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Dual normal *BCK*-algebras

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Abstract

In this paper, by considering the notions of dual right and dual left stabilizers in bounded BCK-algebras, we obtain some related results. After that we investigate the relationship between the dual left(right) stabilizers and dual ideals in bounded BCK-algebras. Then we define a class of special bounded BCK-algebras called dual normal BCK-algebras. Finally we prove that the dual semisimple bounded BCK-algebras and dual J-semisimple bounded BCK-algebras are all dual normals.

Keywords BCK-algebra, Bounded BCK-algebra, Dual right and dual left stabilizers, Dual normal BCK-algebra, Dual semisimple BCK-algebra.
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1 Introduction

The study of BCK-algebras was initiated by Y. Imai and K-Iseki [3] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK-algebras. Dual ideals are important in bounded BCK-algebras. In 1986, the notion of dual ideals in bounded BCK-algebras was introduced by J. Meng [8] and gave certain properties of it. In 1997, Y. Huang and Z. Chen [2] introduced the notions of right and left stabilizers

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and normal BCK-algebras. Now, in this paper we define the dual left and dual right stabilizers and dual normal BCK-algebras, as mentioned in the abstract.

2 Preliminaries

We give herein the basic notions on BCK-algebras. For further information, we refer to the book [11]. By a BCK-algebra we mean an algebra (X, *, 0) of type (2,0) satisfying the following axioms: for every $x, y, z \in X$, (i) ((x * y) * (x * z)) * (z * y) = 0, (ii) (x * (x * y)) * y = 0, (iii) x * x = 0, (iv) $x * y = y * x = 0 \Rightarrow x = y$, (v) 0 * x = 0.

We can define a partial ordering \leq by $x \leq y$ if and only if x * y = 0. In a *BCK*algebra X, the following hold: for all $x, y, z \in X$

A BCK-algebra X is said to be commutative if x * (x * y) = y * (y * x), for all $x, y \in X$. A subalgebra of X is a nonempty subset A of X such that $x * y \in A$, for all $x, y \in A$. A nonempty subset A of X is called an *ideal* of X if it satisfies (i) $0 \in A$ (ii) $(\forall x \in X)(\forall y \in A)$ $(x * y \in A \Rightarrow x \in A)$.

If there is an element 1 of X satisfying $x \leq 1$, for all $x \in X$, then the element 1 is called unit of X. A BCK-algebra with unit is called *bounded*. In a bounded BCK-algebra with unit 1, we denote 1 * x by Nx and $NA = \{Nx \in X | x \in A\}$, for all $\emptyset \neq A \subseteq X$. A bounded BCK-algebra X is called *involutory* if NNx = x, for all $x \in X$.

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- A nonempty subset D of a bounded BCK-algebra X is called a *dual ideal* if
- (i) $1 \in D$

(ii)
$$N(Nx * Ny) \in D$$
 and $y \in D$ imply that $x \in D$, for any $x, y \in X$.

For brevity, we need the following notation in a *BCK*-algebra X: for all $x, y \in X$ and $n \in \aleph$ (natural numbers),

$$x *^{0} y = x, x *^{1} y = x * y, ..., x *^{n+1} y = (x *^{n} y) * y$$

3 Dual stabilizers in bounded *BCK*-algebras

In the sequel let X be a bounded BCK-algebra with unit 1, unless otherwise specified.

Definition 3.1 Let A be a nonempty subset of X. Then the sets

$$DA_l = \{ x \in X | Na * Nx = Na, \forall a \in A \}$$

and

$$DA_r = \{x \in X | Nx * Na = Nx, \forall a \in A\}$$

are called the dual left and dual right stabilizers of A, respectively and the set $DA = DA_l \cap DA_r$ is called the dual stabilizer of A.

For convenience the dual stabilizer, dual left and dual right stabilizers of a single element set $A = \{a\}$ are denoted by DS_a , DL_a and DR_a , respectively.

Theorem 3.2 Let X be an involutory BCK-algebra and A be a nonempty subset of X. Then:

(i) $N(DA_l) = (NA)_l^*$, $N(DA_r) = (NA)_r^*$ and $N(DA) = (NA)^*$, (ii) $N(A_l^*) = D(NA)_l$, $N(A_r^*) = D(NA)_r$ and $N(A^*) = D(NA)$.

Proof (i) By Definition 2.4, we have

$$(NA)_l^* = \{ x \in X | h * x = h, \forall h \in NA \}$$

$$= \{ x \in X | Na * x = Na, \forall a \in A \}$$

Now let $z \in N(DA_l)$. Then z = Nt, for some $t \in DA_l$ and so Na * Nt = Na, for all $a \in A$. Thus Na * z = Na, for all $a \in A$, i.e. $z \in (NA)_l^*$. Therefore $N(DA_l) \subseteq (NA)_l^*$. Now let $z \in (NA)_l^*$. Then Na * z = Na, for all $a \in A$ and so by hypothesis Na * NNz = Na, for all $a \in A$. Hence $Nz \in DA_l$. Since X is involutory, then $z \in N(DA_l)$, i.e. $(NA)_l^* \subseteq N(DA_l)$. Therefore $N(DA_l) = (NA)_l^*$. By similar above argument, we obtain $N(DA_r) = (NA)_r^*$. By hypothesis we have $N(DA) = N(DA_l \cap DA_r) = N(DA_l) \cap N(DA_r) = (NA)_l^* \cap (NA)_r^* = (NA)^*$.

