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Generalized Strong Vector Equilibrium-Like Problems in Banach Spaces[1]

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Abstract

The purpose of this paper is to study the solvability for a class of generalized strong vector equilibrium-like problems in reflexive Banach spaces. Firstly, utilizing Brouwer's fixed point theorem, we prove the solvability for this class of generalized strong vector equilibrium-like problems without monotonicity assumption. Secondly, we introduce the new concept of pseudomonotonicity for vector set-valued mapping and prove the solvability for this class of generalized strong vector equilibrium-like problems for pseudomonotone vector set-valued mapping by using Fan's lemma and Nadler's theorem. Our results extend and improve the corresponding results in this direction.

Keywords: Generalized strong vector equilibrium-like problems; Brouwer's fixed point theorem; Fan's lemma; Nadler's lemma; Pseudomonotone; Solvability.

1 Introduction

In 1980, Giannessi [9] first introduced and studied the vector variational inequality (VVI) in a finite-dimensional Euclidean space, which is a vector-valued version of the variational inequality of Hartman and Stampacchia. Subsequently, many authors investigated vector variational inequalities in abstract spaces and extended vector variational

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inequalities to vector equilibrium problems. As a natural generalization of the vector equilibrium problem, the generalized vector equilibrium-like problem includes various problems, for example, generalized vector variational inequality problem, generalized vector variational-like inequality problem, generalized vector complementarity problem and vector equilibrium problem as special cases, see references [14], [13], [10], [12], [3], [15], [5], and the references therein. Furthermore, as Chen and Hou [2] pointed out that most of the research results in this area touch upon a weak version of VVI and its generalizations and the existence of solutions for strong vector variational inequalities is still an open problem. Later on, Fang and Huang [7] obtained some existences for a class of strong vector variational inequalities (SVVI) and partly answered the open problem proposed by Chen and Hou [2]. Recently, Ceng et al. [4] introduced and studied the solvability for a class of generalized strong vector variational-like inequalities (GSVVLI) in reflexive Banach spaces which include the class of strong vector variational inequalities studied by Fang and Huang in [8] and proved the solvability for the class of generalized strong vector variational-like inequalities by making use of Brouwer's fixed pointed theorem and Fan's and Nadler's Lemmas under without monotonicity assumption and with the pseudomonotonicity of vector multifunctions introducecd by them, respectively.

Motivated and inspired by the above research works, in this paper, we introduce and study a class of generalized strong vector equilibrium-like problems (short for GSVELP). There is no doubt that the class of the GSVELP is more general and includes the GSVVLI which is considered by Ceng et al. in [4] and the SVVI studied by Fang and Huang in [8] as special cases. Firstly, the solvability of the GSVELP without monotonicity is derived by using Brouwer's fixed point theorem. Secondly, we introduce the new concept of pseudomonotonicity and prove the solvability of the GSVELP with the pseudomonotonicity by exploiting Fan's and Nadler's lemmas. The results presented in this paper extend and unify the corresponding results of [4], [8].

2 Preliminaries

Let X be a real reflexive Banach space and let Y be a real Banach space. Let $D \subset X$ be a nonempty, bounded, closed and convex set, let $C \subset Y$ be a nonempty and convex cone with apex at the origin with int$C \neq \emptyset$.

Given C in Y, we can define relations " $\leq_C$ " and " $\not\leq_C$ " as follows:

$$u \leq_C v \Leftrightarrow v - u \in C, u \not\leq_C v \Leftrightarrow v - u \notin C.$$

If " $\leq_C$ " is a partial order, then $(Y, \leq_C)$ is called a Banach space ordered by C. Let $L(X, Y)$ denote the space of all continuous linear maps from X into Y. Given the mappings $T : D \to 2^{L(X,Y)}, A : L(X, Y) \to L(X, Y), f : L(X, Y) \times D \times D \to Y$ and $h : D \to Y$, we consider the generalized strong vector equilibrium-like problem (GSVELP) as follows:

find $u_0 \in D, s_0 \in Tu_0$ such that

$$f(As_0, u_0, v) + h(v) - h(u_0) \notin -(C\backslash\{0\}), \forall\, v \in D.$$

In particular, if we put $f(u, v, w) =< u, \eta(w, v) >$ for all $(u, v, w) \in L(X, Y)\times D\times D$, where $\eta : D \times D \to X$, then the above problem reduces to the following generalized strong vector variational-like inequality problem (GSVVLIP) :

find $u_0 \in D, s_0 \in Tu_0$ such that

$$< As_0, \eta(v, u_0) > +h(v) - h(u_0) \notin -(C\backslash\{0\}), \forall\, v \in D,$$

which was studied by Ceng et al. [9].

