Radial basis functions and FDM for solving fractional diffusion-wave equation

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Abstract

In this work, we apply the radial basis functions for solving the time fractional diffusion-wave equation defined by Caputo sense for $(1 < \alpha \leq 2)$. The problem is discretized in the time direction based on finite difference scheme and is continuously approximated by using the radial basis functions in the space direction which achieves the semi-discrete solution. Numerical results show the accuracy and efficiency of the presented method.

Keywords: Diffusion-wave equation; fractional derivative; radial basis functions; finite difference scheme

1. Introduction

In recent years, fractional calculus has been implemented to express some phenomena in physics and engineering. Also, fractional integral and derivative have been successful to describe many events in fluid mechanics, viscoelasticity, chemical physics, electricity, finance, control theory, biomedical engineering, heat conduction, diffusion problems and other sciences (Kilbas et al., 2006; Podlubny, 1999). Fractional partial differential equations (FPDEs) particularly space- and time-fractional equations, have been widely studied to construe the existence of solution and validity of these problems (Li and Xu, 2009; Zhao et al., Zhuang et al. 2011). In addition, finding the reliable and powerful numerical and analytical methods for solving FPDEs have been the focus in two last decades. According to the mathematical literature, fractional partial differential equations have been developed in many various problems in science and engineering as the Schröinger, telegraph, diffusion and diffusion-wave fractional equation (Li and Xu, 2009; Chen et al., 2010a; Li et al., 2011; Liu et al., 2006; Mohebbi et al., 2013; Zhoa and Li, 2012).

In 2009, Wen *et al*. were pioneers in using the Kansa method for solving the fractional diffusion equation (Chen et al., 2010b) After that the method was expanded for solving the other fractional equations (Mohebbi et al, 2013; Piret and Hanert, 2013; Hosseini et al., 2014; Gu et al., 2010).

In this work, we implement the meshless method for solving time-fractional diffusion-wave equation by using radial basis function (RBF). The time fractional diffusion-wave equation is obtained from the classical diffusion or wave equation by replacing the first or second order time derivative by a fractional derivative of order α with $0 < \alpha < 1$ or $1 < \alpha < 2$. It is observed that as α increases from 0 to 2, the process changes from slow diffusion to classical diffusion and diffusion-wave to classical wave process. In fact, this equation interpolates between the diffusion and the wave equations that behave quite differently regarding their response to a localized disturbance: whereas the diffusion equation describes a process, where a disturbance spreads infinitely fast, the propagation speed of the disturbance is a constant for the wave equation. Fractional diffusion-wave equation has important applications to mathematical physics (Al-Khaled and Momani, 2005; Chen et al., 2012; Giano et al., 1992; Jafari and Daftardar-Gejji, 2006; Jafari and Seifi, 2009; Jiang et al., 2012). The presented method is the coupled RBF and finite difference scheme as is handled in (Chen et al., 2010b; Hosseini et al., 2014; Avazzadeh et al., 2011; Dehghan and Shokri, 2009).

The paper is organized in the following way. In Section 2, the Caputo fractional derivative and RBF methodology as the main ideas for solving fractional diffusion-wave equation are described. In Section 3, the discretization process of the problem in t direction via finite difference scheme is described. Also, using the radial basis functions to

obtain the continuous solution with respect to x is investigated. In Section 4, some numerical examples are demonstrated which confirm the accuracy and applicability of the method. The last section includes some other features of the presented method, conclusion and further ideas for future work.

$$D_*^\alpha F(x) = \begin{cases} \dfrac{1}{\Gamma(k-\alpha)} \displaystyle\int_0^x (x-\xi)^{k-\alpha-1} F^{(k)}(\xi)d\xi, & k-1 < \alpha < k, \\ \\ F^{(k)}(x), & \alpha = k. \end{cases} \quad (1)$$

More properties of the fractional Caputo derivative can be found in (Kilbas et al., 2006; Podlubny, 1999). Also, the further information about fractional calculus and another definitions of fractional derivatives, one can consult the mentioned references.

2.2. Radial basis functions

Considering a finite set of interpolation points $\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$. So the interpolant of u is constructed in the following form

$$(Su)(x) = \sum_{i=1}^N \lambda_i \phi(\|x-x_i\|) + p(x), \quad x \in \mathbb{R}^d, \quad (2)$$

where d is the dimension, $\|.\|$ is the Euclidean norm and $\phi(\|.\|)$ is a radial function (Buhmann, 2003; Cheney and Light, 1999). Also, $p(x)$ is a linear combination of polynomials on $\mathbb{R}^d$ of total degree at most $m-1$ as follows

$$p(x) = \sum_{j=N+1}^{N+l} \lambda_j q_j(x), \qquad l = \binom{m+d-1}{d}. \quad (3)$$

