideal of T. Then the set $f_\gamma = \{x \in T : f(x) > \gamma\}$ is a bi-ideal of T.

Proof: The proof is similar to the proof of Theorem 1.

Theorem 4. Let $2\delta = 1+\gamma$ and A be a non-empty subset of a ternary semigroup T. Then A is a ternary subsemigroup of T if and only if the fuzzy subset f of T defined by:

$$f(x) = \begin{cases} \geq \delta & \text{if } x \in A \\ \leq \gamma & \text{if } x \notin A, \end{cases}$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T, where $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$.

Proof: Let A be a ternary subsemigroup of T.
(1) For $\alpha = \in_\gamma$. Let $x, y, z \in T$ and $t, r, s \in (\gamma, 1]$ be such that $x_t \in_\gamma f$, $y_r \in_\gamma f$ and $z_s \in_\gamma f$. Then $f(x) \geq t > \gamma > \delta$, $f(y) \geq r > \gamma > \delta$ and $f(z) \geq s > \gamma > \delta$. Thus $x, y, z \in A$. Since A is a ternary subsemigroup of T, we have $xyz \in A$, that is, $f(xyz) \geq \delta$.

If $\min\{t,r,s\} \le \delta$, then $f(xyz) \ge \delta \ge \min\{t,r,s\} > \gamma$, which implies that $(xyz)_{\min\{t,r,s\}} \in_\gamma f$.

If $\min\{t,r,s\} > \delta$, then $f(xyz) + \min\{t,r,s\} > \delta + \delta = 2\delta$, which implies that $(xyz)_{\min\{t,r,s\}} q_\delta f$. Hence f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T.

(2) For $\alpha = q_\delta$. Let $x, y, z \in T$ and $t, r, s \in (\gamma, 1]$ be such that $x_t q_\delta f$, $y_r q_\delta f$ and $z_s q_\delta f$. Then $f(x) + t > 2\delta$, $f(y) + r > 2\delta$ and $f(z) + s > 2\delta$, which implies that $f(x) > 2\delta - t \ge 2\delta - 1 = \gamma$, also $f(y) > 2\delta - r \ge 2\delta - 1 = \gamma$ and $f(z) > 2\delta - s \ge 2\delta - 1 = \gamma.$ Thus $x, y, z \in A$ and so $xyz \in A$, which implies that $f(xyz) \ge \delta$.

Now, if, $\min\{t,r,s\} \le \delta$ then $f(xyz) \ge \delta \ge \min\{t,r,s\} > \gamma$, which implies that $(xyz)_{\min\{t,r,s\}} \in_\gamma f$.

If $\min\{t,r,s\} > \delta$, then $f(xyz) + \min\{t,r,s\} > \delta + \delta = 2\delta$, which implies that $(xyz)_{\min\{t,r,s\}} q_\delta f$. Hence f is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T.

(3) For $\alpha = \in_\gamma \vee q_\delta$. Let $x, y, z \in T$ and $t, r, s \in (\gamma, 1]$ be such that $x_t \in_\gamma \vee q_\delta f$, $y_r \in_\gamma \vee q_\delta f$ and $z_s \in_\gamma \vee q_\delta f$, which implies that $x_t \in_\gamma f$ or $x_t q_\delta f$, $y_r \in_\gamma f$ or $y_r q_\delta f$ and $z_s \in_\gamma f$ or $z_s q_\delta f$.

If $x_t \in_\gamma f$, $y_r \in_\gamma f$ and $z_s q_\delta f$, then $f(x) \ge t > \gamma$, $f(y) \ge r > \gamma$ and $f(z) + s > 2\delta$ implies $f(z) > 2\delta - s \ge 2\delta - 1 = \gamma$, which implies that $x, y, z \in A$ and so $xyz \in A$. Analogous to (1) and (2) we obtain $(xyz)_{\min\{t,r,s\}} \in_\gamma \vee q_\delta f$. Hence f is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T. The other cases can be considered similar to this case.

Conversely, assume that f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T, for $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$. Then $A = f$. Thus by Theorem 1, A is a ternary subsemigroup of T.

Corollary 1. Let $2\delta = 1 + \gamma$ and $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$. Then a non-empty subset A of a ternary semigroup T is a ternary subsemigroup of T if and only if the characteristic function C_A of A is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T.

In a similar manner we can prove the following:

Theorem 5. Let $2\delta = 1 + \gamma$ and $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$, and A be a non-empty subset of T. Define a fuzzy subset f of T by:

$$f(x) = \begin{cases} \ge \delta & \text{if } x \in A \\ \le \gamma & \text{if } x \notin A. \end{cases}$$

Then

(1) f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral, two sided) ideal of T if and only if A is a left (right, lateral, two sided) ideal of T.

(2) f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal (generalized bi-ideal) of T if and only if A is a bi-ideal (generalized bi-ideal) of T.

Corollary 2. Let $2\delta = 1 + \gamma$ and $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$. If A is a non-empty subset of T, then

(1) A is a left (right, lateral) ideal of T if and only if the characteristic function C_A of A is a left (right, lateral) ideal of T.

(2) A is a bi-ideal (generalized bi-ideal) of T if and only if the characteristic function C_A of A is a bi-ideal (generalized bi-ideal) of T.

Theorem 6. (1) Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T.

(2) Every $\left(q_{\delta},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy left (right, lateral, two sided) ideal of T is an $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy left (right, lateral, two sided) ideal of T.

(3) Every $\left(q_{\delta},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy bi-ideal (generalized bi-ideal) of T is an $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy bi-ideal (generalized bi-ideal) of T.

Proof: We prove (1). The proofs of (2) and (3) are similar to (1).

