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Dual normal BCK-algebras

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Abstract

In this paper, by considering the notions of dual right and dual left stabilizers in bounded BCK-algebras, we obtain some related results. After that we investigate the relationship between the dual left(right) stabilizers and dual ideals in bounded BCK-algebras. Then we define a class of special bounded BCK-algebras called dual normal BCK-algebras. Finally we prove that the dual semisimple bounded BCK-algebras and dual J-semisimple bounded BCK-algebras are all dual normals.

Keywords BCK-algebra, Bounded BCK-algebra, Dual right and dual left stabilizers, Dual normal BCK-algebra, Dual semisimple BCK-algebra.

1 Introduction

The study of BCK-algebras was initiated by Y. Imai and K-Iseki [3] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK-algebras. Dual ideals are important in bounded BCK-algebras. In 1986, the notion of dual ideals in bounded BCK-algebras was introduced by J. Meng [8] and gave certain properties of it. In 1997, Y. Huang and Z. Chen [2] introduced the notions of right and left stabilizers

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and normal BCK-algebras. Now, in this paper we define the dual left and dual right stabilizers and dual normal BCK-algebras, as mentioned in the abstract.

2 Preliminaries

We give herein the basic notions on BCK-algebras. For further information, we refer to the book [11]. By a BCK-algebra we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$,

(i) $((x * y) * (x * z)) * (z * y) = 0$, (ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = y * x = 0 \Rightarrow x = y$,

(v) $0 * x = 0$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. In a BCK-algebra X, the following hold: for all $x, y, z \in X$

(a) $x * 0 = x$

(b) $x * y \leq x$,

(c) $(x * y) * z = (x * z) * y$,

(d) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$, (e) $x * (x * (x * y)) = x * y$.

A BCK-algebra X is said to be *commutative* if $x * (x * y) = y * (y * x)$, for all $x, y \in X$. A *subalgebra* of X is a nonempty subset A of X such that $x * y \in A$, for all $x, y \in A$. A nonempty subset A of X is called an *ideal* of X if it satisfies (i) $0 \in A$
(ii) $(\forall x \in X)(\forall y \in A)$ $(x * y \in A \Rightarrow x \in A)$.

If there is an element 1 of X satisfying $x \leq 1$, for all $x \in X$, then the element 1 is called *unit* of X. A BCK-algebra with unit is called *bounded* . In a bounded BCK-algebra with unit 1, we denote $1 * x$ by Nx and $NA = \{Nx \in X| \ x \in A\}$, for all $\emptyset \neq A \subseteq X$. A bounded BCK-algebra X is called *involutory* if $NNx = x$, for all $x \in X$.

A nonempty subset D of a bounded BCK-algebra X is called a *dual ideal* if
(i) $1 \in D$
(ii) $N(Nx * Ny) \in D$ and $y \in D$ imply that $x \in D$, for any $x, y \in X$.

For brevity, we need the following notation in a BCK-algebra X: for all $x, y \in X$ and $n \in \aleph$(natural numbers),

$$x *^0 y = x, \quad x *^1 y = x * y, \quad ..., \quad x *^{n+1} y = (x *^n y) * y$$

3 Dual stabilizers in bounded BCK-algebras

In the sequel let X be a bounded BCK-algebra with unit 1 , unless otherwise specified.

Definition 3.1 *Let A be a nonempty subset of X. Then the sets*

$$DA_l = \{x \in X| \ \ Na * Nx = Na, \forall a \in A\}$$

and

$$DA_r = \{x \in X| \ \ Nx * Na = Nx, \forall a \in A\}$$

are called the dual left and dual right stabilizers of A, respectively and the set $DA = DA_l \cap DA_r$ is called the dual stabilizer of A.
For convenience the dual stabilizer, dual left and dual right stabilizers of a single element set $A = \{a\}$ are denoted by DS_a, DL_a and DR_a, respectively.

Theorem 3.2 *Let X be an involutory BCK-algebra and A be a nonempty subset of X. Then:*
(i) $N(DA_l) = (NA)^*_l$, $N(DA_r) = (NA)^*_r$ and $N(DA) = (NA)^*$,
(ii) $N(A^*_l) = D(NA)_l$, $N(A^*_r) = D(NA)_r$ and $N(A^*) = D(NA)$.

Proof (i) By Definition 2.4, we have

$$(NA)^*_l = \{x \in X| \ \ h * x = h, \forall h \in NA\}$$

$$= \{x \in X| \ \ Na * x = Na, \forall a \in A\}$$

Now let $z \in N(DA_l)$. Then $z = Nt$, for some $t \in DA_l$ and so $Na * Nt = Na$, for all $a \in A$. Thus $Na * z = Na$, for all $a \in A$, i.e. $z \in (NA)_l^*$. Therefore $N(DA_l) \subseteq (NA)_l^*$. Now let $z \in (NA)_l^*$. Then $Na*z = Na$, for all $a \in A$ and so by hypothesis $Na*NNz = Na$, for all $a \in A$. Hence $Nz \in DA_l$. Since X is involutory, then $z \in N(DA_l)$, i.e. $(NA)_l^* \subseteq N(DA_l)$. Therefore $N(DA_l) = (NA)_l^*$. By similar above argument, we obtain $N(DA_r) = (NA)_r^*$. By hypothesis we have $N(DA) = N(DA_l \cap DA_r) = N(DA_l) \cap N(DA_r) = (NA)_l^* \cap (NA)_r^* = (NA)^*$.

(ii) The proof is similar to (i).

The following example shows that the condition " X is an involutory BCK-algebra" in the above theorem is necessary.