Moreover, the interpolant Su and additional conditions must be determined to satisfy the system

$$\begin{cases} (Su)(x_i) = u(x_i), & i = 1, 2, \ldots, N, \\ \displaystyle\sum_{i=1}^N \lambda_i q_j(x_i) = 0, & \text{for all } q_j \in \Pi_{m-1}^d, \end{cases} \quad (4)$$

where Π_{m-1}^d denotes the space of all polynomials on $\mathbb{R}^d$ of total degree at most $m-1$. The generalized thin plate splines (GTPS) are defined as follows:

$$\phi(\|x-x_i\|) = \phi(r_i) = r_i^{2m} \log(r_i), \\ i = 1, 2, 3, \ldots, \qquad m = 1, 2, 3, \ldots, \quad (5)$$

2. Basic Definitions

2.1. Fractional derivative

Definition: The Caputo fractional derivative operator of order $\alpha \geq 0$, of a function $F(x)$ is defined as

where $r_i = \|x-x_i\|$. We note that ϕ in Eq.(5) is C^{2m-1} continuous. Therefore, the higher order of partial differential needs the higher order of thin plate splines. Briefly, u and $\mathcal{L}u$ can be approximated as

$$u(x) \simeq \sum_{x_i \in \mathcal{X}} \lambda_i \phi(\|x-x_i\|) + p(x), \qquad x \in \mathbb{R}^d,$$

$$\mathcal{L}u(x) = \sum_{x_i \in \mathcal{X}} \lambda_i \mathcal{L}\phi(\|x-x_i\|) + \mathcal{L}p(x), \qquad x \in \mathbb{R}^d,$$

where $\mathcal{L}$ can be any partial differential operator. The unkown coefficients λ_i can be found by solving the obtained system after substituting Eq.(2) in the proposed problem.

3. Description of the Method

Consider the following time-fractional diffusion-wave equation of order $\alpha (1 < \alpha \leq 2)$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \beta u(x,t) = \gamma \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (6)$$
$$a \leq x \leq b, \quad 0 \leq t \leq T,$$

with the initial conditions

$$u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x), \\ a \leq x \leq b, \quad (7)$$

and the boundary conditions

$$u(a,t) = h_1(t), \quad u(b,t) = h_2(t), \quad t \geq 0, \quad (8)$$

where $a, b, \alpha, g_1(x), g_2(x), h_1(t)$ and $h_2(t)$ are given and $\partial^\alpha u(x,t)/\partial t^\alpha$ represents the Caputo fractional derivative and γ and β are the given constants. According to Eq. (1), $\partial^\alpha u(x,t)/\partial t^\alpha$ can be written as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(2-\alpha)}\int_0^t \dfrac{\partial^2 u(x,\xi)}{\partial \xi^2}(t-\xi)^{1-\alpha}\,\mathrm{d}\xi, & 1 < \alpha < 2, \quad (9) \\[2mm] \dfrac{\partial^2 u(x,\xi)}{\partial \xi^2}, & \alpha = 2. \end{cases}$$

In order to discretize the problem for $(1 < \alpha < 2)$ in time direction, we substitute t^{n+1} into Eq. (9), then the integrals can be reformed as

$$\frac{\partial^\alpha u(x,t^{n+1})}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)}\int_0^{t^{n+1}} \frac{\partial^2 u(x,\xi)}{\partial \xi^2}(t^{n+1}-\xi)^{1-\alpha}\,\mathrm{d}\xi,$$

$$= \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\int_{t^k}^{t^{k+1}} \frac{\partial^2 u(x,\xi)}{\partial \xi^2}(t^{n+1}-\xi)^{1-\alpha}\,\mathrm{d}\xi, \quad (10)$$

where $t^0 = 0$, $t^{n+1} = t^n + \delta t$, $n = 0,1,2,\dots M$. Also, n can be increased to the time length with δt as the time step in which $\delta t.M = T$. Approximation of the second order derivative due to the forward finite difference formulae is defined as

$$\frac{\partial^2 u(x,\xi)}{\partial t^2} \approx \frac{u(x,t^{n+1}) - 2u(x,t^n) + u(x,t^{n-1})}{\delta t^2}. \quad (11)$$

Replacement of Eq. (11) into Eq. (10) gives,

$$\frac{\partial^\alpha u(x,t^{n+1})}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)}\int_0^{t^{n+1}} \frac{\partial^2 u(x,\xi)}{\partial \xi^2}(t^{n+1}-\xi)^{1-\alpha}\,\mathrm{d}\xi,$$

$$= \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\frac{u^{k+1}-2u^k+u^{k-1}}{\delta t^2}\int_{t^k}^{t^{k+1}}(t^{n+1}-\xi)^{\alpha-1}\,\mathrm{d}\xi, \quad (12)$$

where $u^k = u(x,t^k), k = 0,1,\dots,M$. By considering $t^{n+1} - \xi = r$, the integral will be obtained as follows:

$$\int_{t^k}^{t^{k+1}}(t^{n+1}-\xi)^{\alpha-1}\,\mathrm{d}\xi = \frac{-1}{(2-\alpha)}r^{2-\alpha}\Big|_{t^{n-k+1}}^{t^{n-k}}$$

$$= \frac{1}{(2-\alpha)}\delta t^{2-\alpha}[(n-k+1)^{2-\alpha}-(n-k)^{2-\alpha}]. \quad (13)$$