(1) Let f be a $\left(q_{\delta},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroup of T. Let $x,y,z\in T$ and $t,r,s\in(\gamma,1]$ be such that $x_t\in_{\gamma} f$, $y_r\in_{\gamma} f$ and $z_s\in_{\gamma} f$. Then $f(x)\geq t>\gamma$, $f(y)\geq r>\gamma$ and $f(z)\geq s>\gamma$. Suppose $(xyz)_{\min\{t,r,s\}}\overline{\in_{\gamma}\vee q_{\delta}}f$. Then $f(xyz)<\min\{t,r,s\}$ and $f(xyz)+\min\{t,r,s\}\leq 2\delta$. This implies that $f(xyz)+f(xyz)<f(xyz)+\min\{t,r,s\}\leq 2\delta$. This implies that $f(xyz)<\delta$. Now, $\max\{f(xyz),\gamma\}<\min\{f(x),f(y),f(z),\delta\}$. Choose $t_1\in(\gamma,1]$ such that $2\delta-\max\{f(xyz),\gamma\}\geq t_1>2\delta-\min\{f(x),f(y),f(z),\delta\}$, that is

$$2\delta-\max\{f(xyz),\gamma\}=\min\{2\delta-f(xyz),2\delta-\gamma\}$$
$$\geq t_1>\max\{2\delta-f(x),2\delta-f(y),2\delta-f(z),\delta\}.$$

This implies that

$$t_1>2\delta-f(x),t_1>2\delta-f(y),t_1>2\delta-f(z).$$

This implies that

$$f(x)+t_1>2\delta,\ f(y)+t_1>2\delta,\ f(z)+t_1>2\delta$$

and $2\delta-f(xyz)>t_1$. This implies that $f(xyz)+t_1<2\delta$ and $f(xyz)<\delta<t_1$. Thus $x_{t_1}q_{\delta}f, y_{t_1}q_{\delta}f$ and $z_{t_1}q_{\delta}f$ but $(xyz)_{t_1}\overline{\in_{\gamma}\vee q_{\delta}}f$, which is a contradiction. Hence f is an $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroup of S.

Theorem 7. (1) Every $\left(\in_{\gamma}\vee q_{\delta},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroup of a ternary semigroup T is an $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroup of T.

(2) Every $\left(\in_{\gamma}\vee q_{\delta},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy left (right, lateral, two sided) ideal of a ternary semigroup T is an $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy left (right, lateral, two sided) ideal of T.

(3) Every $\left(\in_{\gamma}\vee q_{\delta},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy bi-ideal (generalized bi-ideal) of a ternary semigroup T is an $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy bi-ideal (generalized bi-ideal) of T.

Proof: The proof follows from the fact that if $x_t\in_{\gamma} f$, then $x_t\in_{\gamma}\vee q_{\delta}f$.

The above discussion shows that every (α,β)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of a ternary semigroup T is an $\left(\alpha,\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of T. Also, every $\left(\alpha,\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of T is an $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of T. Thus in the theory of (α,β)-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of T $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroup (left ideal, right ideal, lateral ideal, two sided ideal, bi-ideal, generalized bi-ideal) of T plays a central role.

4. $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ideals

In this section we introduce the concepts of $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy ternary subsemigroups, $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy left (right, lateral) ideals, $\left(\in_{\gamma},\in_{\gamma}\vee q_{\delta}\right)$-fuzzy bi-ideal (generalized bi-ideal) of ternary semigroups and investigate some new results.

Definition 7. A fuzzy subset f of a ternary semigroup T is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup if x_t, y_r, $z_s \in_\gamma f$ implies $(xyz)_{\min\{t,r,s\}} \in_\gamma \vee q_\delta f$ for all $x, y, z \in T$ and $t, r, s \in (\gamma, 1]$.

Definition 8. A fuzzy subset f of a ternary semigroup T is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideal of T if it satisfies:
$z_t \in_\gamma f$ implies $(xyz)_t \in_\gamma \vee q_\delta f$ $((zxy)_t \in_\gamma \vee q_\delta f, (xzy)_t \in_\gamma \vee q_\delta f)$ for all $x, y, z \in T$ and $t \in (\gamma, 1]$.

Definition 9. A fuzzy subset f of a ternary semigroup T is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of T if x_t, y_r, $z_s \in_\gamma f$ implies $(xuyvz)_{\min\{t,r,s\}} \in_\gamma \vee q_\delta f$ for all $x, y, z, u, v \in T$ and $t, r, s \in (\gamma, 1]$.

A fuzzy subset f of a ternary semigroup T is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of T if it is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of T.

Theorem 8. A fuzzy subset f of a ternary semigroup T is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T if and only if $\max\{f(xyz), \gamma\} \geq \min\{f(x), f(y), f(z), \delta\}$.

Proof: Suppose f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of a ternary semigroup T. Assume that
$\max\{f(xyz), \gamma\} < \min\{f(x), f(y), f(z), \delta\}$.
Choose $t \in (\gamma, 1]$ such that
$\max\{f(xyz), \gamma\} < t \leq \min\{f(x), f(y), f(z), \delta\}$.
Then $x_t \in_\gamma f$, $y_t \in_\gamma f$ and $z_t \in_\gamma f$, but $(xyz)_t \overline{\in_\gamma} f$ and $f(xyz) + t < \delta + \delta = 2\delta$. This implies that $(xyz)_t \overline{\in_\gamma \vee q_\delta} f$, which is a contradiction. Hence $\max\{f(xyz), \gamma\} \geq \min\{f(x), f(y), f(z), \delta\}$

Conversely, assume that

$$\max\{f(xyz), \gamma\} \geq \min\{f(x), f(y), f(z), \delta\}.$$

Let $x_t \in_\gamma f$, $y_t \in_\gamma f$ and $z_t \in_\gamma f$. This implies that $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$ and $f(z) \geq t > \gamma$ Since

$$\max\{f(xyz), \gamma\} \geq \min\{f(x), f(y), f(z), \delta\} \geq \min\{t, t, t, \delta\} = \min\{t, \delta\}.$$

If $t \leq \delta$, then $\max\{f(xyz), \gamma\} \geq t$. But $t > \gamma$. So $f(xyz) \geq t > \gamma$. Thus $(xyz)_t \in_\gamma f$.