Example 3.3 *Let* $X = \{0, a, b, c, d, e, f, 1\}$ *and let* $*$ *operation be given by the following table*

$*$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	a	0	0	0	a	0	0	0
b	b	a	0	0	b	a	0	0
c	c	a	a	0	c	a	a	0
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	e	d	d	b	a	0	0
1	1	e	e	d	c	a	a	0

Then $(X, *, 0)$ *is a bounded* BCK*-algebra with unit 1 and it is not involutory, because* $1 * (1 * b) = 1 * e = a \neq b$.

Consider $A = \{c\}$. *Then* $NA = \{d\}$. *Since* $b * d = b$ *and* $d * b = d$, *so* $b \in (NA)_r^* \cap (NA)_l^* = (NA)^*$. *Also we have* $b \notin NX = \{1, e, d, c, a\}$, *thus* $b \notin N(DA_r)$ *and* $b \notin N(DA_l)$. *Therefore* $N(DA_l) \neq (NA)_l^*$, $N(DA_r) \neq (NA)_r^*$ *and* $N(DA) \neq (NA)^*$.

If consider $B = \{d\}$. *Then* $NB = \{c\}$. *Since* $(1 * f) * (1 * c) = a * d = a = 1 * f$ *and* $(1 * c) * (1 * f) = d * a = d = 1 * c$, *then* $f \in D(NB)_r \cap D(NB)_l = D(NB)$. *So* $f \notin NX$ *implies that* $f \notin N(B_r^*)$ *and* $f \notin N(B_l^*)$. *Therefore* $N(B_l^*) \neq D(NB)_l$, $N(B_r^*) \neq D(NB)_r$ *and* $N(B^*) \neq D(NB)$.

Theorem 3.4 *Let* A *be a nonempty subset of* X. *Then* DA_l *is a dual ideal of* X.

Proof Let $N(Nx * Ny) \in DA_l$ and $y \in DA_l$. Then for all $a \in A$

$$
\begin{aligned}
Na = Na * NN(Nx * Ny) &= (Na * Ny) * NN(Nx * Ny) \\
&= (Na * NN(Nx * Ny)) * Ny \\
&= (NNN(Nx * Ny) * a) * Ny \\
&= (N(Nx * Ny) * a) * Ny \\
&= (Na * (Nx * Ny)) * Ny \\
&= (Na * Ny) * (Nx * Ny) \\
&\leq Na * Nx
\end{aligned}
$$

Thus $Na \leq Na * Nx$, for all $a \in A$. Also we have $Na * Nx \leq Na$, so $Na * Nx = Na$, for all $a \in A$, i.e. $x \in DA_l$. Therefore DA_l is a dual ideal of X.

The following example shows that DA_r is not a dual ideal in general.

Example 3.5 *Let* $X = \{0, 1, 2, 3, 4\}$ *and let* $*$ *operation be given by the following* *table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X, *, 0)$ *is a bounded* BCK*-algebra with unit 4. Consider* $A = \{3\}$*, it is easy to check that* $DA_r = \{1, 4\}$*. Now we see that* $4 * ((4 * 3) * (4 * 1)) = 4 \in DA_r$ *and* $1 \in DA_r$ *but* $3 \notin DA_r$*. Hence* DA_r *is not a dual ideal of* X*.*

The following example shows that DA_r and DA_l are not subalgebras of a bounded BCK-algebra X in general.

Example 3.6 *Let* $X = \{0, 1, 2, 3\}$ *and* $*$ *operation be given by the*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then $(X, *, 0)$ *is a bounded* BCK*-algebra with unit 3. Put* $A = \{1, 2\}$*, then* $DA_r = \{3\} = DA_l$*, it is clear that* DA_r *and* DA_l *are not subalgebras of* X*.*

Theorem 3.7 *Let* X *be a commutative bounded* BCK*-algebra and* A *be a nonempty subset of* X*. Then*
(i) $DA_r = DA_l = DA$*,*
(ii) DA_r *and* DA_l *are dual ideals of* X*.*

Proof The proof of (i) follows from the definition of commutative and the proof of (ii) follows from Theorem 3.4 and (i).

Lemma 3.8 *Let $x \in X$. Then $NNx = 1$ if and only if $x = 1$.*

Proof Let $NNx = 1$. Put $1 * x = a$, so by hypothesis we get that $1 * a = 1$. Hence $0 = a * a = (1 * x) * a = (1 * a) * x = 1 * x = a$. Then $1 * x = 0$ and so $x = 1$. The proof of the converse is easy.

Theorem 3.9 *If $0 \in A \subseteq X$, then $DA_l = DA_r = DA = \{1\}$.*

Proof It is clear that $1 \in DA_l \cap DA_r$. Now let $x \in DA_r$, then $Nx * Na = Nx$, for all $a \in A$, and so $0 = Nx * 1 = Nx * N0 = Nx$. Thus $x = 1$. Therefore $DA_r = \{1\}$. Let $x \in DA_l$. Then $Na * Nx = Na$, for all $a \in A$. Hence $N0 * Nx = N0$, i.e. $NNx = 1$ and so by Lemma 3.8, $x = 1$. Thus $DA_l = \{1\}$. Therefore $DA = \{1\}$.

Theorem 3.10 *If $A = \{1\}$, then $DA_l = DA_r = DA = X$.*

Proof Let $z \in X$. Then $Nz = Nz * 0 = Nz * N1$ and $N1 = 0 = 0 * Nz = N1 * Nz$. So $z \in DA_l \cap DA_r = DA$. Therefore $DA_l = DA_r = DA = X$.