(ii) The proof is similar to (i).

The following example shows that the condition " X is an involutory BCK-algebra" in the above theorem is necessary.

Example 3.3 Let $X = \{0, a, b, c, d, e, f, 1\}$ and let * operation be given by the following table

*	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	a	0	0	0	a	0	0	0
b	b	a	0	0	b	a	0	0
c	c	a	a	0	c	a	a	0
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	e	d	d	b	a	0	0
1	1	e	e	d	c	a	a	0

Then (X, *, 0) is a bounded BCK-algebra with unit 1 and it is not involutory, because $1 * (1 * b) = 1 * e = a \neq b$.

Consider $A = \{c\}$. Then $NA = \{d\}$. Since b * d = b and d * b = d, so $b \in (NA)_r^* \cap (NA)_l^* = (NA)^*$. Also we have $b \notin NX = \{1, e, d, c, a\}$, thus $b \notin N(DA_r)$ and $b \notin N(DA_l)$. Therefore $N(DA_l) \neq (NA)_l^*$, $N(DA_r) \neq (NA)_r^*$ and $N(DA) \neq (NA)^*$.

If consider $B = \{d\}$. Then $NB = \{c\}$. Since (1 * f) * (1 * c) = a * d = a = 1 * fand (1 * c) * (1 * f) = d * a = d = 1 * c, then $f \in D(NB)_r \cap D(NB)_l = D(NB)$. So $f \notin NX$ implies that $f \notin N(B_r^*)$ and $f \notin N(B_l^*)$. Therefore $N(B_l^*) \neq D(NB)_l$, $N(B_r^*) \neq D(NB)_r$ and $N(B^*) \neq D(NB)$.

Theorem 3.4 Let A be a nonempty subset of X. Then DA_l is a dual ideal of X.

Proof Let $N(Nx * Ny) \in DA_l$ and $y \in DA_l$. Then for all $a \in A$

$$Na = Na * NN(Nx * Ny) = (Na * Ny) * NN(Nx * Ny)$$

$$= (Na * NN(Nx * Ny)) * Ny$$

$$= (NNN(Nx * Ny) * a) * Ny$$

$$= (N(Nx * Ny) * a) * Ny$$

$$= (Na * (Nx * Ny)) * Ny$$

$$= (Na * Ny) * (Nx * Ny)$$

$$\leq Na * Nx$$

Thus $Na \leq Na * Nx$, for all $a \in A$. Also we have $Na * Nx \leq Na$, so Na * Nx = Na, for all $a \in A$, i.e. $x \in DA_l$. Therefore DA_l is a dual ideal of X.

The following example shows that DA_r is not a dual ideal in general.

Example 3.5 Let $X = \{0, 1, 2, 3, 4\}$ and let * operation be given by the following

table

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then (X, *, 0) is a bounded BCK-algebra with unit 4. Consider $A = \{3\}$, it is easy to check that $DA_r = \{1, 4\}$. Now we see that $4 * ((4 * 3) * (4 * 1)) = 4 \in DA_r$ and $1 \in DA_r$ but $3 \notin DA_r$. Hence DA_r is not a dual ideal of X.

The following example shows that DA_r and DA_l are not subalgebras of a bounded BCK-algebra X in general.

Example 3.6 Let $X = \{0, 1, 2, 3\}$ and * operation be given by the

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then (X, *, 0) is a bounded BCK-algebra with unit 3. Put $A = \{1, 2\}$, then $DA_r = \{3\} = DA_l$, it is clear that DA_r and DA_l are not subalgebras of X.

Theorem 3.7 Let X be a commutative bounded BCK-algebra and A be a nonempty subset of X. Then

 $(i) DA_r = DA_l = DA,$

(ii) DA_r and DA_l are dual ideals of X.

Proof The proof of (i) follows from the definition of commutative and the proof of (ii) follows from Theorem 3.4 and (i).

Lemma 3.8 Let $x \in X$. Then NNx = 1 if and only if x = 1.

Proof Let NNx = 1. Put 1 * x = a, so by hypothesis we get that 1 * a = 1. Hence 0 = a * a = (1 * x) * a = (1 * a) * x = 1 * x = a. Then 1 * x = 0 and so x = 1. The proof of the converse is easy.

Theorem 3.9 If $0 \in A \subseteq X$, then $DA_l = DA_r = DA = \{1\}$.

Proof It is clear that $1 \in DA_l \cap DA_r$. Now let $x \in DA_r$, then Nx * Na = Nx, for all $a \in A$, and so 0 = Nx * 1 = Nx * N0 = Nx. Thus x = 1. Therefore $DA_r = \{1\}$. Let $x \in DA_l$. Then Na * Nx = Na, for all $a \in A$. Hence N0 * Nx = N0, i.e. NNx = 1 and so by Lemma 3.8, x = 1. Thus $DA_l = \{1\}$. Therefore $DA = \{1\}$.

Theorem 3.10 If $A = \{1\}$, then $DA_l = DA_r = DA = X$.

Proof Let $z \in X$. Then Nz = Nz * 0 = Nz * N1 and N1 = 0 = 0 * Nz = N1 * Nz. So $z \in DA_l \cap DA_r = DA$. Therefore $DA_l = DA_r = DA = X$.

Theorem 3.11 Let A be a nonempty subset of X. If $DA_l = X$ or $DA_r = X$ or DA = X, then $A = \{1\}$.