Rearrangement of Eq. (12) and assumption $b_k = (k+1)^{2-\alpha} - (k)^{2-\alpha}$ lead to

$$\frac{\partial^\alpha u(x,t^{n+1})}{\partial t^\alpha} = \frac{\delta t^{-\alpha}}{\Gamma(3-\alpha)}\sum_{k=0}^{n}(u^{n+1}-2u^n+u^{n-1})[(n-k+1)^{2-\alpha}-(n-k)^{2-\alpha}]$$

$$= a_0\sum_{k=0}^{n}b_k(u^{n-k+1}-2u^{n-k}+u^{n-k-1})$$

$$= a_0\left\{u^{n+1}-2u^n+u^{n-1}+\sum_{k=1}^{n}b_k(u^{n-k+1}-2u^{n-k}+u^{n-k-1})\right\}, \quad (14)$$

where $a_0 = \dfrac{\delta t^{-\alpha}}{\Gamma(3-\alpha)}$ and $n = 0,1,2,\dots,M$.

We note that the Eq.(6) in $t = t^{n+1}$ due to θ-weighted finite difference formulation is as follows:

$$\frac{\partial^\alpha u(x,t^{n+1})}{\partial t^\alpha} + \beta u^{n+1} = \gamma\{\theta\nabla^2 u^{n+1} + (1-\theta)\nabla^2 u^n\} + f^{n+1}, (15)$$

where $0 \le \theta \le 1$ is a constant, $\nabla^2 u^n = \nabla^2 u(x,t^n)$ and $f(x,t^{n+1}) = f^{n+1}$. Now, we substitute Eq. (14) into Eq. (15) and obtain

$$\{a_0 + \beta - \gamma\theta\nabla^2\}u^{n+1} = \{2a_0 - \gamma(1-\theta)\nabla^2\}u^n$$

$$-a_0 u^{n-1} + a_0\sum_{k=1}^{n}b_k(u^{n-k+1}-2u^{n-k}+u^{n-k-1}) + f^{n+1}$$

where $n = 0,1,2,\dots,M$. Note that u^{-1} will be observed when $n = 0$ or $k = n$. So, we use the initial conditions to approximate u^{-1} as follows

$$u_t^0 = \frac{u^1 - u^{-1}}{2\delta t}, \quad (16)$$

which concludes $u^{-1} = u^1 - 2\delta t u_t^0$ or $u^{-1} = u^1 - 2\delta t g_2$. Therefore, the obtained recursive equation can be rewritten as

$$\{2a_0 + \beta - \gamma\theta\nabla^2\}u^1 = \{2a_0 - \gamma(1-\theta)\nabla^2\}u^0 \quad (17)$$
$$+ 2a_0\delta t u_t^0 + f^1,$$

and

$$\{a_0 + \beta - \gamma\theta\nabla^2\}u^{n+1} = \{2a_0 - \gamma(1-\theta)\nabla^2\}u^n - a_0 u^{n-1}$$

$$+ a_0\sum_{k=1}^{n-1}b_k(u^{n-k+1}-2u^{n-k}+u^{n-k-1})$$

$$+ 2a_0 b_n(u^1 - u^0 - \delta t u_t^0) + f^{n+1}, \quad (18)$$

at $n = 0$ and $n \ge 1$, respectively. Now we approximate the $u^n(x)$ by radial basis functions as follows:

$$u^n(x) = \sum_{j=1}^{N}\lambda_j^n \varphi(r_j) + \lambda_{N+1}^n x + \lambda_{N+2}^n, \quad (19)$$

where $\lambda_1,\dots,\lambda_N,\lambda_{N+1},\lambda_{N+2}$ are unknowns. So, we consider N collocation points to obtain the

values of coefficients $\lambda_k, \quad k = 1, 2, ... N + 2$ in the interpolant of $u^n(x)$ as

$$u_i^{n+1} = u^{n+1}(x_i) \simeq \sum_{j=1}^{N} \lambda_j^{n+1} \varphi(r_{ij}) + \lambda_{N+1}^{n+1} x_i + \lambda_{N+2}^{n+1}, \quad (20)$$

$$i = 1, 2, ..., N,$$

where $r_{ij} = \| x_i - x_j \|$ when $\| . \|$ is the Euclidean norm, and $\phi(\| . \|)$ is a radial function. The additional conditions can be described as

$$\sum_{j=1}^{N} \lambda_j^{n+1} = \sum_{j=1}^{N} \lambda_j^{n+1} x_j = 0. \quad (21)$$