If $h(u) = 0, \eta(v, u) = v - u$ for all $u, v \in D, A = I$ the identity mapping of $L(X, Y)$ and T is a single-valued mapping, then the GSVELP reduces to the following strong vector variational inequality problem (SVVIP):

find $u_0 \in D$ such that

$$< Tu_0, v - u_0 > \notin -(C\backslash\{0\}), \forall v \in D,$$

which was considered by Fang and Huang [14].

Remark 2.1 *The GSVELP is more general which is a motivation of our writing the paper.*

Now, we recall some definitions and lemmas.

Definition 2.2 *A map $h : D \to Y$ is said to be convex if*

$$h(\lambda u + (1-\lambda)v) \leq_C \lambda h(u) + (1-\lambda)h(v), \forall\, u, v \in D, \lambda \in [0,1].$$

Definition 2.3 *$f(u, v, w)$ is affine with respect to w if, for any given $u, v \in D$,*

$$f(u, v, tw_1 + (1-t)w_2) = tf(u, v, w_1) + (1-t)f(u, v, w_2), \forall\, w_1, w_2 \in D, t \in R,$$

with $w = tw_1 + (1-t)w_2 \in D$.

Definition 2.4 *Let $A : L(X,Y) \to L(X,Y), h : D \to Y$ and $f : L(X,Y) \times D \times D \to Y$ be three mappings. A nonempty compact-valued mapping $T : D \to 2^{L(X,Y)}$ is said to be f-pseudomonotone with respect to A and h if for each $u, v \in D$, the existence of $t \in Tu$ such that*

$$f(At, u, v) + h(v) - h(u) \notin -(C \backslash \{0\}),$$

implies that

$$f(As, u, v) + h(v) - h(u) \in C, \forall\, s \in Tv.$$

Lemma 2.5 *[11] Let $(X, \|\cdot\|)$ be a normed vector space and H be the Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X, induced by a metric d in terms of $d(u,v) = \|u - v\|$, which is defined by*

$$H(U,V) = \max(\sup_{u \in U} \inf_{v \in V} \|u - v\|, \sup_{v \in V} \inf_{u \in U} \|u - v\|),$$

for U and V in $CB(X)$. If U and V are compact sets in X, then for each $u \in U$, there exists $v \in V$ such that

$$\|u - v\| \leq H(U,V).$$

Definition 2.6 *A nonempty compact-valued mapping $T : D \to 2^{L(X,Y)}$ is called H -uniformly continuous if for any given $\epsilon > 0$, there exists $\delta > 0$ such that for any $u, v \in D$ with $\|u - v\| < \delta$, there holds*

$$H(Tu, Tv) < \epsilon,$$

where H is the Hausdorff metric defined on $CB(L(X, Y))$.

3 Solvability of the GSVELP without monotonicity

In this section, we will derive the solvability of the GSVELP without monotonicity assumption by using Brouwer's fixed point theorem. First, recall the following result.

Lemma 3.1 *(Brouwer's fixed point theorem [1]) Let B be a nonempty, compact and convex subset of a finite-dimensional space and let $g : B \to B$ be a continuous mapping. Then there exists $u \in B$ such that $g(u) = u$.*

Now, we state and prove two theorems for the existence results to the GSVELP. It is worth pointing that there are no assumptions of pseudomonotonicity in our existence results.