If $t > \delta$, then $\max\{f(xyz), \gamma\} \geq \delta$. But $\gamma < \delta$. So $f(xyz) \geq \delta$. This implies that $f(xyz) + t > \delta + \delta = 2\delta$. Thus $(xyz)_t q_\delta f$. Therefore, f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T.

If we put $\gamma = 0$ and $\delta = 0.5$ in Theorem 8, we get the following corollary.

Corollary 3. [27] A fuzzy subset f of a ternary semigroup T is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of T if and only if $f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\}$ for all $x, y, z \in T$.

Remark 1. Every fuzzy ternary subsemigroup of a ternary semigroup T is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T but the converse is not true in general.

Example 1. Consider $T = \{-i, 0, i\}$, where T is a ternary semigroup under the usual multiplication of complex numbers. Define a fuzzy subset f of T by:

$$f(-i) = 0.43, \quad f(i) = 0.35, \quad f(0) = 0.6.$$

Then routine calculations show that f is an $(\in_{0.5}, \in_{0.5} \vee q_{0.7})$-fuzzy ternary subsemigroup of T but not a fuzzy ternary subsemigroup of T.

Theorem 9. A fuzzy subset f of a ternary semigroup T is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideal of T if and only if

$$\max\{f(xyz),\gamma\} \geq \min\{f(z),\delta\}\left(\min\{f(x),\delta\},\min\{f(y),\delta\}\right).$$

Proof: The proof is similar to the proof of Theorem 8.

If we put $\gamma = 0$ and $\delta = 0.5$ in Theorem 9, we get the following corollary.

Corollary 4. [27] A fuzzy subset f of a ternary semigroup T is an $(\in, \in \vee q)$-fuzzy left (right, lateral) ideal of T if and only if

$$f(xyz) \geq f(z) \wedge 0.5\left(f(x) \wedge 0.5, f(y) \wedge 0.5\right)$$

for all $x, y, z \in T$.

Example 2. Let T be a ternary semigroup as defined in Example 1. Define a fuzzy subset f of T by:

$$f(-i) = 0.40, \quad f(i) = 0.52, \quad f(0) = 0.31.$$

It is now simple to verify that f is an $(\in_{0.6}, \in_{0.6} \vee q_1)$-fuzzy ideal of T.

Lemma 1. The intersection of any number of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideals of a ternary semigroup T is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideal of T.

Proof: Let $\{f_i\}_{i\in I}$ be a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of T and $x, y, z \in T$. Then

$$\left(\underset{i\in I}{\wedge} f_i\right)(xyz) \vee \gamma = \underset{i\in I}{\wedge} f_i(xyz) \vee \gamma$$
$$\geq \underset{i\in I}{\wedge}\left(f_i(z) \wedge \delta\right) = \left(\underset{i\in I}{\wedge} f_i(z)\right) \wedge \delta = \left(\underset{i\in I}{\wedge} f_i\right)(z) \wedge \delta.$$

This implies that $\left(\underset{i\in I}{\wedge} f_i\right)(xyz) \vee \gamma \geq \left(\underset{i\in I}{\wedge} f_i\right)(z) \wedge \delta$.

Hence $\left(\underset{i\in I}{\wedge} f_i\right)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of T.

In a similar manner we can prove the following lemma.

Lemma 2. The union of any number of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideals of a ternary semigroup T is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideal of T.

Definition 10. Let f be a fuzzy subset of a ternary semigroup T. We define:

$$f_t = \{x \in T : x_t \in_\gamma f\} = \{x \in T : f(x) \geq t > \gamma\} = U(f;t),$$

$$f_t^\delta = \{x \in T : x_t q_\delta f\} = \{x \in T : f(x) + t > 2\delta\},$$

$$[f]_t^\delta = \{x \in T : x_t \in_\gamma \vee q_\delta f\}, \quad \text{for all } t \in (\gamma, 1].$$

Clearly, $[f]_t^\delta = f_t \cup f_t^\delta$.

The next theorem provides the relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup and the crisp ternary subsemigroup of T.

Theorem 10. Let f be a fuzzy subset of a ternary semigroup T. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T if and only if $f_t (\neq \phi)$ is a ternary subsemigroup of T for all $t \in (\gamma, \delta]$.

Proof: Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T and $x, y, z \in f_t$ for some $t \in (\gamma, \delta]$. Then $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, $f(z) \geq t > \gamma$. Now, by hypothesis $\max\{f(xyz),\gamma\} \geq \min\{f(x), f(y), f(z), \delta\} \geq \min\{t,t,t,\delta\} = t$. This implies that $\max\{f(xyz),\gamma\} \geq t$. But $t > \gamma$, so $f(xyz) \geq t$. This implies that $xyz \in f_t$. Thus f_t is a ternary subsemigroup of T.

Conversely, assume that $\phi \neq f_t$ is a ternary subsemigroup of T for all $t \in (\gamma, \delta]$. Suppose there exist $x, y, z \in T$ such that $\max\{f(xyz), \gamma\} < \min\{f(x), f(y), f(z), \delta\}$. Choose $t \in (\gamma, \delta]$ such that $\max\{f(xyz), \gamma\} < t \le \min\{f(x), f(y), f(z), \delta\}$. Then $x, y, z \in f_t$, but $xyz \notin f_t$, which is a contradiction. Hence $\max\{f(xyz), \gamma\} \ge \min\{f(x), f(y), f(z), \delta\}$. Hence, f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T.

In a similar manner we can prove the following theorems.

Theorem 11. A fuzzy subset f of a ternary semigroup T is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideal of T if and only if $f_t(\neq \phi)$ is a left (right, lateral) ideal of T for all $t \in (\gamma, \delta]$.

