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The multi-step homotopy analysis method: A powerful scheme for handling non-linear oscillators

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Abstract

This paper presents approximate analytical solutions for nonlinear oscillators using the multi-step homotopy analysis method (MSHAM). The proposed scheme is only a simple modification of the homotopy analysis method, in which it is treated as an algorithm in a sequence of small intervals (i.e. time step) for finding accurate approximate solutions to the corresponding problems. Several illustrative examples are given to demonstrate the effectiveness of the present method. Figurative comparisons between the MSHAM and the classical fourth-order Runge-Kutta method (RK4) reveal that this modified method is very effective and convenient.

Keywords: Non-linear oscillators; homotopy analysis method; numerical solutions

1. Introduction

Nonlinear oscillatory systems are of crucial importance in all areas of physics and engineering, as well as in other disciplines. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem. There have been many analytical and numerical methods to solve the problems of nonlinear oscillators, such as variational iteration method [1-3], homotopy perturbation method [4-7], Adomian decomposition method [8-9], differential transform method [10-11], harmonic balance based methods [12-13] and the multiple scales method [14] are extensively used to obtain approximate solutions of non-linear oscillatory equations. But these familiar methods are rarely used to solve the equations which contain nonlinear terms. The basic reason is that they become too complex and difficult when applied to nonlinear equations. Recently, Momani et al [15] proposed an analytic method, namely modified homotopy perturbation method (MHPM). The approximate solution of the MHPM displays the periodic behavior which is characteristic of the oscillatory equations. In this paper, we developed a symbolic algorithm to find the solution of linear and nonlinear oscillators by the multi-step homotopy analysis method (MSHAM). The new algorithm is only a simple modification of the homotopy analysis method [16], in which it is treated as an algorithm in a sequence of small intervals (i.e. time step) for finding accurate approximate solutions to the linear and non-linear oscillatory equations. It is found that the corresponding numerical solutions obtained by using HAM are valid only for a short time. While the ones obtained by using MSHAM are more valid and accurate over a longer time, and are in strong agreement with the RK4-5 numerical solutions. The structure of this paper is as follows. In section 2 we describe the MSHAM. In Section 3 we present five examples to show the efficiency and simplicity of the method. Finally, the conclusions are given in Section 4.

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2. MHAM Algorithm

The HAM has been extended by many authors to solve linear and nonlinear problems in terms of convergent series with easily computable components, however it does have some drawbacks: the series solution always converges in a very small region and it has slow convergent rate or is completely divergent in the wider region [17-20]. In this section, we present the basic ideas of the multi-step HAM that have been developed in [21]. To show the basic idea, let us consider the following initial value problem,

$$y''(t) = F(t, y(t), y'(t)), \quad t \geq 0, \qquad (1)$$

subject to the initial conditions

$$y(0)=a, \quad y^{'}(0)=b.$$

Where the function $F(t, y(t), y^{'}(t))$ is an arbitrary linear or nonlinear function of its arguments. With $y'=x$, Eq. (1) is transformed into the system of the first-order differential equations

$$\frac{dx}{dt}=F(t,y(t),x(t)), \quad x(0)=b,$$
$$\frac{dy}{dt}=x(t), \quad y(0)=a,$$
$$t\geq 0. \tag{2}$$

Let $[0,T]$ be the interval over which we want to find the solution of the initial value problem (2). The multi-step approach introduces a new idea for constructing the approximate solution. Assume that the interval $I=[0,T]$ is divided into n-subintervals of equal length Δt, $[t_0,t_1),[t_1,t_2),[t_2,t_3),......,[t_{n-1},t_n]$ with $t_0=0$, $t_n=T$. Let t^* be the initial value for each subinterval and let $x_j(t)$ and $y_j(t)$, $j=1,2,...,n$ be approximate solutions in each subinterval $[t_{j-1},t_j]$, $j=1,2,...,n$ with initial guesses

$$x_1(t^*)=b, \quad x_j(t^*)=x_{j-1}(t_{j-1}),$$
$$y_1(t^*)=a, \quad y_j(t^*)=y_{j-1}(t_{j-1}),$$
$$j=2,3,...,n. \tag{3}$$