Theorem 3.2 *Let D be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and let Y be a real Banach space ordered by a nonempty and convex cone C with apex at the origin and $intC \neq \emptyset$. Let $h : D \to Y$ and $A : L(X, Y) \to L(X, Y)$ be two mappings such that h is convex, and let $f : L(X, Y) \times D \times D \to Y$ be such that $(a) f(\cdot, u, v) = f(\cdot, u, w) + f(\cdot, w, v)$, for all $u, v, w \in D$, and $(b) f(\cdot, \cdot, \cdot)$ is affine in the third variable. Suppose that for given set-valued mapping $T : D \to 2^{L(X,Y)}$, the set $\{u \in D : f(At, u, v) + h(v) - h(u) \in -(C\backslash\{0\}),$ for all $t \in Tu\}$ is weakly open in D for every $v \in D$. Then the GSVELP has a solution.*

Proof *First, notation that condition (a) implies that for each $u, v \in D$,*

$$f(\cdot, u, u) = 0.$$

If the GSVELP does not have a solution, then for every $u_0 \in D$, there exists some $v \in D$ such that

$$f(At, u_0, v) + h(v) - h(u_0) \in -(C\backslash\{0\}), \forall\, t \in Tu_0. \tag{1}$$

For every $v \in D$, define the set N_v as follows:

$$N_v = \{u \in D : f(At, u, v) + h(v) - h(u) \in -(C\backslash\{0\}), \forall\, t \in Tu\}.$$

By the assumption, the set N_v is weakly open in D for every $v \in D$. It is easy to see that the family $\{N_v : v \in D\}$ is an open cover of D in the weak topology of X.

The weak compactness of D implies that there exists a finite set $\{v_1, v_2, \cdots, v_n\} \subseteq D$ such that

$$D = \bigcup_{i=1}^{n} N_{v_i}.$$

Hence there exists a continuous (in the weak topology of X) partition of unity $\{\beta_1, \beta_2, \cdots, \beta_n\}$ subordinated to $\{N_{v_1}, N_{v_2}, \cdots, N_{v_n}\}$ such that $\beta_j(u) \geq 0$, for all $u \in D, j = 1, 2, \cdots, n, \sum_{j=1}^{n} \beta_j(u) = 1, \forall\, u \in D$ and

$$\beta_j(u) \begin{cases} = 0 \quad where \quad u \notin N_{v_j}, \\ > 0 \quad where \quad u \in N_{v_j}. \end{cases}$$

Let $p : D \to X$ be defined as follows:

$$p(u) = \sum_{j=1}^{n} \beta_j(u) v_j, \forall\, u \in D.$$

Since β_j is continuous in the weak topology of X for each j, p is continuous in the weak topology of X. Let $S = co\{v_1, v_2, \cdots, v_n\}$ be the convex hull of $\{v_1, v_2, \cdots, v_n\}$ in D. Then S is a simplex of a finite-dimensional space and p maps S into S. By Lemma 3.1, there exists some $u_0 \in S$ such that $p(u_0) = u_0$. Now for any given $u \in D$, let

$$d(u) = \{j : u \in N_{v_j}\} = \{j : \beta_j(u) > 0\}.$$

Obviously, $d(u) \neq \emptyset$.

Since $u_0 \in S \subseteq D$ is a fixed point of p, we have $p(u_0) = \sum_{j=1}^{n} \beta_j(u_0)v_j$ and hence from the definition of N_v and the convexity of h we derive for each $t \in Tu_0$,

$$\begin{aligned} 0 &= f(At, u_0, u_0) + h(u_0) - h(u_0) \\ &= f(At, u_0, p(u_0)) + h(p(u_0)) - h(u_0) \\ &= f(At, u_0, \textstyle\sum_{j=1}^{n} \beta_j(u_0)v_j) + h(\sum_{j=1}^{n} \beta_j(u_0)v_j) - h(u_0) \\ &\leq_C \textstyle\sum_{j=1}^{n} \beta_j(u_0) f(At, u_0, v_j) + \sum_{j=1}^{n} \beta_j(u_0) h(v_j) - h(u_0) \\ &= \textstyle\sum_{j=1}^{n} \beta_j(u_0)[f(At, u_0, v_j) + h(v_j) - h(u_0)] \leq_{C\backslash\{0\}} 0, \end{aligned}$$

which leads to a contradiction. Therefore, there exist $u^* \in D$ and $s_0 \in Tu^*$ such that

$$f(As_0, u^*, v) + h(v) - h(u^*) \notin -(C\backslash\{0\}), \forall\, v \in D.$$

This completes the proof.