Theorem 3.11 *Let A be a nonempty subset of X. If $DA_l = X$ or $DA_r = X$ or $DA = X$, then $A = \{1\}$.*

Proof Let $DA_r = X$ and $a \in A \subseteq X$. Then $a \in DA_r$ implies that $0 = Na * Na = Na$ and so $a = 1$. Therefore $A = \{1\}$. Similarly, $DA_l = X$ or $DA = X$ implies that $A = \{1\}$.

Theorem 3.12 *Let A be a nonempty subset of X. Then $0 \in DA_l \cup DA_r \cup DA$ if and only if $A = \{1\}$.*

Proof Let $0 \in DA_l \cup DA_r \cup DA$. Then $NNa = 1$, for all $a \in A$, and so by Lemma 3.8 $A = \{1\}$. Conversely, let $A = \{1\}$. Then by Theorem 3.10, it is clear that $0 \in DA_l \cup DA_r \cup DA$.

Theorem 3.13 *Let A be a nonempty subset of X. Then $DA_l(DA_r, DA)$ is a subalgebra of X if and only if $A = \{1\}$.*

Proof The proof follows from Theorems 3.10 and 3.12.

Theorem 3.14 *If $Nx = 1$, for all $x \in X - \{1\}$, then $DA_r = DA_l = DA = \{1\}$, for any nonempty subset $A \neq \{1\}$ of X.*

Proof Straightforward.

Theorem 3.15 *Let $\emptyset \neq A \subseteq X$. If there exists $a \in A$ such that $Na = 1$, then $DA_r = DA_l = DA = \{1\}$.*

Proof Let $x \in DA_r$. Then $Nx*Nt = Nx$, for all $t \in A$. Put $t = a$, then $Nx = Nx*1 = 0$, and so $x = 1$. Thus $DA_r = \{1\}$. Similarly we can get that $DA_l = DA = \{1\}$.

Theorem 3.16 *Let $a \in X - \{0, 1\}$ and $Nx = 1$, for all $x \in X - \{a, 1\}$. Then $DA_r = DA_l = DA = \{1\}$, for any nonempty subset $A \neq \{1\}$ of X.*

Proof First we show that $1 * a = a$. Let $1 * a = c$ and $c \neq a$. Since $1 * a \neq 1$, then $c \neq 1$ and also $a \neq 1$ implies that $c \neq 0$. Then $0 = c * c = (1 * a) * c = (1 * c) * a = 1 * a$, so $a = 1$, which is not true. Thus $1 * a = a$. Now we consider two cases:
Case i: $a \notin A$. If $x \in DA_r$, then $(1 * x) * (1 * b) = 1 * x$, for all $b \in A$. Thus we get that $1 * x = 0$, hence $x = 1$. Therefore $DA_r = \{1\}$. Also $x \in DA_l$ implies that $(1 * b) * (1 * x) = 1 * b$, for all $b \in A$, thus we get that $NNx = 1$. Then $x = 1$, by Lemma 3.8, and so $DA_l = \{1\}$.
Case ii: $a \in A$. If $A \not\subseteq \{a, 1\}$, then similar to above argument we get that $DA_r = DA_l = DA = \{1\}$. If $A = \{a\}$ or $A = \{a, 1\}$. Then $x \in DA_r$ implies that $(1*x)*(1*a) = 1*x$. Since $a \neq 0$, so $x \neq a$. If $x \neq 1$, then we get that $1 * a = 1$, hence $a = 1$, which is not true. Then $x = 1$ and so $DA_r = \{1\}$. Similarly we can get that $DA_l = DA = \{1\}$.

Theorem 3.17 *For all* $x, y, y_1, y_2, ..., y_n \in X$,

$$NN((...((Nx * y_1) * y_2) * ...) * y_n) = (...((Nx * y_1) * y_2) * ...) * y_n$$

also $NN(Nx *^n y) = Nx *^n y$.

Proof It is clear that $NN((...((Nx * y_1) * y_2) * ...) * y_n) \leq (...((Nx * y_1) * y_2) * ...) * y_n$.
Now
$((...((Nx * y_1) * y_2) * ...) * y_n) * NN((...((Nx * y_1) * y_2) * ...) * y_n)$
$= (...(((Nx * NN((...((Nx * y_1) * y_2) * ...) * y_n)) * y_1) * y_2) * ...) * y_n$
$= (...(((NNN((...((Nx * y_1) * y_2) * ...) * y_n) * x) * y_1) * y_2) * ...) * y_n$
$= (...(((N((...((Nx * y_1) * y_2) * ...) * y_n)) * x) * y_1) * y_2) * ...) * y_n$
$= ((...((Nx * y_1) * y_2) * ...) * y_n) * ((...((Nx * y_1) * y_2) * ...) * y_n) = 0$
Therefore $NN((...((Nx * y_1) * y_2) * ...) * y_n) = (...((Nx * y_1) * y_2) * ...) * y_n$.
Put $y_1 = y_2 = ... = y_n = y$, we get that $NN(Nx *^n y) = Nx *^n y$.

Theorem 3.18 *Let* D *be a dual ideal of* X. *Then* $NNx \in D$ *if and only if* $x \in D$.

Proof Let $x \in D$. Since $N(NNNx * Nx) = N(Nx * Nx) = 1 \in D$ and $x \in D$, then $NNx \in D$. Conversely, let $NNx \in D$, since $NNx \leq x$ and so $x \in D$.

Theorem 3.19 *Let* A *be a nonempty subset of* X. *Then*
(i) $[A] \cap DA_r = \{1\}$,
(ii) $DA_r = D[A]_r$,
(iii) if DA_r *is a dual ideal of* X, *then* $DA = DA_r$.