Theorem 12. A fuzzy subset f of a ternary semigroup T is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal (generalized bi-ideal) of T if and only if $f_t(\neq \phi)$ is a bi-ideal (generalized bi-ideal) of T for all $t \in (\gamma, \delta]$.

Theorem 13. Let $2\delta = 1 + \gamma$ and f be a fuzzy subset of a ternary semigroup T. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T if and only if $f_t^\delta(\neq \phi)$ is a ternary subsemigroup of T for all $t \in (\delta, 1]$.

Proof: Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T and $x, y, z \in f_t^\delta$. Then $x_t q_\delta f_t^\delta$, $y_t q_\delta f_t^\delta$, $z_t q_\delta f_t^\delta$. This implies that $f(x) + t > 2\delta$, $f(y) + t > 2\delta$, $f(z) + t > 2\delta$. This implies that $f(x) > 2\delta - t \ge 2\delta - 1 = \gamma$. This implies that $f(x) > \gamma$. Similarly, $f(y) > \gamma$ and $f(z) > \gamma$. Now, by hypothesis $\max\{f(xyz), \gamma\} \ge \min\{f(x), f(y), f(z), \delta\} > \min\{2\delta - t, 2\delta - t, 2\delta - t, \delta\} = 2\delta - t.$ This implies that $f(xyz) > 2\delta - t$. This implies that $f(xyz) + t > 2\delta$. This implies that $xyz \in f_t^\delta$. Hence f_t^δ is a ternary subsemigroup of T.

Conversely, assume that $\phi \ne f_t^\delta$ is a ternary subsemigroup of T for all $t \in (\delta, 1]$. Let $x, y, z \in T$ be such that $\max\{f(xyz), \gamma\} < \min\{f(x), f(y), f(z), \delta\}$. This implies that $2\delta - \min\{f(x), f(y), f(z), \delta\} < 2\delta - \max\{f(xyz), \gamma\}$.
$\max\{2\delta - f(x), 2\delta - f(y), 2\delta - f(z), \delta\} < \min\{2\delta - f(xyz), 2\delta - \gamma\}$.
Take $r \in (\delta, 1]$ such that $\max\{2\delta - f(x), 2\delta - f(y), 2\delta - f(z), \delta\} < r \le \min\{2\delta - f(xyz), 2\delta - \gamma\}$.
Then $2\delta - f(x) < r$, $2\delta - f(y) < r$, $2\delta - f(z) < r$, and $r \le 2\delta - f(xyz)$. This implies that $f(x) + r > 2\delta$, $f(y) + r > 2\delta$, $f(z) + r > 2\delta$ and $f(xyz) + r \le 2\delta$. This implies that $x_r q_\delta f$, $y_r q_\delta f$, $z_r q_\delta f$ but $(xyz)_r \overline{q_\delta} f$, which is a contradiction. Thus $\max\{f(xyz), \gamma\} \ge \min\{f(x), f(y), f(z), \delta\}$. Hence f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T.

In a similar manner we can prove the following theorems.

Theorem 14. Let $2\delta = 1 + \gamma$ and f be a fuzzy subset of a ternary semigroup T. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideal of T if and only if $f_t^\delta(\neq \phi)$ is a left (right, lateral) ideal of T for all $t \in (\delta, 1]$.

Theorem 15. Let $2\delta = 1 + \gamma$ and f be a fuzzy subset of a ternary semigroup T. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal (generalized bi-ideal) of T if and only if $f_t^\delta(\neq \phi)$ is a bi-ideal (generalized bi-ideal) of T for all $t \in (\delta, 1]$.

Theorem 16. Let $2\delta = 1 + \gamma$ and f be a fuzzy subset of a ternary semigroup T. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of T

if and only if $[f]_t^{\delta}(\neq\phi)$ is a ternary subsemigroup of T for all $t\in(\gamma,1]$.

Proof: Let f be an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy ternary subsemigroup of T and $x,y,z\in[f]_t^{\delta}$. Then $x_t\in_{\gamma}\vee q_{\delta}f$, $y_t\in_{\gamma}\vee q_{\delta}f$ and $z_t\in_{\gamma}\vee q_{\delta}f$. This implies that $f(x)\geq t>\gamma$ or $f(x)+t>2\delta$, $f(y)\geq t>\gamma$ or $f(y)+t>2\delta$, and $f(z)\geq t>\gamma$ or $f(z)+t>2\delta$. Thus $f(x)\geq t>\gamma$ or $f(x)>2\delta-t\geq 2\delta-1=\gamma$. Similarly, $f(y)\geq t>\gamma$ or $f(y)>2\delta-t\geq 2\delta-1=\gamma$ and $f(z)\geq t>\gamma$ or $f(z)>2\delta-t\geq 2\delta-1=\gamma$.

If $t\in(\gamma,\delta]$, then $2\delta-t\geq\delta\geq t$. By hypothesis

$$\max\{f(xyz),\gamma\}\geq\min\{f(x),f(y),f(z),\delta\}\geq\min\{t,t,t,\delta\}=t.$$

This implies that $f(xyz)\geq t>\gamma$. This implies that $(xyz)_t\in_{\gamma}f$. Thus $xyz\in[f]_t^{\delta}$.

If $t\in(\delta,1]$, then $2\delta-t<\delta<t$ and so $f(x)>2\delta-t$, $f(y)>2\delta-t$ and $f(z)>2\delta-t$. Again by hypothesis

$$\max\{f(xyz),\gamma\}\geq\min\{f(x),f(y),f(z),\delta\}.$$

This implies that

$$f(xyz)\geq\min\{f(x),f(y),f(z),\delta\}>\min\{2\delta-t,2\delta-t,2\delta-t,\delta\}=2\delta-t.$$

Thus $f(xyz)+t>2\delta$. So $(xyz)_t\,q_{\delta}f$. This implies that $xyz\in[f]_t^{\delta}$. Therefore, $[f]_t^{\delta}$ is a ternary subsemigroup of T.