Theorem 3.3 *Let D be a nonempty, closed and convex subset of a real reflexive Banach space X with $0 \in D$ and let Y be a real Banach space. Let $C \subset Y$ be a point and convex cone with $intC \neq \emptyset$. Let $h : D \to Y$ and $A : L(X,Y) \to L(X,Y)$ be two mappings such that h is convex, and let $f : L(X,Y) \times D \times D \to Y$ be such that $(a) f(\cdot, u, v) = f(\cdot, u, w) + f(\cdot, w, v)$, for all $u, v, w \in D$, and $(b) f(\cdot, \cdot, \cdot)$ is affine in the third variable. Suppose that for given set-valued mapping $T : D \to 2^{L(X,Y)}$, there exists some $r > 0$ such that the following conditions:*

(i) for every $v \in D \cap B_r$, the set $\{u \in D \cap B_r : f(At, u, v) + h(v) - h(u) \in -(C\backslash\{0\})$, for all $t \in Tu\}$ is weakly open in D, where $B_r = \{u \in X : \|u\| \leq r\}$;

(ii) $f(As, 0, v) + h(v) - h(0) \in C\backslash\{0\}$, for all $s \in Tv, v \in D$ with $\|v\| = r$.

Then the GSVELP has a solution.

Proof *First, observe that condition (a) implies that for each $u, v \in D$,*

$$f(\cdot, u, u) = 0, f(\cdot, u, v) + f(\cdot, v, u) = 0.$$

Moreover, according to Theorem 3.1 there exist $u_r \in D \cap B_r$ and $t_r \in Tu_r$ such that

$$f(At_r, u_r, v) + h(v) - h(u_r) \notin -(C\backslash\{0\}), \forall\, v \in D \cap B_r. \tag{2}$$

Putting $v = 0$ in the above inequality, one has

$$f(At_r, u_r, 0) + h(0) - h(u_r) \notin -(C\backslash\{0\}),$$

which implies that

$$f(At_r, 0, u_r) + h(u_r) - h(0) \notin C\backslash\{0\}. \tag{3}$$

Combining condition (ii) with (3.3), we know that $\|u_r\| < r$. For any $w \in D$, choose $\lambda \in (0,1)$ enough small such that $(1-\lambda)u_r + \lambda w \in D \cap B_r$. Putting $v = (1-\lambda)u_r + \lambda w$ in (3.2), one has

$$f(At_r, u_r, (1-\lambda)u_r + \lambda w) + h((1-\lambda)u_r + \lambda w) - h(u_r) \notin -(C\backslash\{0\}). \tag{4}$$

Since h is convex and $f(\cdot,\cdot,\cdot)$ is affine in the third variable, we have

$$\begin{aligned} & f(At_r, u_r, (1-\lambda)u_r + \lambda w) + h((1-\lambda)u_r + \lambda w) - h(u_r) \\ \leq_C \ & (1-\lambda)f(At_r, u_r, u_r) + \lambda f(At_r, u_r, w) + (1-\lambda)h(u_r) + \lambda h(w) - h(u_r) \\ = \ & \lambda[f(At_r, u_r, w) + h(w) - h(u_r)]. \end{aligned}$$

Now we claim that

$$f(At_r, u_r, w) + h(w) - h(u_r) \notin -(C\backslash\{0\}), \forall\, w \in D. \tag{5}$$