Proof (i) Let $x \in [A] \cap DA_r$. Then by $x \in [A]$, there exists $a_1, a_2, ..., a_n \in A$ such that $(...((Nx * Na_1) * Na_2) * ...) * Na_n = 0$, moreover by $x \in DA_r$, $Nx = Nx * Na_n = (Nx * Na_{n-1}) * Na_n = ... = (...((Nx * Na_1) * Na_2) * ...) * Na_n = 0$, so $Nx = 0$, i.e. $x = 1$. Therefore $[A] \cap DA_r = \{1\}$.
(ii) Let $x \in D[A]_r$. Since $A \subseteq [A]$, then $Nx * Na = Nx$, for all $a \in A$. Hence $x \in DA_r$.

On the other hand, suppose that $x \in DA_r$. So $Nx * Na = Nx$, for all $a \in A$. For any $a \in [A]$, there exists $a_1, a_2, ..., a_n \in A$, such that $(...((Na * Na_1) * Na_2) * ...) * Na_n = 0$. By hypothesis we have $(...((Nx * Na_1) * Na_2) * ...) * Na_n = Nx$. So

$$Nx * (Nx * Na) = ((...((Nx * Na_1) * Na_2) * ...) * Na_n) * (Nx * Na)$$

$$= (...(((Nx * (Nx * Na)) * Na_1) * Na_2) * ...) * Na_n$$

$$\leq (...((Na * Na_1) * Na_2) * ...) * Na_n = 0$$

Namely, $Nx \leq Nx * Na$, and so $Nx * Na = Nx$, for all $a \in [A]$. Hence $x \in D[A]_r$.
(iii) It is clear that $DA \subseteq DA_r$. Now let $x \in DA_r$. So by Lemma 3.18 $NNx \in DA_r$ and $NNa \in [A]$, for all $a \in A$. Since $Na*(Na*Nx) \leq Nx$ and $Na*(Na*Nx) \leq Na$, for all $a \in A$, then $NNx \leq N(Na * (Na * Nx))$ and $NNa \leq N(Na * (Na * Nx))$. Thus by hypothesis we get that $N(Na * (Na * Nx)) \in [A] \cap DA_r = \{1\}$, by (i). Then $N(Na * (Na * Nx)) = 1$ and so by Theorem 3.17, $Na * (Na * Nx) = NN(Na * (Na * Nx)) = N1 = 0$, for all $a \in A$. Hence $Na * Nx = Na$, for all $a \in A$. Thus $x \in DA_l$, i.e. $x \in DA_r \cap DA_l = DA$. Therefore $DA = DA_r$.

Theorem 3.20 *Let A and B be nonempty subsets of X. Then*
(i) $A \cap DA_l = \emptyset$ or $\{1\}$, $A \cap DA_r = \emptyset$ or $\{1\}$ and $A \cap DA = \emptyset$ or $\{1\}$,
(ii) if $A \subseteq B$, then $DB_l \subseteq DA_l$, $DB_r \subseteq DA_r$ and $DB \subseteq DA$,
(iii) $A \subseteq D(DA_r)_l \cap D(DA_l)_r$ and $A \subseteq D(DA)$,
(iv) $DA_l = D(D(DA_l)_r)_l$, $DA_r = D(D(DA_r)_l)_r$ and $DA = D(D(DA))$,
(v) $D(A \cup B)_l = DA_l \cap DA_l$, $D(A \cup B)_r = DA_r \cap DA_r$ and $D(A \cup B) = DA \cap DB$,
(vi) $DA_l = \bigcap_{a \in A} DL_a$, $DA_r = \bigcap_{a \in A} DR_a$ and $DA = \bigcap_{a \in A} DS_a$.

Proof (i) Let $A \bigcap DA_l \neq \emptyset$. Then there exists $x \in A \cap DA_l$ and so $Nx * Nx = Nx$, i.e. $Nx = 0$. Therefore $x = 1$. The proof of the other parts is similar.

(ii) Let $x \in DB_l$. Then $Nb * Nx = Nb$, $\forall b \in B$. Since $A \subseteq B$, $Nb * Nx = Nb$, $\forall b \in A$. So $x \in DA_l$. Similarly $DB_r \subseteq DA_r$ and $DB \subseteq DA$.
(iii) Let $a \in A$. Then $Nx * Na = Nx$, $\forall x \in DA_r$ and $Na * Ny = Na$, $\forall y \in DA_l$. So $a \in D(DA_r)_l \cap D(DA_l)_r$. Therefore $A \subseteq D(DA_r)_l \cap D(DA_l)_r$. Since $DA \subseteq DA_r$ and $DA \subseteq DA_l$, then by (ii) $D(DA_r)_l \subseteq D(DA)_l$ and $D(DA_l)_r \subseteq D(DA)_r$. Hence $A \subseteq D(DA_r)_l \cap D(DA_l)_r \subseteq D(DA)_l \cap D(DA)_r = D(DA)$.
(iv) By (iii) we get that $DA_l \subseteq D(D(DA_l)_r)_l$ and $DA_r \subseteq D(D(DA_r)_l)_r$. Also by (ii) and (iii) we have $D(D(DA_r)_l)_r \subseteq DA_r$ and $D(D(DA_l)_r)_l \subseteq DA_l$. Therefore $DA_l = D(D(DA_l)_r)_l$ and $DA_r = D(D(DA_r)_l)_r$, similar to argument in (iii) we can get that $DA = D(D(DA))$.
(v) Since $A, B \subseteq (A \cup B)$, then $D(A \cup B)_l \subseteq DA_l \cap DB_l$. Now let $x \in DA_l \cap DB_l$, then $Na * Nx = Na$, $\forall a \in A$ and $Nb * Nx = Nb$, $\forall b \in B$. Thus $Na * Nx = Na$, $\forall a \in (A \cup B)$ i.e. $x \in D(A \cup B)_l$. Therefore $D(A \cup B)_l = DA_l \cap DA_l$. Similarly $D(A \cup B)_r = DA_r \cap DA_r$, also $D(A \cup B) = D(A \cup B)_r \cap D(A \cup B)_l = (DA_r \cap DB_r) \cap (DA_l \cap DB_l) = DA \cap DB$.
(vi) The proof is similar to the proof of part (v).