Conversely, assume that $[f]_t^{\delta}$ is ternary subsemigroup of T for all $t\in(\gamma,1]$. Let $x,y,z\in T$ be such that $\max\{f(xyz),\gamma\}<\min\{f(x),f(y),f(z),\delta\}$. Choose $t\in(\gamma,1]$ such that $\max\{f(xyz),\gamma\}<t\leq\min\{f(x),f(y),f(z),\delta\}$. Then $x_t\in_{\gamma}f$, $y_t\in_{\gamma}f$ and $z_t\in_{\gamma}f$, but $(xyz)_t\overline{\in_{\gamma}\vee q_{\delta}}f$, which is a contradiction. Hence f is an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy ternary subsemigroup of T.

In a similar manner we can prove the following theorems.

Theorem 17. Let $2\delta=1+\gamma$ and f be a fuzzy subset of a ternary semigroup T. Then f is an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy left (right, lateral) ideal of T if and only if $[f]_t^{\delta}(\neq\phi)$ is a left (right, lateral) ideal of T for all $t\in(\gamma,1]$.

Theorem 18. Let $2\delta=1+\gamma$ and f be a fuzzy subset of a ternary semigroup T. Then f is an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy bi-ideal (generalized bi-ideal) of T if and only if $[f]_t^{\delta}(\neq\phi)$ is a bi-ideal (generalized bi-ideal) of T for all $t\in(\gamma,1]$.

Remark 2. Every $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy ideal of a ternary semigroup T is an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy ternary subsemigroup of T but the converse is not true in general.

Example 3. Consider the ternary semigroup T as in Example 1. Define a fuzzy subset f of T by: $f(-i)=0.60,\quad f(0)=0.40,\quad f(i)=0.87$. Then

$$U(f;t)=\begin{cases} T & \text{if } 0\leq t<0.48\\ \{-i,i\} & \text{if } 0.48\leq t<0.59\\ \{-i\} & \text{if } 0.59\leq t<0.86\\ \phi & \text{if } 0.86\leq t<1.\end{cases}$$

It is now routine to verify that f is an $(\in_{0.48},\in_{0.48}\vee q_{0.59})$-fuzzy ternary subsemigroup but not an $(\in_{0.48},\in_{0.48}\vee q_{0.59})$-fuzzy ideal of T.

Definition 11. A fuzzy subset f of a ternary semigroup T is called an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy quasi-ideal of T if

$$\max\{f(x),\gamma\}\geq\min\{(f\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ f\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ f)(x),\delta\}$$

and

$$\max\{f(x),\gamma\}\geq\min\{(f\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ f\circ\mathbf{T}\circ\mathbf{T})(x),(\mathbf{T}\circ\mathbf{T}\circ f)(x),\delta\}$$

for all $x \in T$, where $\mathbf{T}$ is the fuzzy subset of T mapping every element of T on 1.

Lemma 3. A non-empty subset Q of a ternary semigroup T is a quasi-ideal of T if and only if the characteristic function C_Q of Q is an $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy quasi-ideal of T

Proof: Suppose Q is a quasi-ideal of T and C_Q, the characteristic function of Q. Let $x \in T$. If $x \notin Q$, then $x \notin TTQ$ or $x \notin TQT$ or $x \notin QTT$. If $x \notin TTQ$ or $x \notin TQT$ or $x \notin QTT$, then $\left(\mathbf{T} \circ \mathbf{T} \circ C_Q\right)(x) = 0$ or $\left(\mathbf{T} \circ C_Q \circ \mathbf{T}\right)(x) = 0$ or $\left(C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(x) = 0$ and therefore,

$$\min\left\{\left(\mathbf{T} \circ \mathbf{T} \circ C_Q\right)(x), \left(\mathbf{T} \circ C_Q \circ \mathbf{T}\right)(x), \left(C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(x), \delta\right\} = 0 \le \max\left\{C_Q(x), \gamma\right\}.$$

If $x \in Q$, then

$$\max\left\{C_Q(x), \gamma\right\} = 1 \ge \min\left\{\begin{array}{c}\left(\mathbf{T} \circ \mathbf{T} \circ C_Q\right)(x), \left(\mathbf{T} \circ C_Q \circ \mathbf{T}\right)(x), \\ \left(C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(x), \delta\end{array}\right\}$$

Similarly,

$$\max\left\{C_Q(x), \gamma\right\} = 1 \ge \min\left\{\begin{array}{c}\left(\mathbf{T} \circ \mathbf{T} \circ C_Q\right)(x), \left(\mathbf{T} \circ \mathbf{T} \circ C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(x), \\ \left(C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(x), \delta\end{array}\right\}.$$

Hence, C_Q is an $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy quasi-ideal of T.

Conversely, assume that C_Q is an $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy quasi-ideal of T. We show that Q is a quasi-ideal of T. Let $a \in TTQ \cap TQT \cap QTT$. Then $a \in TTQ$ and $a \in TQT$ and $a \in QTT$. This implies that there exist $x, y, z \in Q$ and $s_1, s_2, s_3, t_1, t_2, t_3 \in T$ such that $a = s_1 t_1 x$ and $a = s_2 y t_2$ and $a = z s_3 t_3$. Thus,

$$\begin{aligned}\left(C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(a) &= \bigvee_{a = pqr} \left\{C_Q(p) \wedge \mathbf{T}(q) \wedge \mathbf{T}(r)\right\} \\ &\ge C_Q(z) \wedge \mathbf{T}(s_3) \wedge \mathbf{T}(t_3) \\ &= C_Q(z) \wedge 1 \wedge 1 = C_Q(z) = 1.\end{aligned}$$

So $\left(C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(a) = 1$. Similarly, $\left(\mathbf{T} \circ \mathbf{T} \circ C_Q\right)(a) = 1$ and $\left(\mathbf{T} \circ C_Q \circ \mathbf{T}\right)(a) = 1$. Now,

$$\begin{aligned}\max\left\{C_Q(a), \gamma\right\} &\ge \min\left\{\left(\mathbf{T} \circ \mathbf{T} \circ C_Q\right)(a), \left(\mathbf{T} \circ \mathbf{T} \circ C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(a), \left(C_Q \circ \mathbf{T} \circ \mathbf{T}\right)(a), \delta\right\} \\ &= \delta.\end{aligned}$$

Thus $C_Q(a) = 1$. This implies that $a \in Q$. Therefore, $TTQ \cap TQT \cap QTT \subseteq Q$. Similarly, we can show that $TTQ \cap TTQTT \cap QTT \subseteq Q$. Hence, Q is a quasi-ideal of T.