Indeed, suppose to the contrary that

$$f(At_r, u_r, w_0) + h(w_0) - h(u_r) \in -(C\backslash\{0\}),$$

for some $w_0 \in D$. Since $-(C\backslash\{0\})$ is a convex cone, we have

$$\lambda[f(At_r, u_r, w_0) + h(w_0) - h(u_r)] \in -(C\backslash\{0\}).$$

Observe that

$$\begin{aligned} & f(At_r, u_r, (1-\lambda)u_r + \lambda w_0) + h((1-\lambda)u_r + \lambda w_0) - h(u_r) \\ = \ & f(At_r, u_r, (1-\lambda)u_r + \lambda w_0) + h((1-\lambda)u_r + \lambda w_0) - h(u_r) \\ - \ & \lambda[f(At_r, u_r, w_0) + h(w_0) - h(u_r)] + \lambda[f(At_r, u_r, w_0) + h(w_0) - h(u_r)] \\ \in \ & -C - (C\backslash\{0\}) = -(C\backslash\{0\}), \end{aligned}$$

which implies that

$$f(At_r, u_r, (1-\lambda)u_r + \lambda w_0) + h((1-\lambda)u_r + \lambda w_0) - h(u_r) \in -(C\backslash\{0\}).$$

This contradicts (3.4). Therefore, (3.5) holds, that is, u_r is a solution of the GSVELP. This completes the proof.

Remark 3.4 *Theorems 2.2 and 2.3 in [4] and Theorem [8] are special cases of Theorems 3.1 and 3.2.*

4 Solvability of the GSVELP with pseudomonotonicity

In this section, we will prove the solvability of the GSVELP with pseudomonotonicity assumption by using Fan's and Nadler's lemmas. First we give some definitions and lemmas.

Definition 4.1 *Let D be a nonempty subset of a topological vector space E. A multivalued map $G : D \to 2^E$ is called a KKM-map if for each finite subset $\{u_1, u_2, \cdots, u_n\} \subseteq D$,*

$$co\{u_1, u_2, \cdots, u_n\} \subseteq \bigcup_{i=1}^{n} G(u_i),$$

where $co\{u_1, u_2, \cdots, u_n\}$ denotes the convex hull of $\{u_1, u_2, \cdots, u_n\}$.

Lemma 4.2 *(Fan's lemma [6]) Let D be an arbitrary nonempty subset of a Hausdorff topological vector space E. Let the multivalued mapping $G : D \to 2^E$ be a KKM-map such that $G(u)$ is closed for all $u \in D$ and $G(v)$ is compact for at least one $v \in D$. Then*

$$\bigcap_{u \in D} G(u) \neq \emptyset.$$

Lemma 4.3 *Let D be a nonempty and convex subset of a real Banach space X and let Y be a real Banach space. Let $C \subset Y$ be a closed, pointed and convex cone with $intC \neq \emptyset$. Let $h : D \to Y$ be convex, and let $A : L(X, Y) \to L(X, Y)$ be continuous.*

Suppose that $f : L(X,Y) \times D \times D \to Y$ *satisfied that* $(a) f(\cdot, u, u) = 0$, *for all* $u \in D$, *and* $(b) f(\cdot,\cdot,\cdot)$ *is continuous in the first variable and is affine in the third variable. Let* $T : D \to 2^{L(X,Y)}$ *be a set-valued mapping which is* H*-uniformly continuous and* f*-pseudomonotone with respect to* A *and* h. *Then the following are equivalent:*

(i) there exist $u^* \in D$ *and* $s^* \in Tu^*$ *such that*

$$f(As^*, u^*, v) + h(v) - h(u^*) \notin -(C\backslash\{0\}), \forall\, v \in D;$$

(ii) there exists $u^* \in D$ *such that*

$$f(At, u^*, v) + h(v) - h(u^*) \in C, \forall\, v \in D, t \in Tv.$$

Proof *Suppose that there exist* $u^* \in D$ *and* $s^* \in Tu^*$ *such that*

$$f(As^*, u^*, v) + h(v) - h(u^*) \notin -(C\backslash\{0\}), \forall\, v \in D.$$