Proof Let $DA_r = X$ and $a \in A \subseteq X$. Then $a \in DA_r$ implies that 0 = Na * Na = Naand so a = 1. Therefore $A = \{1\}$. Similarly, $DA_l = X$ or DA = X implies that $A = \{1\}$.

Theorem 3.12 Let A be a nonempty subset of X. Then $0 \in DA_l \cup DA_r \cup DA$ if and only if $A = \{1\}$.

Proof Let $0 \in DA_l \cup DA_r \cup DA$. Then NNa = 1, for all $a \in A$, and so by Lemma 3.8 $A = \{1\}$. Conversely, let $A = \{1\}$. Then by Theorem 3.10, it is clear that $0 \in DA_l \cup DA_r \cup DA$.

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Theorem 3.13 Let A be a nonempty subset of X. Then $DA_l(DA_r, DA)$ is a subalgebra of X if and only if $A = \{1\}$.

Proof The proof follows from Theorems 3.10 and 3.12.

Theorem 3.14 If Nx = 1, for all $x \in X - \{1\}$, then $DA_r = DA_l = DA = \{1\}$, for any nonempty subset $A \neq \{1\}$ of X.

Proof Straightforward.

Theorem 3.15 Let $\emptyset \neq A \subseteq X$. If there exists $a \in A$ such that Na = 1, then $DA_r = DA_l = DA = \{1\}$.

Proof Let $x \in DA_r$. Then Nx * Nt = Nx, for all $t \in A$. Put t = a, then Nx = Nx * 1 = 0, and so x = 1. Thus $DA_r = \{1\}$. Similarly we can get that $DA_l = DA = \{1\}$.

Theorem 3.16 Let $a \in X - \{0, 1\}$ and Nx = 1, for all $x \in X - \{a, 1\}$. Then $DA_r = DA_l = DA = \{1\}$, for any nonempty subset $A \neq \{1\}$ of X.

Proof First we show that 1 * a = a. Let 1 * a = c and $c \neq a$. Since $1 * a \neq 1$, then $c \neq 1$ and also $a \neq 1$ implies that $c \neq 0$. Then 0 = c * c = (1 * a) * c = (1 * c) * a = 1 * a, so a = 1, which is not true. Thus 1 * a = a. Now we consider two cases:

Case i: $a \notin A$. If $x \in DA_r$, then (1 * x) * (1 * b) = 1 * x, for all $b \in A$. Thus we get that 1 * x = 0, hence x = 1. Therefore $DA_r = \{1\}$. Also $x \in DA_l$ implies that (1 * b) * (1 * x) = 1 * b, for all $b \in A$, thus we get that NNx = 1. Then x = 1, by Lemma 3.8, and so $DA_l = \{1\}$.

Case ii: $a \in A$. If $A \not\subseteq \{a, 1\}$, then similar to above argument we get that $DA_r = DA_l = DA = \{1\}$. If $A = \{a\}$ or $A = \{a, 1\}$. Then $x \in DA_r$ implies that (1 * x) * (1 * a) = 1 * x. Since $a \neq 0$, so $x \neq a$. If $x \neq 1$, then we get that 1 * a = 1, hence a = 1, which is not true. Then x = 1 and so $DA_r = \{1\}$. Similarly we can get that $DA_l = DA = \{1\}$.

Theorem 3.17 For all $x, y, y_1, y_2, ..., y_n \in X$,

$$NN((...((Nx * y_1) * y_2) * ...) * y_n) = (...((Nx * y_1) * y_2) * ...) * y_n$$

also $NN(Nx *^n y) = Nx *^n y.$

Proof It is clear that $NN((...((Nx * y_1) * y_2) * ...) * y_n) \le (...((Nx * y_1) * y_2) * ...) * y_n$. Now

$$\begin{split} &((\dots((Nx*y_1)*y_2)*\dots)*y_n)*NN((\dots((Nx*y_1)*y_2)*\dots)*y_n) \\ &= (\dots(((Nx*NN((\dots((Nx*y_1)*y_2)*\dots)*y_n))*y_1)*y_2)*\dots)*y_n \\ &= (\dots(((NNN((\dots((Nx*y_1)*y_2)*\dots)*y_n)*x)*y_1)*y_2)*\dots)*y_n \\ &= (\dots(((N((\dots((Nx*y_1)*y_2)*\dots)*y_n))*x)*y_1)*y_2)*\dots)*y_n \\ &= ((\dots((Nx*y_1)*y_2)*\dots)*y_n)*((\dots((Nx*y_1)*y_2)*\dots)*y_n) = 0 \\ &\text{Therefore } NN((\dots((Nx*y_1)*y_2)*\dots)*y_n) = (\dots((Nx*y_1)*y_2)*\dots)*y_n. \\ &\text{Put } y_1 = y_2 = \dots = y_n = y, \text{ we get that } NN(Nx*^n y) = Nx*^n y. \end{split}$$

Theorem 3.18 Let D be a dual ideal of X. Then $NNx \in D$ if and only if $x \in D$.

Proof Let $x \in D$. Since $N(NNNx * Nx) = N(Nx * Nx) = 1 \in D$ and $x \in D$, then $NNx \in D$. Conversely, let $NNx \in D$, since $NNx \leq x$ and so $x \in D$.

Theorem 3.19 Let A be a nonempty subset of X. Then (i) $[A] \cap DA_r = \{1\}$, (ii) $DA_r = D[A]_r$, (iii) if DA_r is a dual ideal of X, then $DA = DA_r$.

