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Analytic approximations of solutions to systems of ordinary differential equations with variable coefficients[1]

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Abstract

In this paper, reproducing kernel method is presented to obtain the analytical and approximate solutions to system of ordinary differential equations with variable coefficients. The solution obtained by using the method takes the form of a convergent series with easily computable components. In the mean time, the approximate solution $u_n(x)$ is obtained by the n-term intercept of the analytical solution and is proved to converge to the exact solution. Some numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method indicate the method is simple and effective.

Keywords: analytic approximation; systems of ordinary differential equations; variable coefficients; reproducing kernel

1 Introduction

In this paper, we consider the following system of first order nonlinear ordinary differential equations with variable coefficients in reproducing kernel space

$$\begin{cases} u_i'(x) + \sum\limits_{j=1}^{n} a_{ij}(x)u_j(x) - g_i(u_1, u_2, ..., u_n) = f_i(x), 0 \leq x \leq 1, \\ u_i(0) = 0, i = 1, 2, ..., n, (1) \end{cases} \tag{1.1}$$

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where $u=(u_1,u_2,...,u_n)^\top$, $u_i\in W_2^2[0,1]$, $f_i+g_i\in W_2^1[0,1]$, $f_i(x)$ and $g_i(u_1,u_2,...,u_n)$ are continuous.

As we know, most of higher-order differential equations can be converted into first order differential equations. Ordinary differential equations is an important tool of solving real-world problems. A wide variety of natural phenomena are modelled by first order ordinary differential equations. Ordinary differential equations has been applied to many problems, in physics, engineer, biology and so on. Besides classical numerical method, there are many valid method to obtain numerical solution. Weiming Wang and Zhengqing Li established a mechanical algorithm for solving ordinary differential equations [1]. Luis Lara treated a system of linear ordinary differential equations via an iteration one-step integration method [2]. J.Biaazar and his co-workers studied the equations by Adomian decomposition method [3]. Chen Tang and his co-workers developed various order-explicit multistep schemes of exponential fitting for ordinary differential equations [4]. However, as we know, there are few researchers obtaining the analytical solution of first order ordinary differential equations with variable coefficients involving more than three equations.

Reproducing kernel theory has important application in numerical analysis, differential equation, probability and statistics and so on. Recently, using the RKM, some authors discussed singular linear two-point boundary value problems, singular nonlinear two-point periodic boundary value problems, nonlinear systems of boundary value problems and nonlinear partial differential equations and so on [5-13].

In this paper, we will give the representation of analytical solution to $Eq.(1.1)$ in the reproducing kernel space under the assumption that the solution to $Eq.(1.1)$ is unique.

Put $A_{ii}u_i=u_i'+a_{ii}(x)u_i$, $A_{ij}u_j=a_{ij}(x)u_j$, $(i,j=1,2,...,n)$, $A=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n}\\ A_{21} & A_{22} & \cdots & A_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ A_{n1} & A_{n2} & \cdots & A_{nn}\end{pmatrix}$, then $Eq.$(1.1) can be converted into following form

$$\begin{cases} Au = f(x) + g(u_1, u_2, ..., u_n), 0 \leq x \leq 1, \\ u(0) = 0, \end{cases} \tag{1.2}$$

where $f = (f_1, f_2, ..., f_n)^\top$, $g = (g_1, g_2, ..., g_n)^\top$, $u \in \bigoplus_n W_2^2[0,1]$, $f + g \in \bigoplus_n W_2^1[0,1]$. $\bigoplus_n W_2^2[0,1]$ and $\bigoplus_n W_2^1[0,1]$ are defined in the following section.

2 Several Reproducing Kernel Spaces and Lemmas

In order to solve (1.2) using reproducing kernel method (RKM), we first construct a reproducing kernel Hilbert space $W_2^2[0,1]$ in which every function satisfies the initial conditions of (1.2).