Since T *is* f*-pseudomonotone with respect to* A *and* h,

$$f(At, u^*, v) + h(v) - h(u^*) \in C, \forall\, v \in D, t \in Tv.$$

Conversely, suppose that there exists $u^* \in D$ *such that*

$$f(At, u^*, v) + h(v) - h(u^*) \in C, \forall\, v \in D, t \in Tv.$$

For any given $v \in D$, *we know that* $v_\lambda = \lambda v + (1-\lambda)u^* \in D$, *for all* $\lambda \in (0,1)$ *since* D *is convex. Replacing* v *by* v_λ *in the above inequality, in views of the affinity of* f *with respect to the third variable and the convexity of* h, *one derives for each* $t_\lambda \in Tv_\lambda$,

$$\begin{aligned}
0 \;&\leq_C\; f(At_\lambda, u^*, v_\lambda) + h(v_\lambda) - h(u^*) \\
&=\; f(At_\lambda, u^*, \lambda v + (1-\lambda)u^*) + h(\lambda v + (1-\lambda)u^*) - h(u^*) \\
&\leq_C\; \lambda f(At_\lambda, u^*, v) + (1-\lambda) f(At_\lambda, u^*, u^*) + \lambda h(v) + (1-\lambda)h(u^*) - h(u^*) \\
&=\; \lambda[f(At_\lambda, u^*, v) + h(v) - h(u^*)].
\end{aligned}$$

Hence, we have

$$f(At_\lambda, u^*, v) + h(v) - h(u^*) \in C, \forall\, t_\lambda \in Tv_\lambda, \lambda \in (0,1). \tag{6}$$

Since Tv_λ and Tu^* are compact, it follows from Lemma 2.4 that for each fixed $t_\lambda \in Tv_\lambda$ there exists an $s_\lambda \in Tu^*$ such that

$$\|t_\lambda - s_\lambda\| \leq H(Tv_\lambda, Tu^*).$$

Since Tu^* is compact, without loss of generality, we may assume that $s_\lambda \to s^* \in Tu^*$ as $\lambda \to 0^+$. Since T is H -uniformly continuous and $\|v_\lambda - u^*\| = \lambda\|v - u^*\| \to 0$ as $\lambda \to 0^+$, so $H(Tv_\lambda, Tu^*) \to 0$ as $\lambda \to 0^+$. Thus one has

$$\|t_\lambda - s^*\| \leq \|t_\lambda - s_\lambda\| + \|s_\lambda - s^*\| \leq H(Tv_\lambda, Tu^*) + \|s_\lambda - s^*\| \to 0 \quad as \quad \lambda \to 0^+.$$

Hence, we have $t_\lambda \to s^*$ as $\lambda \to 0^+$. It follows from the continuity of f in the first variable that

$$\|f(At_\lambda, u^*, v) - f(As^*, u^*, v)\| \to 0 \quad as \quad \lambda \to 0^+.$$

Since C is closed, pointed and convex cone, according to (4.1), we have

$$f(As^*, u^*, v) + h(v) - h(u^*) \in C.$$

Therefore,

$$f(As^*, u^*, v) + h(v) - h(u^*) \notin -(C\backslash\{0\}).$$

This completes the proof.

Now we will apply Lemma 4.2 to prove the existence of solution for the GSVELP with pseudomonotonicity assumption.

Theorem 4.4 *Let D be a nonempty, compact and convex subset of a real Banach space X and let Y be a real Banach space. Let $C \subset Y$ be a closed, pointed and convex cone with $intC \neq \emptyset$. Let $h : D \to Y$ be convex and continuous, and let $A : L(X,Y) \to L(X,Y)$ be continuous. Let $f : L(X,Y) \times D \times D \to Y$ be such that $(a) f(\cdot, u, v) = f(\cdot, u, w) + f(\cdot, w, v)$, for all $u, v, w \in D$, and $(b) f(\cdot, \cdot, \cdot)$ is affine and continuous in the third variable and is continuous in the first variable. Let $T : D \to 2^{L(X,Y)}$ be a*

compact-valued mapping which is H-uniformly continuous and f-pseudomonotone with respect A and h. Then the GSVELP has a solution.