By considering Eq. (20) together with Eq. (21) in a matrix form, we obtain

$$[u]^{n+1} = A[\lambda]^{n+1}, \quad (22)$$

where $[u]^{n+1} = [u_1^{n+1} u_2^{n+1} ... u_N^{n+1} 0 0]^T$ and $[\lambda]^{n+1} = [\lambda_1^{n+1} \lambda_2^{n+1} ... \lambda_{N+2}^{n+1}]^T$ and A is an $(N+2) \times (N+2)$ matrix given by

$$A = [a_{ij}] = \begin{bmatrix} \phi_{11} & \cdots & \phi_{1j} & \cdots & \phi_{1N} & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{i1} & \cdots & \phi_{ij} & \cdots & \phi_{iN} & x_i & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \phi_{N1} & \cdots & \phi_{Nj} & \cdots & \phi_{NN} & x_N & 1 \\ x_1 & \cdots & x_j & \cdots & x_N & 0 & 0 \\ 1 & \cdots & 1 & \cdots & 1 & 0 & 0 \end{bmatrix}. \quad (23)$$

In addition, discretization of $\nabla^2 u$ is as follows:

$$\nabla^2 u(x) = \sum_{j=1}^{N} \lambda_j^n \frac{\partial^2}{\partial x^2} \varphi(r_j) = \sum_{j=1}^{N} \psi(r_j), \quad (24)$$

where $r_j = \| x - x_j \|$. Thus, substituting the collocation points gives

$$\nabla^2 u(x_i) = \sum_{j=1}^{N} \psi(r_{ij}), \quad i = 2, ..., N - 1. \quad (25)$$

Reconstruction of Eq. (17) in the matrix form can be illustrated as follows

$$[c]^1 = B[\lambda]^1, \quad (26)$$

in which

$$B = \begin{bmatrix} L(\phi_{11}) & \cdots & L(\phi_{1j}) & \cdots & L(\phi_{1N}) & L(x_1) & L(1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L(\phi_{i1}) & \cdots & L(\phi_{ij}) & \cdots & L(\phi_{iN}) & L(x_i) & L(1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ L(\phi_{N1}) & \cdots & L(\phi_{Nj}) & \cdots & L(\phi_{NN}) & L(x_N) & L(1) \\ x_1 & \cdots & x_j & \cdots & x_N & 0 & 0 \\ 1 & \cdots & 1 & \cdots & 1 & 0 & 0 \end{bmatrix}, \quad (27)$$

where L is an operator defined by Eq. (17) as

$$L(*) = \begin{cases} (2a_0 + \beta - \gamma\theta\nabla^2)(*), & 1 < i < N, \\ (*), & i = 1 \, or \, i = N, \end{cases} \quad (28)$$

and $[c]^1 = [c_1^1, c_2^1, ..., c_N^1, 0, 0]^T$, where

$$c_i^1 = \begin{cases} h_1^1, & i = 1, \\ (2a_0 - \gamma(1-\theta)\nabla^2)u_i^0 + 2a_0 \delta t u_t^0(x_i) + f_i^1, & 1 < i < N, \\ h_2^1, & i = N. \end{cases}$$

Also, for $n \geq 1$,

$$[c]^{n+1} = B[\lambda]^{n+1}, \quad (29)$$

where L and $[c]^{n+1} = [c_1^{n+1}, c_2^{n+1}, ..., c_N^{n+1}, 0, 0]^T$, are obtained by Eq. (18) as

$$L(*) = \begin{cases} (a_0 + \beta - \gamma\theta\nabla^2)(*), & 1 < i < N, \\ (*), & i = 1 \, or \, i = N, \end{cases} \quad (30)$$

and

$$c_i^{n+1} = \left\{ 2a_0 - \gamma(1-\theta)\nabla^2 \right\} u_i^n - a_0 u_i^{n-1}$$

$$+ a_0 \sum_{k=1}^{n-1} b_k \{ (u_i^{n-k+1} - 2u_i^{n-k} + u_i^{n-k-1}) \}$$

$$+ 2a_0 b_n (u_i^1 - u_i^0 - \delta t u_t^0(x_i)) + f_i^{n+1}.$$

and finally considering boundary conditions $c_1^{n+1} = h_1^{n+1}$ and $c_N^{n+1} = h_2^{n+1}$. Obviously the method for $\alpha = 2$ coincides with the coupled method of FDM and radial basis function for integer orders (Avazzadeh et al., 2011; Dehghan and Shokri, 2009; Dehghan and shokri, 2008).

4. Numerical Results

In this section, we investigate practically the applicability and efficiency of the presented method. Thus we implement the method for solving some examples with different parameters. Clearly, increasing the number of collocation points, N, decreasing the length of time step, δt, and growth of m which is order of generalized thin plate splines (GTPS) could improve the results. It is

necessary to emphasize the increase of m does not lead to the increasing of computations or complexity of mathematical operations, but it leads to the ill-conditioning (Buhmann, 2003; Powell, 1994; Schaback, 1996; Wu and Schaback, 1993). Hence, we could increase m restrictively to avoid the ill-conditioning of the obtained system of linear equations. The reported results are obtained by using $\theta = \dfrac{1}{2}$, $m = 4$ and $N = 50$ for different n and α. In this study, the infinity norm of error function is the main criteria for evaluation of accuracy and efficiency of method.