1 The reproducing kernel space $W_2^2[0,1]$

Inner product space $W_2^2[0,1]$ is defined as $W_2^2[0,1] = \{u(x) \mid u, u'$ are absolutely continuous real value functions, $u, u', u'' \in L^2[0,1], u(0) = 0\}$. The inner product in $W_2^2[0,1]$ is given by

$$(u(x), v(x))_{W_2^2} = u(0)v(0) + u(1)v(1) + \int_0^1 u''v''dx, \tag{2.1}$$

$u, v \in W_2^2[0,1]$ and the norm $\| u \|_{W_2^2}$ is denoted by $\| u \|_{W_2^2} = \sqrt{(u,u)_{W_2^2}}$.

Theorem 2.1. *The space $W_2^2[0,1]$ defined as above is a reproducing kernel space. That is, there exists $R_x(y) \in W_2^2[0,1]$,for any $u(y) \in W_2^2[0,1]$ and each fixed $x \in [0,1]$, $y \in [0,1]$, such that $(u(y), R_x(y))_{W_2^2} = u(x)$. The reproducing kernel $R_x(y)$ can denoted by*

$$R_x(y) = \begin{cases} \frac{y\left(x\left(8-3x+x^2\right)+(-1+x)y^2\right)}{6}, & y \leq x \\ \frac{x\left(x^2(-1+y)+y\left(8-3y+y^2\right)\right)}{6}, & y > x. \end{cases} \tag{2.2}$$

For the proof of Theorem (2.1), please can refer to [6-10].

2 The reproducing kernel space $W_2^1[0,1]$

The inner product space $W_2^1[0,1]$ is defined by $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real value function, $u, u' \in L^2[0,1]\}$. The inner product and norm in $W_2^1[0,1]$

are given respectively by

$$(u(x), v(x))_{W_2^1} = u(0)v(0) + \int_0^1 u'v'dx, \ \| u \|_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

where $u(x), v(x) \in W_2^1[0,1]$. In [5], the authors proved that $W_2^1[0,1]$ is a reproducing kernel space and its reproducing kernel is

$$\overline{R}_x(y) = \begin{cases} 1+y, & y \le x \\ 1+x, & y > x. \end{cases}$$

3 Important Lemmas

Suppose $\{X_i\}_{i=1}^n$ are a set of Banach spaces. Define space $\bigoplus_n X_i = \{u = (u_1, u_2, ..., u_n)^\top | u_i \in X_i, i = 1, 2, ..., n\}$. The norm of $\bigoplus_n X_i$ is given by $\| u \| = (\sum_{i=1}^n \| u_i \|)^{\frac{1}{2}}$. Clearly, $\bigoplus_n X_i$ is a Banach space. In particular, take $X_i = W_2^2[0,1], i = 1, 2, ..., n$ and define inner product $(u, v) = \sum_{i=1}^n (u_i, v_i)_{W_2^2}, u, v \in \bigoplus_n W_2^2[0,1]$. Then $\bigoplus_n W_2^2[0,1]$ is a Hilbert space.

Lemma 2.2. *If $A_{ij} : W_2^2[0,1] \to W_2^1[0,1], i, j = 1, 2, ..., n$ is a bounded linear operator, then $A : \bigoplus_n W_2^2[0,1] \to \bigoplus_n W_2^2[0,1]$ is a bounded linear operator.*

Proof. Clearly, A is a linear operator. For $\forall\ u \in \bigoplus_n W_2^2[0,1]$,

$$\begin{aligned} \| Au \| &= (\sum_{i=1}^n \| \sum_{j=1}^n A_{ij} u_j \|^2)^{\frac{1}{2}} \\ &\le [\sum_{i=1}^n (\sum_{j=1}^n \| A_{ij} \| \ \| u_j \|)^2]^{\frac{1}{2}} \\ &\le [\sum_{i=1}^n (\sum_{j=1}^n \| A_{ij} \|^2)(\sum_{j=1}^n \| u_j \|^2)]^{\frac{1}{2}} \\ &= (\sum_{i=1}^n \sum_{j=1}^n \| A_{ij} \|^2)^{\frac{1}{2}} \| u \| . \end{aligned} \tag{2.3}$$

The boundedness of A_{ij} implies that A is bounded. The proof is complete.