Now, we can construct the so-called zeroth-order deformation equations of the system (2) by

$$(1-q)L\left[\phi_{1,j}(t,q)-x_j(t^*)\right]=q\hbar[\frac{d}{dt}\phi_{1,j}(t,q) - F(t,\phi_{1,j}(t,q),\phi_{2,j}(t,q))],$$
$$(1-q)L\left[\phi_{2,j}(t,q)-y_j(t^*)\right]=q\hbar[\frac{d}{dt}\phi_{2,j}(t,q) -\phi_{1,j}(t,q)], \quad j=1,2,...,n, \tag{4}$$

where $q\in[0,1]$ is an embedding parameter, L is an auxiliary linear operator satisfying $L(0)=0$, $\hbar\neq 0$ is an auxiliary parameter and $\phi_{i,j}(t,q)$, $i=1,2$, $j=1,2,...,n$, is an unknown function. Obviously when $q=0$, we have

$$\phi_{1,1}(t,0)=b, \quad \phi_{1,j}(t,0)=x_{j-1}(t_{j-1}),$$
$$\phi_{2,1}(t,0)=a, \quad \phi_{2,j}(t,0)=y_{j-1}(t_{j-1}),$$
$$j=2,3,...,n,$$

and when $q=1$, we have

$$\phi_{1,j}(t,1)=x_j(t),$$
$$\phi_{2,j}(t,1)=y_j(t), \quad j=1,2,...,n.$$

Expanding $\phi_{i,j}(t,q)$, $i=1,2$, $j=1,2,...,n$, in Taylor series with respect to q, one has

$$\phi_{1,j}(t,q)=x_j(t^*)+\sum_{m=1}^{\infty}x_{j,m}(t)\,q^m,$$
$$\phi_{2,j}(t,q)=y_j(t^*)+\sum_{m=1}^{\infty}y_{j,m}(t)\,q^m,$$
$$j=1,2,...,n, \tag{5}$$

where

$$x_{j,m}(t)=\frac{1}{m!}\frac{\partial^m\phi_{1,j}(t,q)}{\partial q^m}\Big|_{q=0},$$
$$y_{j,m}(t)=\frac{1}{m!}\frac{\partial^m\phi_{2,j}(t,q)}{\partial q^m}\Big|_{q=0},$$
$$j=1,2,...,n. \tag{6}$$

If the auxiliary parameter $\hbar$, and the initial guesses $x_j(t^*)$ and $y_j(t^*)$ are so properly chosen, then the series (5) converges at $q=1$, and one has

$$x_j(t)=x_j(t^*)+\sum_{m=1}^{\infty}x_{j,m}(t),$$
$$y_j(t)=y_j(t^*)+\sum_{m=1}^{\infty}y_{j,m}(t),$$
$$j=1,2,...,n, \tag{7}$$

Define the vectors

$$\vec{x}_{j,m}=\left\{x_{j,0}(t),x_{j,1}(t),...,x_{j,m}(t)\right\},$$
$$\vec{y}_{j,m}=\left\{y_{j,0}(t),y_{j,1}(t),...,y_{j,m}(t)\right\},$$

Differentiate system (4) with respect to the embedding parameter q, then setting $q=0$ and dividing them by m!, finally using (6), we have the so-called mth-order deformation equations

$$L\left[x_{j,m}(t)-\chi_m x_{j,m-1}(t)\right] = \hbar R^1_{j,m}(\vec{x}_{j,m-1}(t)),$$
$$L\left[y_{j,m}(t)-\chi_m y_{j,m-1}(t)\right] = \hbar R^2_{j,m}(\vec{y}_{j,m-1}(t)), \quad (8)$$

subject to the initial conditions $x_{j,m}(0)=y_{j,m}(0)=0,\quad j=1,2,...,n,\quad m=1,2,3,...$ where

$$R^1_{j,m}(\vec{u}_{j,m-1}(t))=x'_{j,m-1}(t)-\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[F(t,\phi_{j,1}(t,q),\phi_{j,2}(t,q))]\Big|_{q=0},$$
$$R^2_{j,m}(\vec{v}_{j,m-1}(t))=y'_{j,m-1}(t)-x_{j,m-1}(t)$$
$$j=1,2,...n. \quad (9)$$