Example 1. Consider the fractional diffusion-wave equation Eq. (6) in $[0,1]$, $\gamma = \pi$, $\beta = 1$ and $f(x,t)$ which is compatible with the exact solution as follows

$$u(x,t) = t^2 \sinh(x),$$

The semi-discrete approximated solution and error functions are shown in Fig. 1. Also, the root mean square of error(RMSE) for random points is reported in Table 1 for different value of n and α. Furthermore, the approximated value at $t = 1$, where we have the most accumulated error, for different value of n and α are reported in Table 2. Note that the differences between L_∞-error and RMS-error shown in Table 1 confirm the error accumulates when n is increasing. Also, the absolute error functions illustrated in Fig. 1 are increase with respect to t which construe accumulation of error with respect to n.
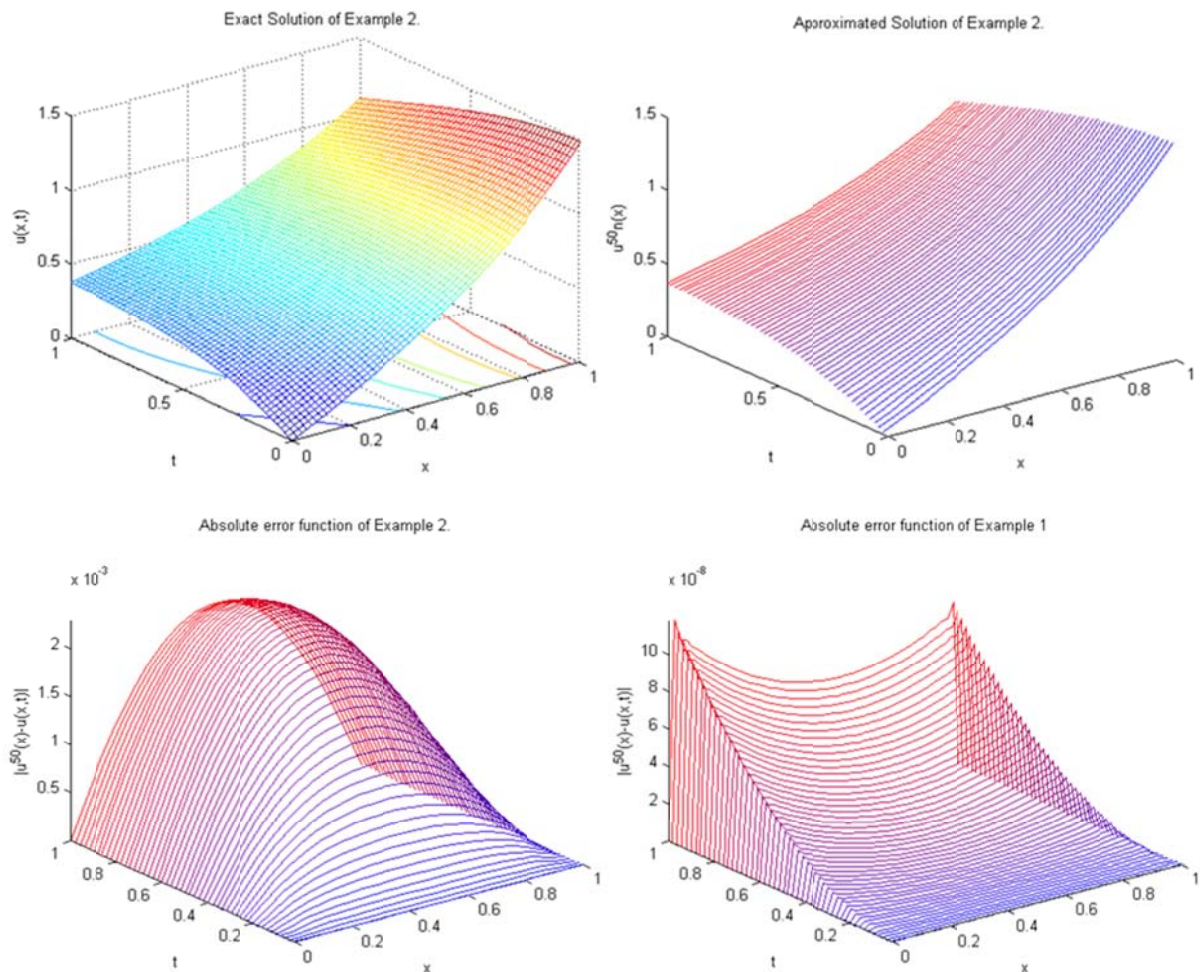

Fig. 1. Exact solutions, semi-discrete approximated functions and absolute error functions with $n = 50$ and $\alpha = 1.5$ for Examples 1 and 2 are illustrated.