Proof *It follows that condition (a) we have for each $u, v \in D$,*

$$f(\cdot, u, u) = 0, f(\cdot, u, v) + f(\cdot, v, u) = 0.$$

We define two multivalued maps $F, G : D \to 2^D$ as follows:

$$\begin{aligned} F(v) = &\quad \{u \in D : f(At, u, v) + h(v) - h(u) \notin -(C\backslash\{0\}) \quad for \quad some \quad t \in Tu\}, \forall\, v \in D, \\ G(v) = &\quad \{u \in D : f(As, u, v) + h(v) - h(u) \in C, \forall\, s \in Tv\}, \forall\, v \in D. \end{aligned} \tag{7}$$

Obviously, both $F(v)$ and $G(v)$ are nonempty since $v \in F(v) \cap G(v)$ for all $v \in D$. We claim that F is a KKM mapping. If this is false, then there exists a finite set $\{v_1, v_2, \cdots, v_n\} \subseteq D$ and $\alpha_i \geq 0, i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \alpha_i = 1$ such that

$$\overline{v} = \sum_{i=1}^{n} \alpha_i v_i \notin \bigcup_{i=1}^{n} F(v_i).$$

Hence for any $\bar{t} \in T\overline{v}$ one has

$$f(A\bar{t}, \overline{v}, v_i) + h(v_i) - h(\overline{v}) \in -(C\backslash\{0\}), i = 1, 2, \cdots, n.$$

Since $f(\cdot, \cdot, \cdot)$ is affine in the third variable and h is convex, it follows that

$$\begin{aligned} 0 &= f(A\bar{t}, \overline{v}, \overline{v}) + h(\overline{v}) - h(\overline{v}) \\ &= f(A\bar{t}, \overline{v}, \textstyle\sum_{i=1}^n \alpha_i v_i) + h(\sum_{i=1}^n \alpha_i v_i) - h(\overline{v}) \\ &\leq_C \textstyle\sum_{i=1}^n \alpha_i f(A\bar{t}, \overline{v}, v_i) + \sum_{i=1}^n \alpha_i h(v_i) - h(\overline{v}) \\ &= \textstyle\sum_{i=1}^n \alpha_i [f(A\bar{t}, \overline{v}, v_i) + h(v_i) - h(\overline{v})] \leq_{C\backslash\{0\}} 0, \end{aligned}$$

which leads to a contradiction. So F is a KKM mapping. Furthermore, it is clear that $F(v) \subseteq G(v)$ for every $v \in D$ since T is f-pseudomonotone with respect to A and h. Thus, G is also a KKM mapping. Now we claim that $G(v) \subseteq D$ is closed. Indeed, suppose $\{u_n\} \subseteq G(v)$ is a sequence such that u_n converges to $\overline{u} \in D$. Then we derive for each $t \in Tv$,

$$-[f(At, v, u_n) + h(u_n) - h(v)] = f(At, u_n, v) + h(v) - h(u_n) \in C, \forall\, n.$$

Since $h : D \to Y$ be convex and continuous and $f(\cdot,\cdot,\cdot)$ is continuous in the third variable, hence we have

$$-[f(At, v, u_n) + h(u_n) - h(v)] \to -[f(At, v, \overline{u}) + h(\overline{u}) - h(v)] \quad as \quad n \to \infty.$$

Also, since C is closed,

$$-[f(At, v, \overline{u}) + h(\overline{u}) - h(v)] \in C.$$

Thus we get

$$f(At, \overline{u}, v) + h(v) - h(\overline{u}) \in C, \forall\, t \in Tv,$$

and so $\overline{u} \in G(v)$. This shows that $G(v)$ is closed for each $v \in D$. Since $D \subseteq X$ is compact and so is $G(v)$. According to Lemma 4.1,

$$\bigcap_{v \in D} G(v) \neq \emptyset.$$

This implies that there exists $u^* \in D$ such that

$$f(At, u^*, v) + h(v) - h(u^*) \in C, \forall\, v \in D, t \in Tv.$$

Therefore, by Lemma 4.2 we know that the GSVELP has a solution.