And

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \quad (10)$$

Select the auxiliary linear operator $L=\frac{d}{dt}$, then the mth-order deformation equations (8) can be written in the form

$$x_{j,m}(t)=\chi_m x_{j,m-1}(t)+\hbar\int_{t_{j-1}}^{t}R^1_{j,m}(\vec{x}_{j,m-1}(\tau))\,d\tau,$$
$$y_{j,m}(t)=\chi_m y_{j,m-1}(t)+\hbar\int_{t_{j-1}}^{t}R^2_{j,m}(\vec{y}_{j,m-1}(\tau))\,d\tau,$$
$$j=1,2,...n. \quad (11)$$

The solutions of system (2) in each subinterval $[t_{j-1},t_j],\ j=1,2,...,n,$ have the form

$$x_j(t)=\sum_{m=0}^{\infty}x_{j,m}(t),\quad y_j(t)=\sum_{m=0}^{\infty}y_{j,m}(t),$$
$$j=1,2,...n, \quad (12)$$

and the solution of system (2) for $[0,T]$ is given by

$$x(t)=\sum_{j=1}^{n}\chi_r\,x_j(t),\qquad y(t)=\sum_{j=1}^{n}\chi_r\,y_j(t), \quad (13)$$

where

$$\chi_r=\begin{cases}1, & t\in[t_{j-1},t_j),\\ 0, & t\notin[t_{j-1},t_j).\end{cases}$$

Finally, the solution of the initial value problem (1) is $y(t)=\sum_{j=1}^{n}\chi_r\,y_j(t).$

3. Numerical experiments

To demonstrate the effectiveness of the proposed algorithm as an approximate tool for solving linear and nonlinear oscillatory systems, we apply the proposed algorithm, the multi-step HAM, to five oscillator equations.

Example 3.1 Consider the following linear equation

$$y''(t)+0.5y(t)=1,\qquad t>0, \quad (14)$$

subject to the initial conditions

$$y(0)=0,\ y'(0)=0.$$

Let $y'=x$, then Eq. (14) is transformed into the following system

$$\frac{dx}{dt}=1-0.5\,y(t),\qquad x(0)=0,$$
$$\frac{dy}{dt}=x(t),\qquad y(0)=0. \quad (15)$$

In this example, we apply the proposed algorithm on the interval $[0,100]$. We choose to divide the interval $[0,100]$ to subintervals with time step $\Delta t=0.1.$ So we start with initial approximation

$$x_1(t^*)=0,\quad x_j(t^*)=x_{j-1}(t_{j-1})=b_j,$$
$$y_1(t^*)=0,\quad y_j(t^*)=y_{j-1}(t_{j-1})=a_j,$$
$$j=2,3,...,n. \quad (16)$$

Where t^* is the initial value for each subinterval. In view of the algorithm presented in the previous section, we have the mth-order deformation equation (11), where

$$R^1_{j,m}(\vec{x}_{j,m-1}(t)) = x'_{j,m-1}(t) + 0.5y_{j,m-1}(t) - (1-\chi_m),$$
$$R^2_{j,m}(\vec{y}_{j,m-1}(t)) = y'_{j,m-1}(t) - x_{j,m-1}(t),$$
$$j = 1,2,...n. \tag{17}$$

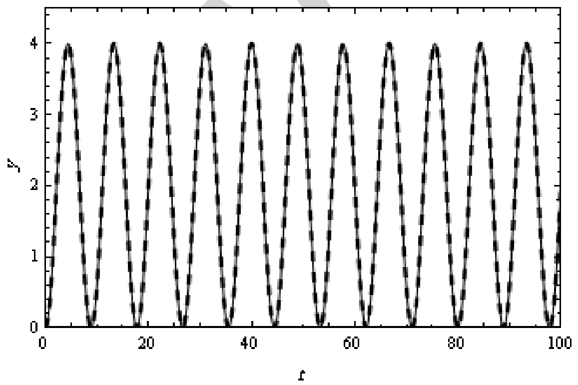
Now, according to the multi-step HAM, the series solution for system (15) is given by,