Theorem 19. Every $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy right (left, lateral) ideal of a ternary semigroup T is an $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy quasi-ideal of T.

Proof: Let f be an $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy right ideal of T and $x \in T$. Then

$$\begin{aligned}\left(f \circ \mathbf{T} \circ \mathbf{T}\right)(x) &= \bigvee_{x = uvw} \left(f(u) \wedge \mathbf{T}(v) \wedge \mathbf{T}(w)\right) \\ &= \bigvee_{x = uvw} f(u).\end{aligned}$$

This implies that

$$\begin{aligned}\left(f \circ \mathbf{T} \circ \mathbf{T}\right)(x) \wedge \delta &= \left(\bigvee_{x = uvw} f(u)\right) \wedge \delta = \bigvee_{x = uvw} \left(f(u) \wedge \delta\right) \\ &\le \bigvee_{x = uvw} f(uvw) \vee \gamma = f(x) \vee \gamma.\end{aligned}$$

This implies that $f(x) \vee \gamma \ge \left(f \circ \mathbf{T} \circ \mathbf{T}\right)(x) \wedge \delta$. This implies that

$$\max\left\{f(x), \gamma\right\} \ge \min\left\{\left(f \circ \mathbf{T} \circ \mathbf{T}\right)(x), \left(\mathbf{T} \circ f \circ \mathbf{T}\right)(x), \left(\mathbf{T} \circ \mathbf{T} \circ f\right)(x), \delta\right\}.$$

Similarly,

$$\max\left\{f(x), \gamma\right\} \ge \min\left\{\left(f \circ \mathbf{T} \circ \mathbf{T}\right)(x), \left(\mathbf{T} \circ \mathbf{T} \circ f \circ \mathbf{T} \circ \mathbf{T}\right)(x), \left(\mathbf{T} \circ \mathbf{T} \circ f\right)(x), \delta\right\}.$$

Hence f is an $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy quasi-ideal of T.

Similarly, we can prove the case of left and lateral ideal of T.

The converse of Theorem 19 does not hold in general, as shown in the following example.

Example 4. Let

$$T = \left\{\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & -1\end{pmatrix}\right\},$$

where T is a ternary semigroup under ternary matrix multiplication. Let f be a fuzzy subset of T defined by:

$$f(x)=\begin{cases} 0.7 \text{ if } x=\begin{pmatrix}0&0\\0&0\end{pmatrix}, \\ 0.5 \text{ if } x=\begin{pmatrix}0&-1\\0&0\end{pmatrix}, \\ 0.4 \text{ if } x=\begin{pmatrix}-1&0\\0&0\end{pmatrix}, \\ 0.3 \text{ if } x=\begin{pmatrix}0&0\\-1&0\end{pmatrix}, \\ 0.2 \text{ if } x=\begin{pmatrix}0&0\\0&-1\end{pmatrix}. \end{cases}$$

Then we have the following:

$$U(f;t)=\begin{cases} T & \text{if } t\le 0.2 \\ \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&-1\\0&0\end{pmatrix},\begin{pmatrix}-1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\-1&0\end{pmatrix}\right\} & \text{if } 0.2<t\le 0.3 \\ \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&-1\\0&0\end{pmatrix},\begin{pmatrix}-1&0\\0&0\end{pmatrix}\right\} & \text{if } 0.3<t\le 0.4 \\ \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&-1\\0&0\end{pmatrix}\right\} & \text{if } 0.4<t\le 0.5 \\ \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\} & \text{if } 0.5<t\le 0.7 \\ \phi & \text{if } 0.7<t. \end{cases}$$

This shows that f is an $(\in_{0.4},\in_{0.4}\vee q_{0.7})$-fuzzy quasi-ideal, but neither $(\in_{0.4},\in_{0.4}\vee q_{0.7})$-fuzzy left nor $(\in_{0.4},\in_{0.4}\vee q_{0.7})$-fuzzy right, nor $(\in_{0.4},\in_{0.4}\vee q_{0.7})$-fuzzy lateral ideal of T.

Theorem 20. Every $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy quasi-ideal of a ternary semigroup T is an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy bi-ideal of T.

Proof: Let f be an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy quasi-ideal of T and $u,v,x,y,z\in T$. Then

$$\max\{f(xyz),\gamma\}\ge\min\{(f\circ\mathbf{T}\circ\mathbf{T})(xyz),(\mathbf{T}\circ f\circ\mathbf{T})(xyz),(\mathbf{T}\circ\mathbf{T}\circ f)(xyz),\delta\}$$

$$\begin{aligned}\max\{f(xyz),\gamma\}&\ge(f\circ\mathbf{T}\circ\mathbf{T})(xyz)\wedge(\mathbf{T}\circ f\circ\mathbf{T})(xyz)\wedge(\mathbf{T}\circ\mathbf{T}\circ f)(xyz)\wedge\delta\\ &=\left[\bigvee_{xyz=uvw}\{f(u)\wedge\mathbf{T}(v)\wedge\mathbf{T}(w)\}\right]\wedge\left[\bigvee_{xyz=pqr}\{\mathbf{T}(p)\wedge f(q)\wedge\mathbf{T}(r)\}\right]\\ &\quad\wedge\left[\bigvee_{xyz=abc}\{\mathbf{T}(a)\wedge\mathbf{T}(b)\wedge f(c)\}\right]\wedge\delta\\ &\ge\{f(x)\wedge\mathbf{T}(y)\wedge\mathbf{T}(z)\}\wedge\{\mathbf{T}(x)\wedge f(y)\wedge\mathbf{T}(z)\}\\ &\quad\wedge\{\mathbf{T}(x)\wedge\mathbf{T}(y)\wedge f(z)\}\wedge\delta\\ &=f(x)\wedge f(y)\wedge f(z)\wedge\delta.\end{aligned}$$