The following example shows that A is a dual ideal, but DA_r is not a dual ideal.

Example 3.21 *Let* $X = \{0, 1, 2, 3, 4\}$ *in which* $*$ *is defined by the table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	3	3	0	0
4	4	3	3	1	0

Then $(X, *, 0)$ *is a bounded* BCK*-algebra with unit* 4. *Also* $A = \{3, 4\}$ *is a dual ideal,* *but* $DA_r = \{1, 2, 4\}$ *is not a dual ideal, because* $4 * ((4 * 3) * (4 * 2)) = 4 \in DA_r$ *and* $2 \in DA_r$ *but* $3 \notin DA_r$.

Theorem 3.22 *If* A *is a dual ideal of* X*, then* DA *is a dual ideal of* X.

Proof Let $N(Nx * Ny) \in DA$ and $y \in DA$. Then by Theorem 3.4 we get that $x \in DA_l$. Since A is a dual ideal, then $NNa \in A$, for all $a \in A$. Also $x \in DA_l$ implies that $NNx \in DA_l$. We have $NNx \leq N(Nx*(Nx*Na))$ and $NNa \leq N(Nx*(Nx*Na))$. Then by hypothesis we get that $N(Nx * (Nx * Na)) \in A \cap DA_l = \{1\}$, by Theorem 3.20(i) and so $N(Nx * (Nx * Na)) = 1$, for all $a \in A$. Hence, by Theorem 3.17 , $Nx = Nx * Na$, i.e. $x \in DA_r$. Therefore $x \in DA$.

Theorem 3.23 *Let* A *and* B *be two dual ideals of* X*. Then* $A \cap B = \{1\}$ *if and only if* $A \subseteq DB$.

Proof Let $A \cap B = \{1\}$ and $a \in A$. Since $NNa \leq N(Na * (Na * Nb))$ and $NNb \leq N(Na * (Na * Nb))$, for all $b \in B$, also $NNa \in A$ and $NNb \in B$, thus by hypothesis we get that $N(Na * (Na * Nb)) \in A \cap B = \{1\}$. Then $N(Na * (Na * Nb)) = 1$ and so $Na * (Na * Nb) = 0$, by Theorem 3.17. Therefore $Na = Na * Nb$, for all $b \in B$. Similarly, we can get that $Nb * Na = Nb$, for all $b \in B$, then $a \in DB$.
Conversely, let $A \subseteq DB$. Consider $x \in A \cap B$. Then $Nx * Nb = Nx$, for all $b \in B$ and so $0 = Nx * Nx = Nx$. Thus $x = 1$. Therefore $A \cap B = \{1\}$.

Theorem 3.24 *A be a dual ideal of* X*. Then* $DA_l = DA \subseteq DA_r$*. In particular, if* DA_r *is a dual ideal of* X*,* $DA = DA_l = DA_r$.

Proof We have $DA \subseteq DA_l$. Since DA_l and A are dual ideals and $DA_l \cap A = \{1\}$ by Theorem 3.20(i), then $DA_l \subseteq DA$, by Theorem 3.23. Therefore $DA_l = DA \subseteq DA_r$. In particular, if DA_r is a dual ideal of X, then by Theorem 3.19(iii) we get that $DA = DA_r$. Therefore $DA = DA_l = DA_r$.

4 Dual normal BCK-algebras

Definition 4.1 *A bounded BCK-algebra X is called dual normal, if the dual right stabilizer DR_a of any element $a \in X$ is a dual ideal of X.*

The following theorem follows from Theorems 3.10, 3.14 and 3.16.

Theorem 4.2 *Under each of the following conditions, X is dual normal.*
(i) $Nx = 1$, for all $x \in X - \{1\}$,
(ii) $a \in X - \{0, 1\}$ and $Nx = 1$, for all $x \in X - \{a, 1\}$.

According to Theorem 3.7 any commutative bounded BCK-algebra is a dual normal BCK-algebra, but the converse may not be true.

Example 4.3 *Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by the table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	1	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

Then $(X, *, 0)$ is a bounded BCK-algebra with unit 4 and it is dual normal, by Theorem 4.2(i). But X is not commutative, because $4 * (4 * 2) = 0 \neq 2 = 2 * (2 * 4)$.

Theorem 4.4 *The following statements are equivalent:*
(i) X is dual normal,
(ii) $DR_a \subseteq DL_a$, $\forall a \in X$,
(iii) $DR_a = DL_a$, $\forall a \in X$,
(v) $Nx * Ny = Nx$ implies $Ny * Nx = Ny$, $\forall x, y \in X$.