Theorem 4.5 *Let D be a nonempty, unbounded, closed and convex subset of a real Banach space X with $0 \in D$ and let Y be a real Banach space. Let $C \subset Y$ be a closed, pointed and convex cone with $intC \neq \emptyset$. Let $h : D \to Y$ be convex and continuous and let $A : L(X,Y) \to L(X,Y)$ be continuous. Let $f : L(X,Y) \times D \times D \to Y$ be such that $(a) f(\cdot, u, v) = f(\cdot, u, w) + f(\cdot, w, v)$, for all $u, v, w \in D$, and $(b) f(\cdot,\cdot,\cdot)$ is affine and continuous in the third variable and is continuous in the first variable. Let $T : D \to 2^{L(X,Y)}$ be a compact-valued mapping which is H-uniformly continuous and f-pseudomonotone with respect to A and h. If there exists some $r > 0$ such that*

$$f(At, 0, v) + h(v) - h(0) \in C \backslash \{0\}, \forall\, t \in Tv, v \in D \quad with \quad \|v\| = r,$$

then the GSVELP is solvable.

Proof *According to Theorem 4.1 there exist* $u_r \in D \cap B_r$ *and* $s_r \in Tu_r$ *such that*

$$f(As_r, u_r, v) + h(v) - h(u_r) \notin -(C\backslash\{0\}), \forall\, v \in D \cap B_r.$$

Since the remainder of the proof is similar to that of Theorem 3.2, we omit it. This completes the proof.

Remark 4.6 *Theorems 4.1 and 4.2 extend Theorems 3.3 and 3.4 in [4].*

References

[1] Brouwer L.E.J. (1911) "Beweis der invarianz des n-dimensionalen gebiets," Mathematische Annalen, 71(3), 305-313.

[2] Chen G.Y., Hou S.H. "Existence of solutions for vector variational inequalities," in Vector Variational Inequalities and Vector Equilibria, F. Giannessi, Ed., vol. 38 of Nonconvex Optimization and its Applications, pp. 73-86, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000 .

[3] Ceng L.C., Guu S.M., Yao J.C. "Generalized vector equilibrium-like problems without pseudomonotonicity in Banach spaces," Journal of Inequalities and Applications, doi: 10.1155/2007/61794.

[4] Ceng L.C., Lin Y.C., Yao J.C. "On generalized strong vector variational-like inequalities in Banach spaces," Journal of Inequalities and Applications, doi:10.1155/2007/94092.

[5] Ceng L.C., Yao J.C. (2006) "An existence result for generalized vector equilibrium problems without pseudomonotonicity," Applied Mathematics Letters, 19(12), 1320-1326.

[6] Fan K. (1961) "A generalization of Tychonoff's fixed point theorem," Mathematische Annalen, 142(3), 305-310.

[7] Fang Y.P., Huang N.J. (2002) "On the strong vector variational inequalities," Research Report, Department of Mathematics, Sichuan University, Sichuan, China.

[8] Fang Y.P., Huang N.J. (2006) "Strong vector variational inequalities in Banach spaces," Applied Mathematics Letters, 19, 362-368.

[9] Gianessi F. "Theorems of alternative, quadratic programs and complementarity problems," in Variational Inequalities and Complementarity Problems (Proc. Internat. School, Erice, 1978), R.W.Cottle, F.Giannessi, and J.-L. Lions, Eds, pp. 151-186, John Wiley and Sons, Chichester, UK, 1980.

[10] Konnov I.V., Yao J.C. (1997) "On the generalized vector variational inequality problem," Journal of Mathematical Analysis and Applications, 206(1), 42-58.

[11] Nadler Jr. S.B. (1969) "Multi-valued contraction mappings," Pacific Journal of Mathematics, 30, 475-488.

[12] Yang X.Q. (1993) "Vector complementarity and minimal element problems," Journal of Optimization Theory and Applications, 77(3), 483-495.

[13] Yang X.Q. (1997) "Vector variational inequality and vector pseudolinear optimization," Journal of Optimization Theory and Applications, 95(3), 729-734.

[14] Yang X.Q., Goh C.J. (1997) "On vector variational inequalities: application to vector equilibria," Journal of Optimization Theory and Applications, 95(2), 431-443.

[15] Zhao Y.L., Quan X.Z. (2006) "On the existence of solutions to generalized vector variational-like inequalities," Nonlinear Analysis: Theory, Methods and Applications, 64, 2075-2083.