This implies that

$$\max\{f(xyz),\gamma\}\ge\min\{f(x),f(y),f(z),\delta\}$$

also,

$$\{f(xuyvz),\gamma\}\ge\min\left\{\begin{matrix}(f\circ\mathbf{T}\circ\mathbf{T})(xuyvz),(\mathbf{T}\circ\mathbf{T}\circ f\circ\mathbf{T}\circ\mathbf{T})(xuyvz),\\(\mathbf{T}\circ\mathbf{T}\circ f)(xuyvz),\delta\end{matrix}\right\}$$

$$\begin{aligned}\{f(xuyvz),\gamma\}&\ge(f\circ\mathbf{T}\circ\mathbf{T})(xuyvz)\wedge(\mathbf{T}\circ\mathbf{T}\circ f\circ\mathbf{T}\circ\mathbf{T})(xuyvz)\\ &\quad\wedge(\mathbf{T}\circ\mathbf{T}\circ f)(xuyvz)\wedge\delta\\ &=\left[\bigvee_{xuyvz=s_1s_2s_3}\{f(s_1)\wedge\mathbf{T}(s_2)\wedge\mathbf{T}(s_3)\}\right]\\ &\quad\wedge\left[\bigvee_{xuyvz=s_4s_5s_6}\{(\mathbf{T}\circ\mathbf{T}\circ f)(s_4)\wedge\mathbf{T}(s_5)\wedge\mathbf{T}(s_6)\}\right]\\ &\quad\wedge\left[\bigvee_{xuyvz=r_1r_2r_3}\{\mathbf{T}(r_1)\wedge\mathbf{T}(r_2)\wedge f(r_3)\}\right]\wedge\delta\end{aligned}$$

$$\begin{aligned}&=\left[\bigvee_{xuyvz=s_1s_2s_3}\{f(s_1)\wedge\mathbf{T}(s_2)\wedge\mathbf{T}(s_3)\}\right]\\ &\wedge\left[\bigvee_{xuyvz=s_4s_5s_6}\{\bigvee_{s_4=abc}(\mathbf{T}(a)\wedge\mathbf{T}(b)\wedge f(c))\wedge\mathbf{T}(s_5)\wedge\mathbf{T}(s_6)\}\right]\\ &\wedge\left[\bigvee_{xuyvz=r_1r_2r_3}\{\mathbf{T}(r_1)\wedge\mathbf{T}(r_2)\wedge f(r_3)\}\right]\wedge\delta\\ &=\left[\bigvee_{xuyvz=s_1s_2s_3}\{f(s_1)\wedge\mathbf{T}(s_2)\wedge\mathbf{T}(s_3)\}\right]\\ &\wedge\left[\bigvee_{xuyvz=(abc)s_5s_6}(\mathbf{T}(a)\wedge\mathbf{T}(b)\wedge f(c))\wedge\mathbf{T}(s_5)\wedge\mathbf{T}(s_6)\right]\\ &\wedge\left[\bigvee_{xuyvz=r_1r_2r_3}\{\mathbf{T}(r_1)\wedge\mathbf{T}(r_2)\wedge f(r_3)\}\right]\wedge\delta\end{aligned}$$

$$\begin{aligned}&\{f(x)\wedge\mathbf{T}(uyv)\wedge\mathbf{T}(z)\}\wedge\{\mathbf{T}(x)\wedge\mathbf{T}(u)\wedge f(y)\wedge\mathbf{T}(v)\wedge\mathbf{T}(z)\}\\ &\wedge\{\mathbf{T}(x)\wedge\mathbf{T}(uyv)\wedge f(z)\}\wedge\delta\\ &=f(x)\wedge f(y)\wedge f(z)\wedge\delta.\end{aligned}$$

This implies that

$$\max\{f(xuyvz),\gamma\}\ge\min\{f(x),f(y),f(z),\delta\}.$$

Hence, f is an $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy bi-ideal of T.

Definition 12. Let f,g and h be fuzzy subsets of a ternary semigroup T. We define the fuzzy subsets $(f)_{\gamma}^{\delta}$, $f\wedge_{\gamma}^{\delta}g$, $f\vee_{\gamma}^{\delta}g$ and $f\circ_{\gamma}^{\delta}g\circ_{\gamma}^{\delta}h$ as follows:

(1) $(f)_{\gamma}^{\delta}(x)=(f(x)\vee\gamma)\wedge\delta$

(2) $\left(f \wedge_{\gamma}^{\delta} g\right)(x)=\left(\left(f(x) \wedge g(x)\right) \vee \gamma\right) \wedge \delta$

(3) $\left(f \vee_{\gamma}^{\delta} g\right)(x)=\left(\left(f(x) \vee g(x)\right) \vee \gamma\right) \wedge \delta$

(4) $\left(f \circ_{\gamma}^{\delta} g \circ_{\gamma}^{\delta} h\right)(x)=\left((f \circ g \circ h)(x) \vee \gamma\right) \wedge \delta$

for all $x \in T$.

Lemma 4. Let f, g and h be fuzzy subsets of a ternary semigroup T. Then the following hold.

(1) $\left(f \wedge_{\gamma}^{\delta} g\right)=(f)_{\gamma}^{\delta} \wedge(g)_{\gamma}^{\delta}$

(2) $\left(f \vee_{\gamma}^{\delta} g\right)=(f)_{\gamma}^{\delta} \vee(g)_{\gamma}^{\delta}$

(3) $f \circ_{\gamma}^{\delta} g \circ_{\gamma}^{\delta} h=(f)_{\gamma}^{\delta} \circ(g)_{\gamma}^{\delta} \circ(h)_{\gamma}^{\delta}$.