Proof $(i) \to (ii)$ Since X is dual normal, then for all $a \in X$, DR_a is a dual ideal of X. So

$$DR_a = D[a]_r, \quad \text{by Theorem 3.19(ii)}$$

$$= D[a]_l, \quad \text{by Theorem 3.24}$$

$$\subseteq DL_a, \quad \text{by Theorem 3.20(ii)}$$

$(ii) \to (iii)$ For any $x \in DL_a$, we have $Na * Nx = Na$, thus $a \in DR_x$. By (ii) we have $DR_x \subseteq DL_x$, hence $a \in DL_x$. So $Nx * Na = Nx$, i.e. $x \in DR_a$. Hence $DL_a \subseteq DR_a$. Therefore $DR_a = DL_a$.
$(iii) \to (v)$ Assume that $x, y \in X$ and $Nx * Ny = Nx$. Then $x \in DR_y = DL_y$ and so $Ny * Nx = Ny$.
$(v) \to (i)$ By hypothesis we have

$$DR_a = \{x \in X \mid Nx * Na = Nx\} = \{x \in X \mid Na * Nx = Na\} = DL_a$$

Since DL_a is a dual ideal, then DR_a is too. Therefore X is dual normal.

Definition 4.5 *A bounded BCK -algebra X is called dual semisimple if every dual ideal A of X is a sub-summand of X, i.e. there exists a dual ideal B of X such that $A \bigcap B = \{1\}$ and $X = [A \bigcup B]$.*

Consider Example 3.6. It is easy to check that $A_1 = \{1, 3\}$, $A_2 = \{3, 2\}$, $A_3 = \{3\}$ and $A_4 = \{0, 1, 2, 3\}$ are the only dual ideals of X. Also $A_1 \cap A_2 = \{3\}$ and $[A_1 \bigcup A_2] = [\{1, 2, 3\}] = X$. Thus X is dual semisimple.

Theorem 4.6 *Every dual semisimple bounded BCK-algebra is dual normal.*

Proof Let $a \in X$. Then there exists a dual ideal B of X such that $[a] \cap B = \{1\}$ and $X = [[a] \bigcup B]$. By $[a] \cap B = \{1\}$ and Theorem 3.23 we get that $B \subseteq D[a] \subseteq D[a]_r$.

Since by Theorem 3.19(ii) $D[a]_r = DR_a$, then $B \subseteq DR_a$. Let $x \in DR_a$. Then by $x \in X = [[a] \bigcup B]$ there exists $n \in \aleph$ such that $N(Nx *^n Na) \in B$. Since $x \in DR_a$, then $Nx{*^n}Na = Nx$. Hence $NNx \in B$ and so $x \in B$ by Lemma 3.18. Thus $B = DR_a$. Therefore X is dual normal.

Theorem 4.7 *Let $[a]$ be a sub-summand of X, for any $a \in X$. Then DR_a is a sub-summand of X, for any $a \in X$.*

Proof Let $a \in X$. Then there exists dual ideal B of X such that $X = [[a] \bigcup B]$ and $[a] \cap B = \{1\}$. By the proof of Theorem 4.6, we get that $B = DR_a$. Therefore DR_a is a sub-summand of X.

Theorem 4.8 *et X be a finite bounded BCK-algebra. Then the following are equivalent:*
(i) X is dual normal,
(ii) $[a]$ is a sub-summand of X, for any $a \in X$
(iii) DR_a is a sub-summand of X, for any $a \in X$
(iv) There exists a fixed natural number n such that $Nx * (Ny *^n Nx) = Nx$, for all $x, y \in X$,
(v) There exists a fixed natural number n such that $Nx*(Nx*Ny) \leq Ny*(Ny*^n Nx)$, for all $x, y \in X$,
(vi) X is dual semisimpel.

Proof (i)$\rightarrow$ (ii) Let $a \in X$. Since for all $x \in X$, $... \leq Nx *^n Na \leq ... \leq Nx *^2 Na \leq Nx * Na$ and X is finite, then there exists $n \in \aleph$ such that $Nx *^n Na = Nx *^{n+1} Na$. This show that by Theorem 3.17, $NN(Nx *^n Na) * Na = NN(Nx *^n Na)$ and so $N(Nx *^n Na) \in DR_a$, Since DR_a is a dual ideal, then by Theorem 2.3, we get that $x \in [[a] \cup DR_a]$ i.e. $X = [[a] \cup DR_a]$. Also by Theorem 3.19(i), $[a] \cap DR_a = \{1\}$. Therefore $[a]$ is a sub-summand of X.
(ii)$\rightarrow$ (iii) It is proved in Theorem 4.7.

(iii) $\to$ (iv) Let $x, y \in X$. Since DR_x is a sub-summand of X, then there exists a dual ideal A of X such that $A \cap DR_x = \{1\}$ and $X = [DR_x \cup A]$. By argument in the last part we get that there exists $m = m(x,y) \in \aleph$ such that $N(Ny *^m Nx) \in DR_x$. Consider $T = \{n = n(u,v) \in \aleph| \ \ N(Nu *^n Nv) \in DR_v, u, v \in X\}$. Since X is finite, then T is a finite set and so it has the greatest element, say n. Clearly $N(Ny *^n Nx) \in DR_x$. Now $x \in X = [A \cup DR_x]$ implies that by Theorem 2.2 there exists $x_1, x_2, ..., x_n \in DR_x$ such that $N((...((Nx * Nx_1) * Nx_2) * ...) * Nx_n) \in A$. On the other hand since $[x] \cap DR_x = \{1\}$, by Theorem 3.23 $[x] \subseteq D(DR_x)$, and thus by $x \in [x] \subseteq D(DR_x)$ we get that $Nx * Nx_i = Nx$, for all $1 \leq i \leq n$. Hence $NNx \in A$, so by Lemma 3.18, $x \in A$. Since $A \cap DR_x = \{1\}$, then by Theorem 3.23 $A \subseteq D(DR_x)$, so $x \in A$ and $N(Ny *^n Nx) \in DR_x$ implies that $Nx * NN(Ny *^n Nx) = Nx$. Therefore by Theorem 3.17, $Nx * (Ny *^n Nx) = Nx$.