Table 1. The root mean square (RMS) and infinity norm of error
function with different n and α for Example 1.

| | $\alpha = 1.25$ | | $\alpha = 1.5$ | | $\alpha = 1.75$ | | $\alpha = 1.95$ | |
n	L_{∞}-error	RMS-error	L_{∞}-error	RMS-error	L_{∞}-error	RMS-error	L_{∞}-error	RMS-error
10	1.5565 E-4	5.8833 E-07	1.5557 E-4	5.8291 E-07	1.5559 E-4	5.7850 E-07	1.5573 E-4	5.7586 E-07
20	1.5564 E-4	5.5403 E-07	1.5556 E-4	5.4850 E-07	1.5565 E-4	5.4481 E-07	1.5595 E-4	5.4277 E-07
50	1.5563 E-4	5.3333 E-07	1.5556 E-4	5.2782 E-07	1.5569 E-4	5.2526 E-07	1.5619 E-4	5.2413 E-07
100	1.5563 E-4	5.2639 E-07	1.5555 E-4	5.2092 E-07	1.5570 E-4	5.1889 E-07	1.5630 E-4	5.1812 E-07

Table 2. Numerical results for Example 1 with different n and α at $t = 1$

| | $\alpha = 1.5$ | | | $\alpha = 1.25$ | | $\alpha = 1.75$ | | Exact Solution |
x	$n = 10$	$n = 20$	$n = 50$	$n = 10$	$n = 50$	$n = 10$	$n = 50$	$u(x,1)$
0.1	0.09950933	0.09951043	0.09951107	0.09950226	0.09950368	0.09951844	0.09952297	0.10016675
0.2	0.20079972	0.20080174	0.20080291	0.20078691	0.20078954	0.20081683	0.20082524	0.20133600
0.3	0.30402685	0.30402960	0.30403116	0.30400968	0.30401323	0.30405039	0.30406191	0.30452029
0.4	0.41029949	0.41030268	0.41030448	0.41027962	0.41028373	0.41032718	0.41034071	0.41075232
0.5	0.52065615	0.52065950	0.52066138	0.52063540	0.52063970	0.52068530	0.52069953	0.52109530
0.6	0.63621617	0.63621936	0.63622115	0.63619639	0.63620048	0.63624390	0.63625745	0.63665358
0.7	0.75812244	0.75812517	0.75812672	0.75810541	0.75810893	0.75814606	0.75815761	0.75858370
0.8	0.88761746	0.88761947	0.88762062	0.88760481	0.88760740	0.88763465	0.88764309	0.88810598
0.9	1.02593218	1.02593326	1.02593389	1.02592522	1.02592662	1.02594135	1.02594589	1.02651672
1.0	1.17520119	1.17520119	1.17520119	1.17520119	1.17520119	1.17520119	1.17520119	1.17520119

Example 2. Consider the fractional diffusion-wave equation Eq.(6) in $[0,1]$, $\gamma = 1$, $\beta = 1$ and $f(x,t)$ which is compatible with the exact solution as follows

$$u(x,t) = \frac{t+x}{1+\sin(x)}\exp(x-t),$$

The approximated solution and error functions are shown in Fig. 1. Also, the error functions at $t = 0.5$ and $t = 1$ are illustrated for different α in Fig. 2 for both Examples 1 and 2. Observably the error value is often increasing when α grows up. Also, the approximated value at $t = 1$ for different n and α are reported in Table 3.

5. Conclusion

In this study, we implemented the RBF method for solving the fractional diffusion-wave equation and numerical results show the validity and accuracy of the method. According to the method, the solution is approximated continuously with respect to space direction by using the radial basis function. Obviously, the results could be improved by refinement of meshsize in the both of time and space directions. Moreover, description of the proposed method shows the method is flexible for different boundary conditions. However it is necessary to investigate the initial conditions