$$x_j(t) = b_j - \hbar(\hbar^2+3\hbar+3)(1-\frac{a_j}{2})(t-t^*)$$
$$-b_j\frac{\hbar^2}{2}(\hbar+\frac{3}{2})(t-t^*)^2 + \frac{\hbar^3}{12}(1-\frac{a_j}{2})(t-t^*)^3 + ...,$$
$$y_j(t) = a_j - b_j\hbar(\hbar^2+3\hbar+3)(t-t^*)$$
$$+\hbar^2(\hbar+\frac{3}{2})(1-\frac{a_j}{2})(t-t^*)^2 + b_j\frac{\hbar^3}{12}(t-t^*)^3 + ..., \tag{18}$$

and the multi-step series solution of the problem (14) in each subinterval $[t_{j-1}, t_j],\ j = 1,2,...,n,$ has the form

$$y_j(t) = a_j - b_j\hbar(\hbar^2+3\hbar+3)(t-t^*)$$
$$+\hbar^2(\hbar+\frac{3}{2})(1-\frac{a_j}{2})(t-t^*)^2 + b_j\frac{\hbar^3}{12}(t-t^*)^3 + ..., \tag{19}$$

Figure 1 shows the displacement and phase diagram of the MSHAM when $\hbar = -1$ and the exact solution ($y(t) = 2(1-\cos(\frac{t}{\sqrt{2}}))$) of the oscillatory equation (14). It can be seen that the results from the MSHAM match the results of the exact solution very well; therefore, the proposed method is very efficient and an accurate method that can be used to provide analytical solutions for linear systems of differential equations.



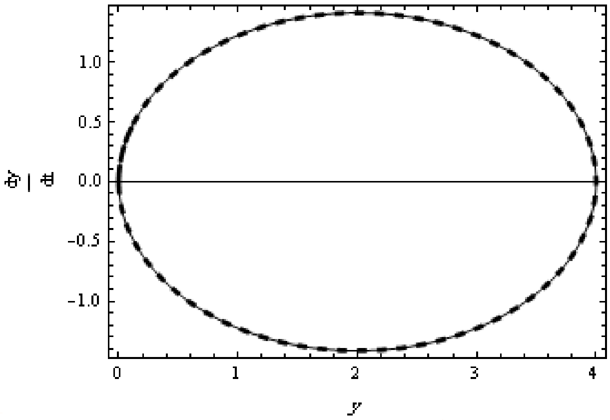


Fig. 1. The displacement and phase plane for Example 3.1: MSHAM solution (Solid line), the exact solution (Dotted line)

Example 3.2. Consider the following nonlinear equation

$$y''(t) + 2y(t) + y^2(t) = 0, \qquad t \geq 0, \tag{20}$$

subject to the initial conditions

$$y(0) = 0.1,\ y'(0) = 0.$$

Momani et al. [15] derived a numerical solution for the above equation using the modified homotopy perturbation method. Let $y' = x$, then Eq. (20) is transformed into the following system

$$\frac{dx}{dt} = -2y(t) - y^2(t), \qquad x(0) = 0,$$
$$\frac{dy}{dt} = x(t), \qquad y(0) = 0.1. \tag{21}$$

In this example, the interval $[0,100]$ is divided to subintervals with time step $\Delta t = 0.1.$ So we start with initial approximation

$$x_1(t^*) = 0, \qquad x_j(t^*) = x_{j-1}(t_{j-1}) = b_j,$$
$$y_1(t^*) = 0.1, \quad y_j(t^*) = y_{j-1}(t_{j-1}) = a_j,$$
$$j = 2,3,...,n. \tag{22}$$

In view of the algorithm presented in the previous section, we have the mth-order deformation equation (11), where

$$R^1_{j,m}(\vec{x}_{j,m-1}(t)) = x'_{j,m-1}(t) + 2y_{j,m-1}(t) + \sum_{i=0}^{m-1} y_{j,i}(t) y_{j,m-i-1}(t),$$

$$R^2_{j,m}(\vec{y}_{j,m-1}(t)) = y'_{j,m-1}(t) - x_{j,m-1}(t), \quad j = 1,2,\ldots n. \tag{23}$$