Proof (i) Let $x \in [A] \cap DA_r$. Then by $x \in [A]$, there exists $a_1, a_2, ..., a_n \in A$ such that $(...((Nx * Na_1) * Na_2) * ...) * Na_n = 0$, moreover by $x \in DA_r$, $Nx = Nx * Na_n = (Nx * Na_{n-1}) * Na_n = ... = (...((Nx * Na_1) * Na_2) * ...) * Na_n = 0$, so Nx = 0, i.e. x = 1. Therefore $[A] \cap DA_r = \{1\}$. (ii) Let $x \in D[A]_r$. Since $A \subseteq [A]$, then Nx * Na = Nx, for all $a \in A$. Hence $x \in DA_r$.

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On the other hand, suppose that $x \in DA_r$. So Nx * Na = Nx, for all $a \in A$. For any $a \in [A]$, there exists $a_1, a_2, ..., a_n \in A$, such that $(...((Na * Na_1) * Na_2) * ...) * Na_n = 0$. By hypothesis we have $(...((Nx * Na_1) * Na_2) * ...) * Na_n = Nx$. So

$$Nx * (Nx * Na) = ((...((Nx * Na_1) * Na_2) * ...) * Na_n) * (Nx * Na)$$

$$= (...(((Nx * (Nx * Na)) * Na_1) * Na_2) * ...) * Na_n$$

$$\leq (...((Na * Na_1) * Na_2) * ...) * Na_n = 0$$

Namely, $Nx \leq Nx * Na$, and so Nx * Na = Nx, for all $a \in [A]$. Hence $x \in D[A]_r$. (iii) It is clear that $DA \subseteq DA_r$. Now let $x \in DA_r$. So by Lemma 3.18 $NNx \in DA_r$ and $NNa \in [A]$, for all $a \in A$. Since $Na * (Na * Nx) \leq Nx$ and $Na * (Na * Nx) \leq Na$, for all $a \in A$, then $NNx \leq N(Na * (Na * Nx))$ and $NNa \leq N(Na * (Na * Nx))$. Thus by hypothesis we get that $N(Na * (Na * Nx)) \in [A] \cap DA_r = \{1\}$, by (i). Then N(Na * (Na * Nx)) = 1 and so by Theorem 3.17, Na * (Na * Nx) = NN(Na * (Na * Nx)) = N1 = 0, for all $a \in A$. Hence Na * Nx = Na, for all $a \in A$. Thus $x \in DA_l$, i.e. $x \in DA_r \cap DA_l = DA$. Therefore $DA = DA_r$.

Theorem 3.20 Let A and B be nonempty subsets of X. Then (i) $A \cap DA_l = \emptyset$ or $\{1\}$, $A \cap DA_r = \emptyset$ or $\{1\}$ and $A \cap DA = \emptyset$ or $\{1\}$, (ii) if $A \subseteq B$, then $DB_l \subseteq DA_l$, $DB_r \subseteq DA_r$ and $DB \subseteq DA$, (iii) $A \subseteq D(DA_r)_l \cap D(DA_l)_r$ and $A \subseteq D(DA)$, (iv) $DA_l = D(D(DA_l)_r)_l$, $DA_r = D(D(DA_r)_l)_r$ and DA = D(D(DA)), (v) $D(A \cup B)_l = DA_l \cap DA_l$, $D(A \cup B)_r = DA_r \cap DA_r$ and $D(A \cup B) = DA \cap DB$, (vi) $DA_l = \bigcap_{a \in A} DL_a$, $DA_r = \bigcap_{a \in A} DR_a$ and $DA = \bigcap_{a \in A} DS_a$.

Proof (i) Let $A \cap DA_l \neq \emptyset$. Then there exists $x \in A \cap DA_l$ and so Nx * Nx = Nx, i.e. Nx = 0. Therefore x = 1. The proof of the other parts is similar.

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(ii) Let $x \in DB_l$. Then Nb * Nx = Nb, $\forall b \in B$. Since $A \subseteq B$, Nb * Nx = Nb, $\forall b \in A$. So $x \in DA_l$. Similarly $DB_r \subseteq DA_r$ and $DB \subseteq DA$.

(iii) Let $a \in A$. Then Nx * Na = Nx, $\forall x \in DA_r$ and Na * Ny = Na, $\forall y \in DA_l$. So $a \in D(DA_r)_l \cap D(DA_l)_r$. Therefore $A \subseteq D(DA_r)_l \cap D(DA_l)_r$. Since $DA \subseteq DA_r$ and $DA \subseteq DA_l$, then by (ii) $D(DA_r)_l \subseteq D(DA)_l$ and $D(DA_l)_r \subseteq D(DA)_r$. Hence $A \subseteq D(DA_r)_l \cap D(DA_l)_r \subseteq D(DA)_l \cap D(DA)_r = D(DA)$.

(iv) By (iii) we get that $DA_l \subseteq D(D(DA_l)_r)_l$ and $DA_r \subseteq D(D(DA_r)_l)_r$. Also by (ii) and (iii) we have $D(D(DA_r)_l)_r \subseteq DA_r$ and $D(D(DA_l)_r)_l \subseteq DA_l$. Therefore $DA_l = D(D(DA_l)_r)_l$ and $DA_r = D(D(DA_r)_l)_r$, similar to argument in (iii) we can get that DA = D(D(DA)).