(iv)$\to$ (v) $Nx * (Nx * Ny) = (Nx * (Ny *^n Nx)) * (Nx * Ny) \leq Ny * (Ny *^n *Nx)$, for all $x, y \in X$.

(v)$\to$ (vi) Let A be a dual ideal of X and $x \in DA_r$. Then $Nx * Na = Nx$, for all $a \in A$ and so $Nx *^n Na = Nx$. Hence by (v) we get that $Na * (Na * Nx) \leq Nx * (Nx *^n Na) = Nx * Nx = 0$, that is, $Na = Na * Nx$ and so $x \in DA_l$. Therefore $DA_r \subseteq DA_l$. Since A is a dual ideal, then by Theorem 3.24, $DA_l \subseteq DA_r$ and so $DA_r = DA_l$. Hence by Theorem 3.4 DA_r is a dual ideal of X. Since X is finite , we suppose that $A = \{a_1, a_2, ..., a_k\}$ and $Nx *^{n_i} Na_i = Nx *^{n_i+1} Na_i$, where $x \in X$, $a_i \in A$ and $n_i = n_i(x, a_i) \in \aleph$, for all $1 \leq i \leq k$. Put $y = N((...((Nx *^{n_1} Na_1) *^{n_2} Na_2) *^{n_3} ...) *^{n_k} Na_k$ by hypothesis and Theorem 3.17 $Ny = (...((Nx *^{n_1} Na_1) *^{n_2} Na_2) *^{n_3} ...) *^{n_k} Na_k)$ and so $Ny = Ny * Na$, for all $a \in A$, that is $y \in DA_r$ also $((...((Nx *^{n_1} Na_1) *^{n_2} Na_2) *^{n_3} ...) *^{n_k} Na_k) * Ny = 0$, hence $x \in [A \cup DA_r]$ i.e $X = [A \cup DA_r]$. Also $A \cap DA_r = \{1\}$, by Theorem 3.20(i). Therefore X is dual semisimple.

(vi)$\to$ (i) It is proved in Theorem 4.6.

Open problem. Is any infinite dual normal BCK-algebra a dual semisimple?

Definition 4.9 *The dual J-radical , denoted by $DJ(X)$, of a bounded BCK-algebra X means the intersection of all maximal dual ideals of X. By Zorn's Lemma the collection of maximal dual ideals of X is nonempty. If $DJ(X) = \{1\}$, then X is called dual J-semisimple.*

Consider bounded BCK-algebra $X = \{0, 1, 2, 3, 4\}$ in Example 3.5. We can see that the only dual ideals of X are $\{3, 4\}$, $\{4\}$ and X, so $DJ(X) = \{3, 4\}$.
Bounded BCK-algebra $X = \{0, 1, 2, 3\}$ in Example 3.6 is a dual J-semisimple, because the only dual ideals on X are $\{2, 3\}$, $\{1, 3\}$, $\{3\}$ and X, so $DJ(X) = \{3\}$.

Theorem 4.10 *Every dual J-semisimple bounded BCK-algebra is dual normal.*

Proof On the contrary, let X do not be dual normal. Then $\exists\ a \in X$ such that DR_a is not a dual ideal. Thus $DR_a \subset [DR_a]$(proper containing). We show that $[DR_a] \cap [a] \neq \{1\}$.
If $[DR_a] \cap [a] = \{1\}$, then by Theorem 3.23 $[DR_a] \subseteq D[a] \subseteq D[a]_r$. Since by Theorem 3.19 $DR_a = D[a]_r$, then we get that $[DR_a] \subseteq DR_a$, which is impossible. Thus $[DR_a] \cap [a] \neq \{1\}$. We choose $1 \neq b \in [a] \cap [DR_a]$. Let M be a maximal dual ideal of X. We consider the following cases:
Case (i): $a \in M$, since $b \in [a] \subseteq M$, then $b \in M$.
Case (ii): $a \notin M$, then by maximality of M, $X = [M \cup \{a\}]$. We show that $DR_a \subseteq M$. Let $x \in DR_a \subseteq X$. Then there exists $n \in \aleph$ such that $N(Nx *^n Na) \in M$ and so by $x \in DR_a$, we get that $NNx \in M$. Thus by Lemma 3.18, $x \in M$ and so $DR_a \subseteq M$. Hence $b \in M$, since $b \in [DR_a] \subseteq M$. This show that $1 \neq b \in DJ(X)$, a contradiction with $DJ(X) = \{1\}$.

Open problem. Is any dual normal BCK-algebra a dual J-semisimple?

Theorem 4.11 *Let Y be a bounded BCK-algebra with unit 1. Then Y is a dual normal BCK-algebra if and only if every subalgebra X of Y containing 1, is dual normal BCK-algebra .*

Proof ($\Leftarrow$) It is clear.
($\Rightarrow$) Let $a \in X$ and let DR_a and $DR_a^{'}$ be the dual right stabilizers of element a with respect to X and Y, respectively . i.e.

$$DR_a = \{x \in X | \ \ Nx{*}Na = Nx\}$$

and

$$DR_a^{'} = \{x \in Y | \ \ Nx{*}Na = Nx\}.$$

Then $DR_a = DR_a^{'} \bigcap X$. Now we show that DR_a is a dual ideal of X. Let $x, y \in X, N(Nx{*}Ny) \in DR_a$ and $y \in DR_a$. Since $DR_a \subseteq DR_a^{'}$ and $DR_a^{'}$ is a dual ideal of Y, so $x \in DR_a^{'}$. Also $x \in DR_a^{'} \bigcap X = DR_a$ implies DR_a is a dual ideal of X. Therefore X is dual normal.