Now, according to the multi-step HAM, the series solution for system (21) is given by,

$$\begin{aligned} x_j(t) &= b_j + a_j\hbar(\hbar^2 + 3\hbar + 3)(a_j + 2)(t - t^*) \\ &- b_j\hbar^2(2\hbar + 3)(a_j + 1)(t - t^*)^2 \\ &- \frac{\hbar^3}{3}(a_j(a_j^2 - 3a_j + 2) + b_j^2)(t - t^*)^3 + \ldots, \\ y_j(t) &= a_j - b_j\hbar(\hbar^2 + 3\hbar + 3)(t - t^*) \\ &- a_j\hbar^2(\hbar + \frac{3}{2})(a_j + 2)(t - t^*)^2 \\ &+ \frac{\hbar^3}{3} b_j(a_j + 1)(t - t^*)^3 + \ldots, \end{aligned} \tag{24}$$

and the multi-step series solution of the problem (20) in each subinterval $[t_{j-1}, t_j],\ j = 1,2,\ldots,n,$ has the form

$$\begin{aligned} y_j(t) &= a_j - b_j\hbar(\hbar^2 + 3\hbar + 3)(t - t^*) \\ &- a_j\hbar^2(\hbar + \frac{3}{2})(a_j + 2)(t - t^*)^2 \\ &+ \frac{\hbar^3}{3} b_j(a_j + 1)(t - t^*)^3 + \ldots, \end{aligned} \tag{25}$$

Figure 2 shows the displacement and phase diagram of the MSHAM when $\hbar = -1$ and the fourth-order Runge-Kutta method of the nonlinear oscillatory equation (20). The results from the MSHAM match the results of the Runge-Kutta method is a very well; therefore, the proposed method is very efficient and accurate method that can be used to provide analytical solutions for nonlinear systems of differential equations. Also, the results of our computations are in excellent agreement with the results obtained by the numerical solution of Momani et al. [12] Using modified homotopy perturbation method.

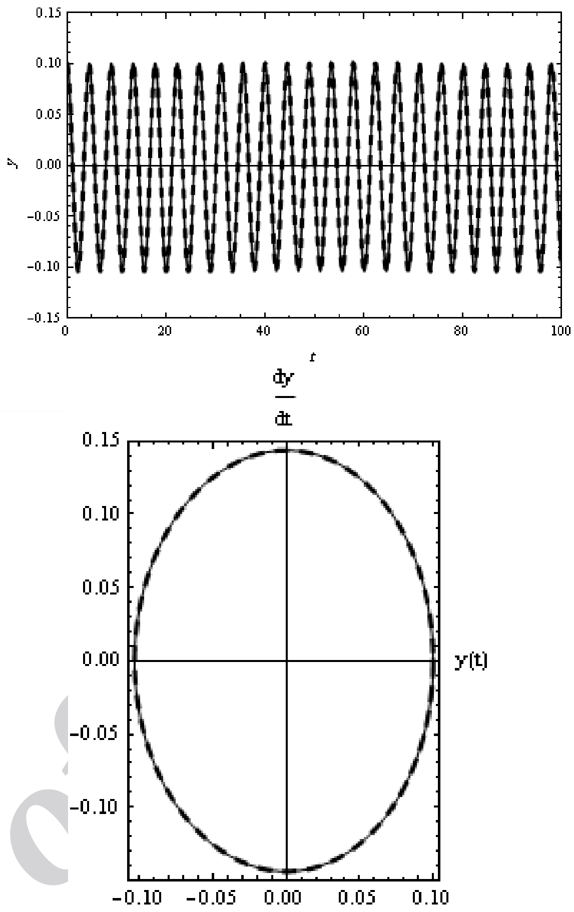


Fig. 2. The displacement and phase plane for Example 3.2: MSHAM solution (Solid line), Runge-Kutta method solution (Dotted line)

Example 3.3. Consider the following nonlinear Van der Pol equation

$$y''(t) + y(t) + 0.1y^2(t)y'(t) = 0, \quad t \geq 0, \tag{26}$$

subject to the initial conditions

$$y(0) = 1, \ y'(0) = 0.$$

The solution of this equation is expected to oscillate with decreasing amplitude to zero. Momani et al. [15] derived a numerical solution for the above equation using the modified homotopy perturbation method.