(v) Since $A, B \subseteq (A \cup B)$, then $D(A \cup B)_l \subseteq DA_l \cap DB_l$. Now let $x \in DA_l \cap DB_l$, then Na * Nx = Na, $\forall a \in A$ and Nb * Nx = Nb, $\forall b \in B$. Thus Na * Nx = Na, $\forall a \in (A \cup B)$ i.e. $x \in D(A \cup B)_l$. Therefore $D(A \cup B)_l = DA_l \cap DA_l$. Similarly $D(A \cup B)_r = DA_r \cap DA_r$, also $D(A \cup B) = D(A \cup B)_r \cap D(A \cup B)_l = (DA_r \cap DB_r) \cap (DA_l \cap DB_l) = DA \cap DB$.

(vi) The proof is similar to the proof of part (v).

The following example shows that A is a dual ideal, but DA_r is not a dual ideal.

Example 3.21 Let $X = \{0, 1, 2, 3, 4\}$ in which * is defined by the table

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	3	3	0	0
4	4	3	3	1	0

Then (X, *, 0) is a bounded BCK-algebra with unit 4. Also $A = \{3, 4\}$ is a dual ideal,

but $DA_r = \{1, 2, 4\}$ is not a dual ideal, because $4 * ((4 * 3) * (4 * 2)) = 4 \in DA_r$ and $2 \in DA_r$ but $3 \notin DA_r$.

Theorem 3.22 If A is a dual ideal of X, then DA is a dual ideal of X.

Proof Let $N(Nx * Ny) \in DA$ and $y \in DA$. Then by Theorem 3.4 we get that $x \in DA_l$. Since A is a dual ideal, then $NNa \in A$, for all $a \in A$. Also $x \in DA_l$ implies that $NNx \in DA_l$. We have $NNx \leq N(Nx*(Nx*Na))$ and $NNa \leq N(Nx*(Nx*Na))$. Then by hypothesis we get that $N(Nx * (Nx * Na)) \in A \cap DA_l = \{1\}$, by Theorem 3.20(i) and so N(Nx * (Nx * Na)) = 1, for all $a \in A$. Hence, by Theorem 3.17, Nx = Nx * Na, i.e. $x \in DA_r$. Therefore $x \in DA$.

Theorem 3.23 Let A and B be two dual ideals of X. Then $A \cap B = \{1\}$ if and only if $A \subseteq DB$.

Proof Let $A \cap B = \{1\}$ and $a \in A$. Since $NNa \leq N(Na * (Na * Nb))$ and $NNb \leq N(Na * (Na * Nb))$, for all $b \in B$, also $NNa \in A$ and $NNb \in B$, thus by hypothesis we get that $N(Na * (Na * Nb)) \in A \cap B = \{1\}$. Then N(Na * (Na * Nb)) = 1 and so Na * (Na * Nb) = 0, by Theorem 3.17. Therefore Na = Na * Nb, for all $b \in B$. Similarly, we can get that Nb * Na = Nb, for all $b \in B$, then $a \in DB$. Conversely, let $A \subseteq DB$. Consider $x \in A \cap B$. Then Nx * Nb = Nx, for all $b \in B$ and so 0 = Nx * Nx = Nx. Thus x = 1. Therefore $A \cap B = \{1\}$.

Theorem 3.24 A be a dual ideal of X. Then $DA_l = DA \subseteq DA_r$. In particular, if DA_r is a dual ideal of X, $DA = DA_l = DA_r$.

Proof We have $DA \subseteq DA_l$. Since DA_l and A are dual ideals and $DA_l \cap A = \{1\}$ by Theorem 3.20(i), then $DA_l \subseteq DA$, by Theorem 3.23. Therefore $DA_l = DA \subseteq DA_r$. In particular, if DA_r is a dual ideal of X, then by Theorem 3.19(iii) we get that $DA = DA_r$. Therefore $DA = DA_l = DA_r$.

4 Dual normal *BCK*-algebras

Definition 4.1 A bounded BCK-algebra X is called dual normal, if the dual right stabilizer DR_a of any element $a \in X$ is a dual ideal of X.

The following theorem follows from Theorems 3.10, 3.14 and 3.16.

Theorem 4.2 Under each of the following conditions, X is dual normal.

- (i) Nx = 1, for all $x \in X \{1\}$,
- (*ii*) $a \in X \{0, 1\}$ and Nx = 1, for all $x \in X \{a, 1\}$.

According to Theorem 3.7 any commutative bounded BCK-algebra is a dual normal BCK-algebra, but the converse may not be true.

Example 4.3 Let $X = \{0, 1, 2, 3, 4\}$ in which * is defined by the table

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	1	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

Then (X, *, 0) is a bounded BCK-algebra with unit 4 and it is dual normal, by Theorem 4.2(i). But X is not commutative, because $4 * (4 * 2) = 0 \neq 2 = 2 * (2 * 4)$.

Theorem 4.4 The following statements are equivalent:

- (i) X is dual normal,
- (*ii*) $DR_a \subseteq DL_a, \forall a \in X$,
- (*iii*) $DR_a = DL_a, \forall a \in X,$
- (v) Nx * Ny = Nx implies $Ny * Nx = Ny, \forall x, y \in X$.