correspondingly to Eq. (16) in the description of method. In fact, we can modify the first and N th row of the matrix in Eq. (27) with regard to the non-classic conditions. Therefore, the method is applicable to solve the large class of different type of fractional diffusion-wave equations. At last, implementation of the proposed method leads to the only linear equations systems through the recursive equation Eq. (18). Hence, the method can be considered fast, simple and efficient.
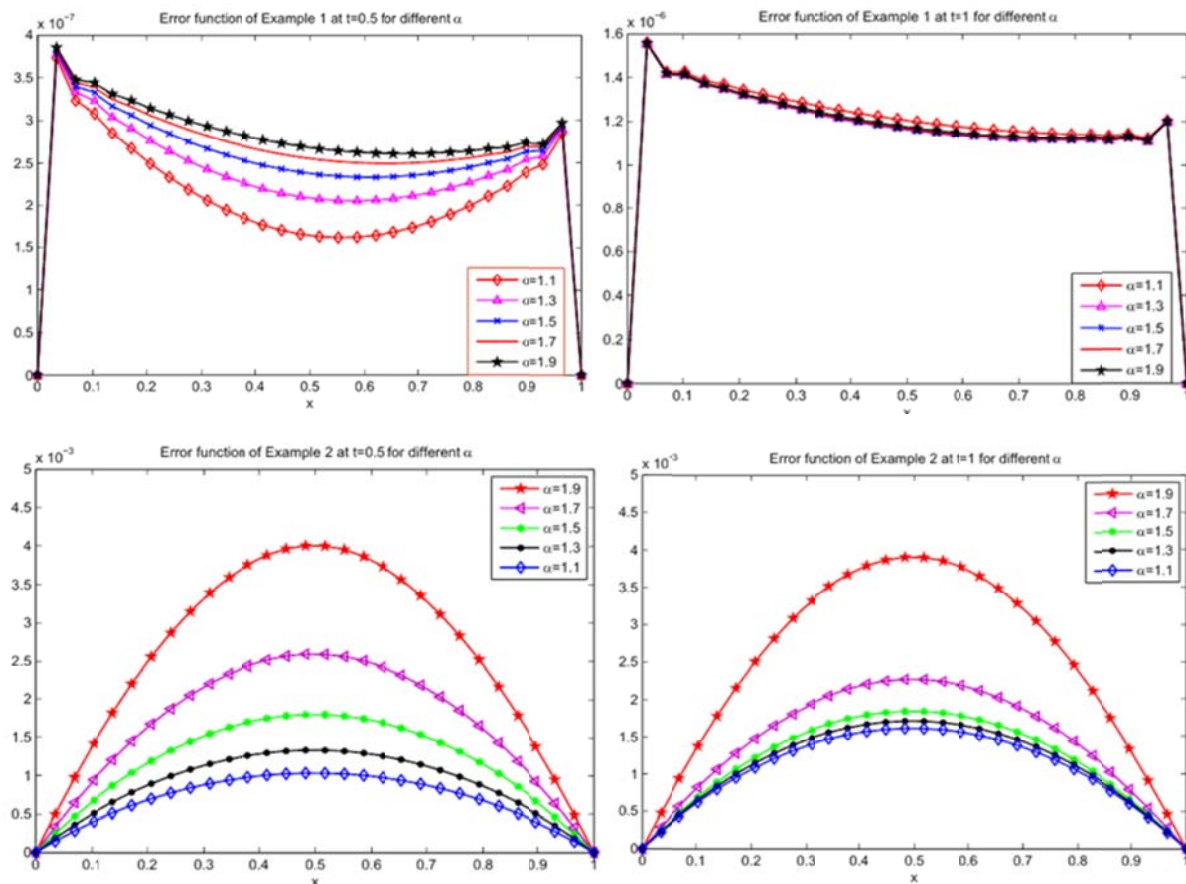
Fig. 2. Error functions for $n = 30$ and different value of α at $t = 0.5$ and $t = 1$ for Examples 1 and 2

Table 3. Numerical results for Example 2 with different n and α at $t = 1$

x	$\alpha = 1.5$			$\alpha = 1.25$		$\alpha = 1.75$		Exact Solution
	$n = 10$	$n = 20$	$n = 50$	$n = 10$	$n = 50$	$n = 10$	$n = 50$	$u(x,1)$
0.1	0.40895817	0.40720504	0.40614181	0.40864415	0.40628597	0.40999119	0.40614032	0.40663123
0.2	0.45485343	0.45177388	0.44991249	0.45424043	0.45006669	0.45680611	0.44995190	0.44982777
0.3	0.50516789	0.50114666	0.49872304	0.50430596	0.49886598	0.50786205	0.49878619	0.49830245
0.4	0.56096238	0.55638669	0.55363223	0.55993940	0.55376386	0.56412999	0.55370517	0.55299132
0.5	0.62327837	0.61853079	0.61567384	0.62220364	0.61580062	0.62660031	0.61574779	0.61496571
0.6	0.69338696	0.68884735	0.68611469	0.69237620	0.68624375	0.69652791	0.68618364	0.68546782
0.7	0.77273043	0.76877289	0.76638792	0.77188962	0.76652521	0.77537889	0.76644488	0.76595148
0.8	0.86306665	0.86006075	0.85824447	0.86247729	0.85838943	0.86496883	0.85827832	0.85813033
0.9	0.96633572	0.96463816	0.96360914	0.96603881	0.96374173	0.96733276	0.96360589	0.96403586
1.0	1.08608825	1.08608825	1.08608825	1.08608825	1.08608825	1.08608825	1.08608825	1.08608825

References

Al-Khaled, K., & Momani, S. (2005). An approximate solution for a fractional diffusion-wave equation using the decomposition method. *Applied Mathematics and Computation,* 165(2), 473–483.