Let $y' = x$, then Eq. (26) is transformed into the following system

$$\begin{aligned} \frac{dx}{dt} &= -y(t) - 0.1x(t)y^2(t), \qquad x(0) = 0, \\ \frac{dy}{dt} &= x(t), \qquad y(0) = 1. \end{aligned} \tag{27}$$

Divide the interval $[0,100]$ to subintervals with time step $\Delta t = 0.1$. Then start with initial approximation

$$x_1(t^*) = 0, \qquad x_j(t^*) = x_{j-1}(t_{j-1}) = b_j,$$
$$y_1(t^*) = 1, \qquad y_j(t^*) = y_{j-1}(t_{j-1}) = a_j,$$
$$j = 2,3,\ldots,n. \tag{28}$$

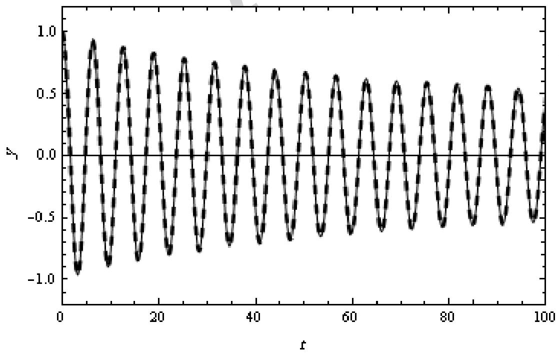
In view of the algorithm presented in the previous section, we have the mth-order deformation equation (11), where

$$R^1_{j,m}(\vec{x}_{j,m-1}(t)) = x'_{j,m-1}(t) + y_{j,m-1}(t)$$
$$+0.1\sum_{i=0}^{m-1} x_{j,m-i-1}(t)\sum_{n=0}^{i} y_n(t) y_{i-n}(t),$$
$$R^2_{j,m}(\vec{y}_{j,m-1}(t)) = y'_{j,m-1}(t) - x_{j,m-1}(t),$$
$$j = 1,2,\ldots n. \tag{29}$$

Now, according to the multi-step HAM, the series solution for the Van der Pol equation (26) in each subinterval $[t_{j-1}, t_j],\ j = 1,2,\ldots,n,$ has the form

$$y_j(t) = a_j - b_j\hbar(\hbar^2 + 3\hbar + 3)(t - t^*)$$
$$-a_j\hbar^2(\hbar + \frac{3}{2})(\frac{a_j b_j}{10} + 1)(t - t^*)^2 + \frac{\hbar^3}{3}((b_j - \frac{a_j^3}{10})$$
$$+\frac{a_j b_j}{5}(b_j - \frac{a_j^3}{20}))(t - t^*)^3 + \ldots, \tag{30}$$

Figure 3 shows the displacement and phase diagram of the MSHAM when $\hbar = -1$ and the fourth-order Runge-Kutta method of the nonlinear Van der Pol equation (26). Also, the results of our computations are in excellent agreement with the results obtained by the numerical solution of Momani et al. [15] Using modified homotopy perturbation method.



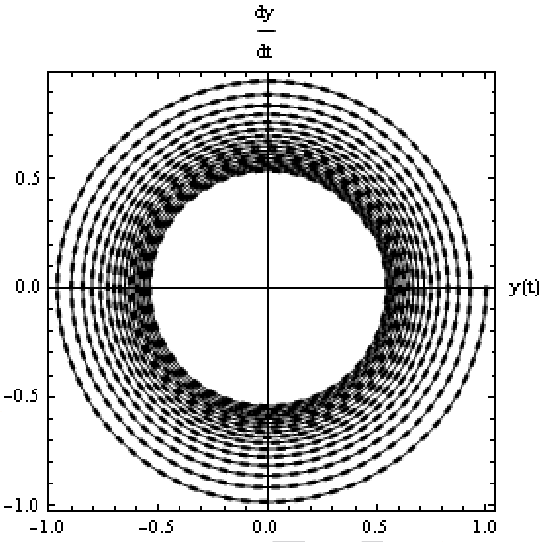


Fig. 3. The displacement and phase plane for Example 3. 3: MSHAM solution (Solid line), Runge-Kutta method solution (Dotted line)

Example 3.4. Consider the following nonlinear equation

$$y''(t) + y(t) + 0.45y^2(t) - y(t)y'(t) = 0,$$
$$t \ge 0, \tag{31}$$

subject to the initial conditions

$$y(0) = 0.1,\ \ y'(0) = 0.$$

Let $y' = x$, then Eq. (31) is transformed into the following system

$$\frac{dx}{dt} = -y(t) - 0.45y^2(t) + x(t)y(t),\ x(0) = 0,$$
$$\frac{dy}{dt} = x(t),\ y(0) = 0.1. \tag{32}$$