Proof $(i) \to (ii)$ Since X is dual normal, then for all $a \in X$, DR_a is a dual ideal of X. So

$$DR_a = D[a]_r$$
, by Theorem 3.19(ii)

 $= D[a]_l$, by Theorem 3.24

 $\subseteq DL_a$, by Theorem 3.20(ii)

 $(ii) \rightarrow (iii)$ For any $x \in DL_a$, we have Na * Nx = Na, thus $a \in DR_x$. By (ii) we have $DR_x \subseteq DL_x$, hence $a \in DL_x$. So Nx * Na = Nx, i.e. $x \in DR_a$. Hence $DL_a \subseteq DR_a$. Therefore $DR_a = DL_a$. $(iii) \rightarrow (v)$ Assume that $x, y \in X$ and Nx * Ny = Nx. Then $x \in DR_y = DL_y$ and so Ny * Nx = Ny.

 $(v) \rightarrow (i)$ By hypothesis we have

$$DR_a = \{x \in X \mid Nx * Na = Nx\} = \{x \in X \mid Na * Nx = Na\} = DL_a$$

Since DL_a is a dual ideal, then DR_a is too. Therefore X is dual normal.

Definition 4.5 A bounded BCK -algebra X is called dual semisimple if every dual ideal A of X is a sub-summand of X, i.e. there exists a dual ideal B of X such that $A \cap B = \{1\}$ and $X = [A \cup B]$.

Consider Example 3.6. It is easy to check that $A_1 = \{1,3\}$, $A_2 = \{3,2\}$, $A_3 = \{3\}$ and $A_4 = \{0,1,2,3\}$ are the only dual ideals of X. Also $A_1 \cap A_2 = \{3\}$ and $[A_1 \cup A_2] = [\{1,2,3\}] = X$. Thus X is dual semisimple.

Theorem 4.6 Every dual semisimple bounded BCK-algebra is dual normal.

Proof Let $a \in X$. Then there exists a dual ideal B of X such that $[a] \cap B = \{1\}$ and $X = [[a] \bigcup B]$. By $[a] \cap B = \{1\}$ and Theorem 3.23 we get that $B \subseteq D[a] \subseteq D[a]_r$.

Since by Theorem 3.19(ii) $D[a]_r = DR_a$, then $B \subseteq DR_a$. Let $x \in DR_a$. Then by $x \in X = [[a] \cup B]$ there exists $n \in \aleph$ such that $N(Nx *^n Na) \in B$. Since $x \in DR_a$, then $Nx *^n Na = Nx$. Hence $NNx \in B$ and so $x \in B$ by Lemma 3.18. Thus $B = DR_a$. Therefore X is dual normal.

Theorem 4.7 Let [a] be a sub-summand of X, for any $a \in X$. Then DR_a is a sub-summand of X, for any $a \in X$.

Proof Let $a \in X$. Then there exists dual ideal B of X such that $X = [[a] \cup B]$ and $[a] \cap B = \{1\}$. By the proof of Theorem 4.6, we get that $B = DR_a$. Therefore DR_a is a sub-summand of X.

Theorem 4.8 et X be a finite bounded BCK-algebra. Then the following are equivalent:

(i) X is dual normal,

(ii) [a] is a sub-summand of X, for any $a \in X$

(iii) DR_a is a sub-summand of X, for any $a \in X$

(iv) There exists a fixed natural number n such that $Nx * (Ny *^n Nx) = Nx$, for all $x, y \in X$,

(v) There exists a fixed natural number n such that $Nx * (Nx * Ny) \le Ny * (Ny *^n Nx)$, for all $x, y \in X$,

(vi) X is dual semisimpel.

Proof (i) \rightarrow (ii) Let $a \in X$. Since for all $x \in X$, ... $\leq Nx *^n Na \leq ... \leq Nx *^2 Na \leq Nx * Na$ and X is finite, then there exists $n \in \aleph$ such that $Nx *^n Na = Nx *^{n+1} Na$. This show that by Theorem 3.17, $NN(Nx *^n Na) * Na = NN(Nx *^n Na)$ and so $N(Nx *^n Na) \in DR_a$, Since DR_a is a dual ideal, then by Theorem 2.3, we get that $x \in [[a] \cup DR_a]$ i.e. $X = [[a] \cup DR_a]$. Also by Theorem 3.19(i), $[a] \cap DR_a = \{1\}$. Therefore [a] is a sub-summand of X.

(ii) \rightarrow (iii) It is proved in Theorem 4.7.

(iii) \rightarrow (iv) Let $x, y \in X$. Since DR_x is a sub-summand of X, then there exists a dual ideal A of X such that $A \cap DR_x = \{1\}$ and $X = [DR_x \cup A]$. By argument in the last part we get that there exists $m = m(x, y) \in \aleph$ such that $N(Ny *^m Nx) \in DR_x$. Consider $T = \{n = n(u, v) \in \aleph| \ N(Nu *^n Nv) \in DR_v, u, v \in X\}$. Since X is finite, then T is a finite set and so it has the greatest element, say n. Clearly $N(Ny *^n Nx) \in DR_x$. Now $x \in X = [A \cup DR_x]$ implies that by Theorem 2.2 there exists $x_1, x_2, ..., x_n \in DR_x$ such that $N((...((Nx * Nx_1) * Nx_2) * ...) * Nx_n) \in A$. On the other hand since $[x] \cap DR_x = \{1\}$, by Theorem 3.23 $[x] \subseteq D(DR_x)$, and thus by $x \in [x] \subseteq D(DR_x)$ we get that $Nx * Nx_i = Nx$, for all $1 \le i \le n$. Hence $NNx \in A$, so by Lemma 3.18, $x \in A$. Since $A \cap DR_x = \{1\}$, then by Theorem 3.23 $A \subseteq D(DR_x)$, so $x \in A$ and $N(Ny *^n Nx) \in DR_x$ implies that $Nx * NN(Ny *^n Nx) = Nx$.