Proof: The proofs of (1) and (2) are obvious.
(3) Let $x \in T$. Consider,

$$\begin{aligned}
\left(f \circ_{\gamma}^{\delta} g \circ_{\gamma}^{\delta} h\right)(x) &= \left((f \circ g \circ h)(x) \vee \gamma\right) \wedge \delta \\
&= \left\{\left(\bigvee_{x=pqr}\left(f(p) \wedge g(q) \wedge h(r)\right)\right) \vee \gamma\right\} \wedge \delta \\
&= \left[\bigvee_{x=pqr}\left\{(f(p) \vee \gamma) \wedge \delta\right\} \wedge\left\{(g(q) \vee \gamma) \wedge \delta\right\} \wedge\left\{(h(r) \vee \gamma) \wedge \delta\right\}\right] \\
&= \left[\bigvee_{x=pqr}(f)_{\gamma}^{\delta}(p) \wedge(g)_{\gamma}^{\delta}(q) \wedge(h)_{\gamma}^{\delta}(r)\right] \\
&= \left((f)_{\gamma}^{\delta} \circ(g)_{\gamma}^{\delta} \circ(h)_{\gamma}^{\delta}\right)(x).
\end{aligned}$$

Hence $f \circ_{\gamma}^{\delta} g \circ_{\gamma}^{\delta} h=(f)_{\gamma}^{\delta} \circ(g)_{\gamma}^{\delta} \circ(h)_{\gamma}^{\delta}$.

Theorem 21. Let f be an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy right ideal, g an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy lateral ideal and h an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$ -fuzzy left ideal of a ternary semigroup T. Then $f \circ_{\gamma}^{\delta} g \circ_{\gamma}^{\delta} h \leq f \wedge_{\gamma}^{\delta} g \wedge_{\gamma}^{\delta} h$.

Proof: Let f be an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy right ideal, g an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy lateral ideal and h an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy left ideal of T and $x \in T$. Consider,

$$\begin{aligned}
\left(f \circ_{\gamma}^{\delta} g \circ_{\gamma}^{\delta} h\right)(x) &= \left((f \circ g \circ h)(x) \vee \gamma\right) \wedge \delta \\
&= \left\{\left(\bigvee_{x=uvw}\left(f(u) \wedge g(v) \wedge h(w)\right)\right) \vee \gamma\right\} \wedge \delta \\
&= \left[\left\{\bigvee_{x=uvw}\left(f(u) \wedge \delta\right) \wedge\left(g(v) \wedge \delta\right) \wedge\left(h(w) \wedge \delta\right)\right\} \vee \gamma\right] \wedge \delta \\
&\leq \left[\left\{\bigvee_{x=uvw}\left(f(uvw) \vee \gamma\right) \wedge\left(g(uvw) \vee \gamma\right) \wedge\left(h(uvw) \vee \gamma\right)\right\} \vee \gamma\right] \wedge \delta \\
&= \left\{\left(\bigvee_{x=uvw} f(uvw) \wedge g(uvw) \wedge h(uvw)\right) \vee \gamma\right\} \wedge \delta \\
&= \left(\left(f(x) \wedge g(x) \wedge h(x)\right) \vee \gamma\right) \wedge \delta \\
&= \left(f \wedge_{\gamma}^{\delta} g \wedge_{\gamma}^{\delta} h\right)(x).
\end{aligned}$$

Hence $f \circ_{\gamma}^{\delta} g \circ_{\gamma}^{\delta} h \leq f \wedge_{\gamma}^{\delta} g \wedge_{\gamma}^{\delta} h$.

Theorem 22. Let f be an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy right ideal and g an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy left ideal of a ternary semigroup T. Then $f \circ_{\gamma}^{\delta} \mathbf{T} \circ_{\gamma}^{\delta} g \leq f \wedge_{\gamma}^{\delta} g$.

Proof: Suppose f is an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy right ideal and g an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy left ideal of T. Let $x \in T$. Consider

$$\begin{aligned}
\left(f \circ_{\gamma}^{\delta} \mathbf{T} \circ_{\gamma}^{\delta} g\right)(x) &= \left\{(f \circ \mathbf{T} \circ g) \vee \gamma\right\} \wedge \delta \\
&= \left\{\bigvee_{x=uvw}\left(f(u) \wedge \mathbf{T}(v) \wedge g(w)\right) \vee \gamma\right\} \wedge \delta \\
&= \left[\left\{\bigvee_{x=uvw}\left(f(u) \vee \gamma\right) \wedge\left(g(w) \vee \gamma\right)\right\} \vee \gamma\right] \wedge \delta \\
&\leq \left[\left\{\bigvee_{x=uvw}\left(f(uvw) \wedge \delta\right) \wedge\left(g(uvw) \wedge \delta\right)\right\} \vee \gamma\right] \wedge \delta \\
&= \left\{\left(f(x) \wedge g(x)\right) \vee \gamma\right\} \wedge \delta=\left(f \wedge_{\gamma}^{\delta} g\right)(x).
\end{aligned}$$

Thus, $f \circ_{\gamma}^{\delta} \mathbf{T} \circ_{\gamma}^{\delta} g \leq f \wedge_{\gamma}^{\delta} g$.

The proof of the following theorem is straightforward and we omit the detail.

Theorem 23. A non-empty subset A of a ternary semigroup T is a left (right, lateral, two sided) ideal of T if and only if $\left(C_{A}\right)_{\gamma}^{\delta}$ is an $\left(\in_{\gamma}, \in_{\gamma} \vee q_{\delta}\right)$-fuzzy left (right, lateral, two sided) ideal of T, where C_{A} is the characteristic function of A.

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