Also, divide the interval $[0,100]$ to subintervals with time step $\Delta t = 0.1$. We start with initial approximation

$$x_1(t^*) = 0, \qquad x_j(t^*) = x_{j-1}(t_{j-1}) = b_j,$$
$$y_1(t^*) = 0.1, \qquad y_j(t^*) = y_{j-1}(t_{j-1}) = a_j,$$
$$j = 2,3,\ldots,n. \tag{33}$$

In view of the algorithm presented in the previous section, we have the mth-order deformation equation (11), where

$$R^1_{j,m}(\vec{x}_{j,m-1}(t)) = x'_{j,m-1}(t) + y_{j,m-1}(t)$$
$$+0.45\sum_{i=0}^{m-1} y_{j,i}(t) y_{j,m-i-1}(t)$$
$$-\sum_{i=0}^{m-1} x_{j,i}(t) y_{j,m-i-1}(t),$$
$$R^2_{j,m}(\vec{y}_{j,m-1}(t)) = y'_{j,m-1}(t) - x_{j,m-1}(t),$$
$$j = 1,2,\ldots n. \qquad (34)$$

Now, according to the multi-step HAM, the series solution for the nonliear equation (31) in each subinterval $[t_{j-1}, t_j],\ j = 1,2,\ldots,n,$ has the form

$$y_j(t) = a_j - b_j \hbar(\hbar^2 + 3\hbar + 3)(t - t^*)$$
$$-a_j \hbar^2(\hbar + \frac{3}{2})(\frac{9}{20}a_j - b_j + 1)(t - t^*)^2$$
$$+\hbar^3(\frac{1}{6}(1 - b_j)(b_j + a_j^2) + \frac{3}{20}a_j(b_j + \frac{a_j^2}{2}))(t - t^*)^3 + \ldots, \qquad (35)$$

Figure 4 shows the comparison between the MSHAM solution when $\hbar = -1$ and the numerical integration results obtained by RK4 method for the displacement and phase diagram of nonlinear equation (31).

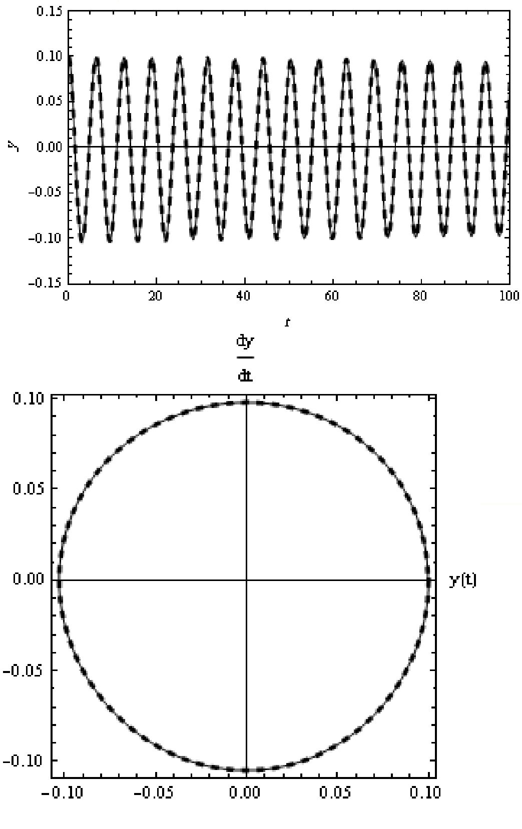


Fig. 4. The displacement and phase plane for Example 3.4: MSHAM solution (Solid line), Runge-Kutta method solution (Dotted line)

Example 3.5. Consider the oscillatory equation

$$y''(t) + \frac{y^3(t)}{1 + y^2(t)} = 0, \qquad (36)$$

subject to the initial conditions

$$y(0) = 0, \quad y'(0) = 0.3. \qquad (37)$$

By using the transformation $y' = x$, we get the following systems of differential equations

$$x'(t) + x'(t)\,y^2(t) + y^3(t) = 0, \quad x(0) = 0.3,$$
$$y'(t) = x(t), \ \ y(0) = 0. \qquad (38)$$