 $(iv) \to (v) Nx * (Nx * Ny) = (Nx * (Ny *^n Nx)) * (Nx * Ny) \le Ny * (Ny *^n * Nx), \text{ for all } x, y \in X.$

 $(\mathbf{v}) \rightarrow (\mathbf{v}i)$ Let A be a dual ideal of X and $x \in DA_r$. Then Nx * Na = Nx, for all $a \in A$ and so $Nx *^n Na = Nx$. Hence by (\mathbf{v}) we get that $Na * (Na * Nx) \leq Nx * (Nx *^n Na) =$ Nx * Nx = 0, that is, Na = Na * Nx and so $x \in DA_l$. Therefore $DA_r \subseteq DA_l$. Since Ais a dual ideal, then by Theorem 3.24, $DA_l \subseteq DA_r$ and so $DA_r = DA_l$. Hence by Theorem 3.4 DA_r is a dual ideal of X. Since X is finite, we suppose that $A = \{a_1, a_2, ..., a_k\}$ and $Nx *^{n_i} Na_i = Nx *^{n_i+1} Na_i$, where $x \in X$, $a_i \in A$ and $n_i = n_i(x, a_i) \in \aleph$, for all $1 \leq i \leq k$. Put $y = N((...((Nx *^{n_1} Na_1) *^{n_2} Na_2) *^{n_3} ...) *^{n_k} Na_k$ by hypothesis and Theorem 3.17 $Ny = (...((Nx *^{n_1} Na_1) *^{n_2} Na_2) *^{n_3} ...) *^{n_k} Na_k)$ and so Ny = Ny * Na, for all $a \in A$, that is $y \in DA_r$ also $((...((Nx *^{n_1} Na_1) *^{n_2} Na_2) *^{n_3} ...) *^{n_k} Na_k) * Ny = 0$, hence $x \in [A \cup DA_r]$ i.e $X = [A \cup DA_r]$. Also $A \cap DA_r = \{1\}$, by Theorem 3.20(i). Therefore X is dual semisimple.

 $(vi) \rightarrow (i)$ It is proved in Theorem 4.6.

Open problem. Is any infinite dual normal *BCK*-algebra a dual semisimple?

Definition 4.9 The dual J-radical, denoted by DJ(X), of a bounded BCK-algebra X means the intersection of all maximal dual ideals of X. By Zorn's Lemma the collection of maximal dual ideals of X is nonempty. If $DJ(X) = \{1\}$, then X is called dual J-semisimple.

Consider bounded *BCK*-algebra $X = \{0, 1, 2, 3, 4\}$ in Example 3.5. We can see that the only dual ideals of X are $\{3, 4\}$, $\{4\}$ and X, so $DJ(X) = \{3, 4\}$.

Bounded *BCK*-algebra $X = \{0, 1, 2, 3\}$ in Example 3.6 is a dual J-semisimple, because the only dual ideals on X are $\{2, 3\}$, $\{1, 3\}$, $\{3\}$ and X, so $DJ(X) = \{3\}$.

Theorem 4.10 Every dual J-semisimple bounded BCK-algebra is dual normal.

Proof On the contrary, let X do not be dual normal. Then $\exists a \in X$ such that DR_a is not a dual ideal. Thus $DR_a \subset [DR_a]$ (proper containing). We show that $[DR_a] \cap [a] \neq \{1\}.$

If $[DR_a] \cap [a] = \{1\}$, then by Theorem 3.23 $[DR_a] \subseteq D[a] \subseteq D[a]_r$. Since by Theorem 3.19 $DR_a = D[a]_r$, then we get that $[DR_a] \subseteq DR_a$, which is impossible. Thus $[DR_a] \cap [a] \neq \{1\}$. We choose $1 \neq b \in [a] \cap [DR_a]$. Let M be a maximal dual ideal of X. We consider the following cases:

Case (i): $a \in M$, since $b \in [a] \subseteq M$, then $b \in M$.

Case (ii): $a \notin M$, then by maximality of M, $X = [M \cup \{a\}]$. We show that $DR_a \subseteq M$. Let $x \in DR_a \subseteq X$. Then there exists $n \in \aleph$ such that $N(Nx *^n Na) \in M$ and so by $x \in DR_a$, we get that $NNx \in M$. Thus by Lemma 3.18, $x \in M$ and so $DR_a \subseteq M$. Hence $b \in M$, since $b \in [DR_a] \subseteq M$. This show that $1 \neq b \in DJ(X)$, a contradiction with $DJ(X) = \{1\}$.

Open problem. Is any dual normal *BCK*-algebra a dual *J*-semisimple?

Theorem 4.11 Let Y be a bounded BCK-algebra with unit 1. Then Y is a dual normal BCK-algebra if and only if every subalgebra X of Y containing 1, is dual normal BCK-algebra .

Proof (\Leftarrow) It is clear.

 (\Rightarrow) Let $a \in X$ and let DR_a and DR'_a be the dual right stabilizers of element a with respect to X and Y, respectively. i.e.

$$DR_a = \{ x \in X | Nx * Na = Nx \}$$

and

$$DR'_a = \{ x \in Y | Nx * Na = Nx \}.$$

Then $DR_a = DR'_a \cap X$. Now we show that DR_a is a dual ideal of X. Let $x, y \in X, N(Nx*Ny) \in DR_a$ and $y \in DR_a$. Since $DR_a \subseteq DR'_a$ and DR'_a is a dual ideal of Y, so $x \in DR'_a$. Also $x \in DR'_a \cap X = DR_a$ implies DR_a is a dual ideal of X. Therefore X is dual normal.