Avazzadeh, Z., Beygi Rizi, Z., Maalek Ghaini, F. M., & Loghmani, G. B. (2011). A semi-discrete scheme for solving nonlinear hyperbolic-type partial integro-

differential equations using radial basis functions. *Journal of Mathematical Physics*, 52(6), 1–15.

Buhmann, M. D. (2003). *Radial Basis Functions: Theory and Implementations*. Cambridge University Press.

Chen, W., Sun, H., Zhang, X., & Korošakb, D. (2010a). Anomalous diffusion modeling by fractal and fractional derivatives. *Journal of Computers and Mathematics with Applications*, 59(5), 1754–1758.

Chen, W., Le, Y., & Sun, H. (2010b). Fractional diffusion equation by the Kansa method. *Computers & Mathematics with Application*, 59(5), 1614–1620.

Chen, J., Liu, F., Anh, V., Shen, S., Liu, Q., & Liao, C. (2012). The analytical solution and numerical solution of the fractional diffusion-wave equation with damping. *Applied Mathematics and Computation*, 219(4), 1737–1748.

Cheney, W., & Light, W. (1999). *A Course in Approximation Theory*. William Allan, New York.

Dehghan, M., & Shokri, A. (2008). A Numerical Method for Solving the Hyperbolic Telegraph Equation. *Numerical Methods for Partial Differential Equations*, 24(4), 1080–1093.

Dehghan, M., & Shokri, A. (2009). Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions. *Journal of Computational and Applied Mathematics*, 230(2), 400–410.

Ginoa, M., Cerbelli, S., & Roman, H. E. (1992). Fractional diffusion equation and relaxation in complex viscoelastic materials. *Physica A*, 191(1), 449–453.

Gu, Y. T., Zhuang, P., & Liu, F. (2010). An advanced implicit meshless approach for the nonlinear anomalous subdiffusion equation. *Computer Modeling in Engineering & Science,* 56(3), 303–334.

Hosseini, V. R., Chen, W., & Avazzadeh, Z. (2014). Numerical solution of fractional telegraph equation by using radial basis functions. *Engineering Analysis with Boundary Elements*, 38(1), 31–39.

Jafari, H., & Daftardar-Gejji, V. (2006). Solving, linear and nonlinear fractional diffusion and wave equations by Adomian decomposition. *Applied Mathematics and Computation,* 180(2), 488–497.

Jafari, H., & Seifi, S. (2009). Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation. *Commun Nonlinear Sci Numer Simulat*, 14(5), 2006–2012.

Jiang, H., Liu, F., Turner, I., & Burrage, K. (2012). Analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations in a finite domain. *Computers & Mathematics with Applications*, 64(10), 3377–3388.

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Application of Fractional Differential Equations*. Elsevier, Amsterdam.

Li, X., & Xu, C. (2009). A space-time spectral method for the time fractional diffusion equation. *SIAM Journal on Numerical Analysis*, 47(3), 2108–2131.

Li, C., Zhao, Z., & Chen, Y. (2011). Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion. *Computers & Mathematics with Applications*, 62(3), 855–875.

Liu, F., Zhuang, P., Anh, V., & Turner, I. (2006). A fractional-order implicit difference approximation for the space time fractional diffusion equation. *ANZIAM Journal*, 47, 48–68.

Mohebbi, A., Abbaszadeh, M., & Dehghan, M. (2013). The use of a meshless technique based on collocation and radial basis functions for solving the time fractional nonlinear Schrödinger equation arising in quantum mechanics. *Engineering Analysis with Boundary Elements*, 37(2), 475–485.

Piret, C., & Hanert, E. (2013). A radial basis functions method for fractional diffusion equations. *Journal of Computational Physics*, 238(4), 71–81.

Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, San Diego, CA.

Powell, M. J. D. (1994). The uniform convergence of thin plate spline interpolation in two dimensions. *Numerische Mathematik*, 68(1), 107–128.

Schaback, R. (1996). Approximation by radial basis functions with finitely many centers. *Constructive Approximation*, 12(3), 331–340.

Wu, Z. M., & Schaback, R. (1993). Local error estimates for radial basis function interpolation of scattered data. *IMA Journal of Numerical Analysis*, 13(2), 13–27.

Zhao, J., Xiao, J., & Xu, Y. (2013). Stability and convergence of an effective finite element method for multi term fractional partial differential equations. *Abstract and Applied Analysis*, doi:10.1155/2013/857205.

Zhoa, Z., & Li, C. (2012). Fractional difference/finite element approxmation for the time-space fractional telegraph equation. *Journal of Applied Mathematices and Computation*, 219(6), 2975–2988.

Zhuang, P., Gu, Y. T., Liu, F., Turner, I., & Yarlagadda, P. K .D. V. (2011). Time-dependent fractional advection-diffusion equations by an implicit MLS meshless method. *International Journal for Numerical Methods in Engineering*, 88(13), 1346–1362.