Let $(X_i, *_i, 0_i)(i \in I)$ be an indexed family of BCK-algebras and $\prod\limits_{i \in I} X_i$ be the set of all mapping $f : I \longrightarrow \bigcup\limits_{i \in I} X_i$ and $f(i) \in X_i$ for all $i \in I$.

For $f, g \in \prod_{i \in I} X_i$, we define $f{*}g$ by
$(f{*}g)(i) = f(i){*}_i g(i)$, for all $i \in I$ and 0 by $0(i) = 0_i$.

Then $(\prod\limits_{i \in I} X_i, *, 0)$ is a BCK-algebra. Also $\prod\limits_{i \in I} X_i$ is bounded if and only if every X_i is bounded.

Theorem 4.12 *Let $\{I_i\}_{i \in I}$ be an indexed family of subsets of bounded BCK-algebras $X_i (i \in I)$. Then* *(i) if every I_i is a dual ideal of $X_i(i \in I)$, then $\prod_{i\in I} I_i$ is a dual ideal of $\prod_{i\in I} X_i$,*

(ii) if $\prod_{i\in I} I_i$ is a dual ideal of $\prod_{i\in I} X_i$, then every I_i is a dual ideal of X_i.

Proof (i) Let I_i be a dual ideal of X_i ,$\forall i \in I$ and let $N(Nx{*}Ny) \in \prod_{i\in I} I_i$ and $y \in \prod_{i\in I} I_i$. Then $1_i{*_i}((1_i{*_i}x(i)){*_i}(1_i{*_i}y(i))) = 1(i){*_i}((1(i){*_i}x(i)){*_i}(1(i){*_i}y(i)) = 1{*}((1{*}x){*}(1{*}y))(i) \in I_i$ and $y(i) \in I_i$, for all $i \in I$. So $x(i) \in I_i$. This shows $x \in \prod_{i\in I} I_i$.

(ii) Suppose that $\prod_{i\in I} I_i$ is a dual ideal of $\prod_{i\in I} X_i$. Without loss of generality we show that I_j is a dual ideal of X_j. If $1_j{*_j}((1_j{*_j}x_1){*_j}(1_j{*_j}y_1)) \in I_j$ and $y_1 \in I_j$, we define x and y as follows

$$x(i) = \begin{cases} x_1 & \text{if } i = j \\ 1_i & \text{if } i \neq j \end{cases}$$

$$y(i) = \begin{cases} y_1 & \text{if } i = j \\ 1_i & \text{if } i \neq j \end{cases}$$

and so

$$N(Nx{*}Ny)(i) = \begin{cases} 1_j{*_j}((1_j{*_j}x_1){*_j}(1_j{*_j}y_1)) & \text{if } i = j \\ 1_i & \text{if } i \neq j \end{cases}$$

Then $N(Nx{*}Ny) \in \prod_{i\in I} I_i$ and $y \in \prod_{i\in I} I_i$. Thus $x \in \prod_{i\in I} I_i$, i.e. $x_1 \in I_j$.

Theorem 4.13 *Let $(X_i, {*_i}, 0_i, 1_i)(i \in I)$ be an indexed family of bounded BCK-algebras. Then every X_i is dual normal BCK-algebra if and only if $\prod_{i\in I} X_i$ is dual normal BCK-algebra.*

Proof $(\Rightarrow)$ Let $f \in \prod_{i\in I} X_i$. Then

$$DR_f = \{g \in \prod_{i\in I} X_i \mid 1{*}g = (1{*}g){*}(1{*}f)\}$$

$$= \{g \in \prod_{i\in I} X_i \mid 1(i){*_i}g(i) = (1(i){*_i}g(i)){*_i}(1(i){*_i}f(i)), \forall i \in I\}$$

$$= \{g \in \prod_{i\in I} X_i \mid 1_i{*_i}g(i) = (1_i{*_i}g(i)){*_i}(1_i{*_i}f(i)), \forall i \in I\}$$

$$= \{g \in \prod_{i\in I} X_i \mid g(i) \in DR_{f_{(i)}}\} = \prod_{i\in I} DR_{f_{(i)}}$$

Since $DR_{f_{(i)}}$ is a dual ideal of X_i,$\forall i \in I$ then by Theorem 4.12 we get that $DR_f = \prod_{i\in I} DR_{f_{(i)}}$ is a dual ideal of $\prod_{i\in I} X_i$.

($\Leftarrow$) Let $a_j \in X_j$. Then we define $f : I \longrightarrow \bigcup_{i\in I} X_i$ such that

$$f(i) = \begin{cases} a_j & \text{if } i = j \\ 1_j & \text{if } i \neq j \end{cases}$$

So $f \in \prod_{i\in I} X_i$. Similar to above we have $DR_f = \prod_{i\in I} DR_{f_{(i)}}$ where,

$$DR_{f(i)} = \begin{cases} DR_{a_j} & \text{if } i = j \\ X_i & \text{if } i \neq j \end{cases}$$

Since $\prod_{i\in I} X_i$ is dual normal , then DR_f is a dual ideal of $\prod_{i\in I} X_i$. So by Theorem 4.12 DR_{a_j} is a dual ideal of X_j. Therefore X_j is dual normal.

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