The simple idea is to divide the interval $[0,100]$ to subintervals with time step $\Delta t = 0.1.$ So we start with initial approximation

$$x_1(t^*) = 0.3, \qquad x_j(t^*) = x_{j-1}(t_{j-1}) = b_j,$$
$$y_1(t^*) = 0, \quad y_j(t^*) = y_{j-1}(t_{j-1}) = a_j,$$
$$j = 2,3,\ldots,n. \qquad (39)$$

In view of the algorithm presented in the previous section, we have the mth-order deformation equation (11), where

$$R^1_{j,m}(\vec{x}_{j,m-1}(t)) = x'_{j,m-1}(t)$$
$$+\sum_{i=0}^{m-1} x'_{j,m-i-1}(t) \sum_{n=0}^{i} y_n(t) y_{i-n}(t)$$
$$+\sum_{i=0}^{m-1} y_{j,m-i-1}(t) \sum_{n=0}^{i} y_n(t) y_{i-n}(t),$$
$$R^2_{j,m}(\vec{y}_{j,m-1}(t)) = y'_{j,m-1}(t) - x_{j,m-1}(t),$$
$$j = 1,2,\ldots n. \qquad (40)$$

Now, according to the multi-step HAM, the series solution for the nonliear equation (36) in each subinterval $[t_{j-1}, t_j],\ j = 1,2,\ldots,n,$ has the form

$$y_j(t) = a_j - b_j \hbar(\hbar^2 + 3\hbar + 3)(t - t^*)$$
$$-a_j^3 \hbar^2(\hbar(1 + \frac{a_j^2}{2}) + \frac{3}{2})(t - t^*)^2$$
$$+\frac{1}{2}a_j^2 b_j \hbar^3 (t - t^*)^3 + \ldots, \qquad (41)$$

Figure 5 shows the comparison between the MSHAM solution when $\hbar = -1$ and the numerical integration results obtained by RK4 method for the displacement and phase diagram of nonlinear equation (36). From Fig. 5, it is obvious that the solution obtained by the present method is nearly identical with that given by RK4 method.

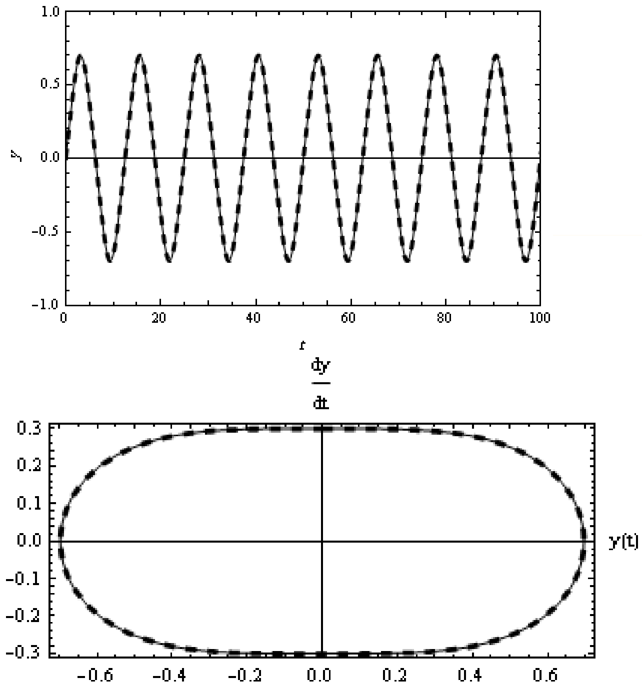


Fig. 5. The displacement and phase plane for Example 3.5: MSHAM solution (Solid line), Runge-Kutta method solution (Dotted line)

4. Conclusions

In this work, we proposed an efficient modification of the HAM which introduces an efficient tool for solving linear and nonlinear oscillatory equations. Comparisons of the results obtained by using the MSHAM with that obtained by the fourth-order Runge-Kutta method reveal that the approximate solutions obtained by HAM are only valid for a small time, while the ones obtained by MSHAM are highly accurate and valid for a long time. Finally, we can see that the method considered here is very simple in its principle and we think that the method has great potential and can be applied to other strongly nonlinear oscillators.

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