Let $(X_i, *_i, 0_i)(i \in I)$ be an indexed family of *BCK*-algebras and $\prod_{i \in I} X_i$ be the set of all mapping $f: I \longrightarrow \bigcup_{i \in I} X_i$ and $f(i) \in X_i$ for all $i \in I$.

For $f,g \in \prod_{i \in I} X_i$, we define f * g by $(f*g)(i) = f(i)*_ig(i)$, for all $i \in I$ and 0 by $0(i) = 0_i$.

Then $(\prod_{i \in I} X_i, *, 0)$ is a *BCK*-algebra. Also $\prod_{i \in I} X_i$ is bounded if and only if every X_i is bounded.

Theorem 4.12 Let $\{I_i\}_{i \in I}$ be an indexed family of subsets of bounded BCK-algebras $X_i (i \in I)$. Then

(i) if every I_i is a dual ideal of $X_i (i \in I)$, then $\prod_{i \in I} I_i$ is a dual ideal of $\prod_{i \in I} X_i$, (ii) if $\prod_{i \in I} I_i$ is a dual ideal of $\prod_{i \in I} X_i$, then every I_i is a dual ideal of X_i .

Proof (i) Let I_i be a dual ideal of X_i , $\forall i \in I$ and let $N(Nx*Ny) \in \prod_{i \in I} I_i$ and $y \in \prod_{i \in I} I_i$. Then $1_i*_i((1_i*_ix(i)))*_i(1_i*_iy(i))) = 1(i)*_i((1(i)*_ix(i))*_i(1(i)*_iy(i))) = 1*((1*x)*(1*y))(i) \in I_i$ and $y(i) \in I_i$, for all $i \in I$. So $x(i) \in I_i$. This shows $x \in \prod_{i \in I} I_i$.

(ii) Suppose that $\prod_{i \in I} I_i$ is a dual ideal of $\prod_{i \in I} X_i$. Without loss of generality we show that I_j is a dual ideal of X_j . If $1_j *_j((1_j *_j x_1) *_j(1_j *_j y_1)) \in I_j$ and $y_1 \in I_j$, we define x and y as follows

$$x(i) = \begin{cases} x_1 & \text{if } i = j \\ 1_i & \text{if } i \neq j \end{cases}$$
$$y(i) = \begin{cases} y_1 & \text{if } i = j \\ 1_i & \text{if } i \neq j \end{cases}$$

and so

$$N(Nx*Ny)(i) = \begin{cases} 1_{j}*_{j}((1_{j}*_{j}x_{1})*_{j}(1_{j}*_{j}y_{1})) & \text{if } i = j \\ 1_{i} & \text{if } i \neq j \end{cases}$$

Then $N(Nx*Ny) \in \prod_{i \in I} I_i$ and $y \in \prod_{i \in I} I_i$. Thus $x \in \prod_{i \in I} I_i$, i.e. $x_1 \in I_j$.

Theorem 4.13 Let $(X_i, *_i, 0_i, 1_i)(i \in I)$ be an indexed family of bounded BCKalgebras. Then every X_i is dual normal BCK-algebra if and only if $\prod_{i \in I} X_i$ is dual normal BCK-algebra.

Proof (\Rightarrow) Let $f \in \prod_{i \in I} X_i$. Then

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$$DR_{f} = \{g \in \prod_{i \in I} X_{i} \mid 1 * g = (1 * g) * (1 * f)\}$$

$$= \{g \in \prod_{i \in I} X_{i} \mid 1(i) *_{i}g(i) = (1(i) *_{i}g(i)) *_{i}(1(i) *_{i}f(i)), \forall i \in I\}$$

$$= \{g \in \prod_{i \in I} X_{i} \mid 1_{i} *_{i}g(i) = (1_{i} *_{i}g(i)) *_{i}(1_{i} *_{i}f(i)), \forall i \in I\}$$

$$= \{g \in \prod_{i \in I} X_{i} \mid g(i) \in DR_{f_{(i)}}\} = \prod_{i \in I} DR_{f_{(i)}}$$

Since $DR_{f_{(i)}}$ is a dual ideal of $X_i, \forall i \in I$ then by Theorem 4.12 we get that $DR_f = \prod_{i \in I} DR_{f_{(i)}}$ is a dual ideal of $\prod_{i \in I} X_i$.

 $(\Leftarrow) \text{ Let } a_j \in X_j. \text{ Then we define } f: I \longrightarrow \bigcup_{i \in I} X_i \text{ such that}$ $f(i) = \begin{cases} a_j & \text{ if } i = j \\ 1_j & \text{ if } i \neq j \end{cases}$

So $f \in \prod_{i \in I} X_i$. Similar to above we have $DR_f = \prod_{i \in I} DR_{f_{(i)}}$ where,

$$DR_{f(i)} = \begin{cases} DR_{a_j} & \text{if } i = j \\ X_i & \text{if } i \neq j \end{cases}$$

Since $\prod_{i \in I} X_i$ is dual normal, then DR_f is a dual ideal of $\prod_{i \in I} X_i$. So by Theorem 4.12 DR_{a_j} is a dual ideal of X_j . Therefore X_j is dual normal.

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