Lemma 2.3. *If $A_{ij} : W_2^2[0,1] \to W_2^1[0,1], i, j = 1, 2, ..., n$ are bounded linear opera-*

tors, then the adjoint operator A^* *of* A *is* $\begin{pmatrix} A_{11}^* & A_{21}^* & \cdots & A_{n1}^* \\ A_{12}^* & A_{22}^* & \cdots & A_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n}^* & A_{2n}^* & \cdots & A_{nn}^* \end{pmatrix}$, *where* A_{ij}^* *is the adjoint operator of* A_{ij}.

Proof. For $\forall u \in \bigoplus_n W_2^2[0,1], v \in \bigoplus_n W_2^1[0,1]$,

$$\begin{aligned} (Au, v) &= \sum_{i=1}^{n}(\sum_{j=1}^{n} A_{ij}u_j, v_i) = \sum_{i=1}^{n}\sum_{j=1}^{n}(A_{ij}u_j, v_i) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(u_j, A_{ij}^* v_i) = \sum_{j=1}^{n}(u_j, \sum_{i=1}^{n} A_{ij}^* v_i) \\ &= (u, \begin{pmatrix} A_{11}^* & A_{21}^* & \cdots & A_{n1}^* \\ A_{12}^* & A_{22}^* & \cdots & A_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n}^* & A_{2n}^* & \cdots & A_{nn}^* \end{pmatrix} v). \end{aligned} \tag{2.4}$$

From (2.4), it follows that $A^* = \begin{pmatrix} A_{11}^* & A_{21}^* & \cdots & A_{n1}^* \\ A_{12}^* & A_{22}^* & \cdots & A_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n}^* & A_{2n}^* & \cdots & A_{nn}^* \end{pmatrix}$. □

□

3 Analytic approximation of solutions to Eq.(1.2)

In this section, we will give the representation of analytical solution of *Eq.*(1.2) and implementation method in the reproducing kernel space $\bigoplus_n W_2^2[0,1]$.

In view of Lemma(2.1), it is clear that $A : \bigoplus_n W_2^2[0,1] \to \bigoplus_n W_2^2[0,1]$ is a bounded linear operator. Put $\varphi_{ij}(x) = \overline{R}_{x_i}(x)\overrightarrow{e_j} = (0, 0, ..., \overline{R}_{x_i}(x), 0, ..., 0)^\top$ and $\psi_{ij}(x) = A^*\varphi_{ij}(x)$, where $\overline{R}_x(y)$ is the reproducing kernel of $W_2^1[0,1]$ and A^* is the adjoint operator of A . The normal orthogonal system $\{\overline{\psi}_{ij}(x)\}_{(1,1)}^{(\infty,n)}$ of $\bigoplus_n W_2^2[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\psi_{ij}(x)\}_{(1,1)}^{(\infty,n)}$,

$$\overline{\psi}_{ij}(x) = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(x), i = 1, 2, \cdots. \tag{3.1}$$

Theorem 3.1. *For Eq.*(1.2)*, if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$*, then* $\{\psi_{ij}(x)\}_{(1,1)}^{(\infty,n)}$ *is the complete system of* $\bigoplus_n W_2^2[0,1]$.

Proof. For each fixed $u(x) \in \bigoplus_n W_2^2[0,1]$, let $(u(x), \psi_{ij}(x)) = 0, (i = 1, 2, \cdots)$, which means that,

$$(Au(x), \varphi_{ij}(x)) = 0. \tag{3.2}$$

Note that

$$u(x) = \sum_{j=1}^{n} u_j(x)\overrightarrow{e_j} = \sum_{j=1}^{n}(u(\cdot), R_x(\cdot)\overrightarrow{e_j})\overrightarrow{e_j}.$$

Hence, $Au(x_i) = \sum_{j=1}^{n}(Au(y), \varphi_{ij}(y))\overrightarrow{e_j} = 0(i = 1, 2, \cdots)$. Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, we must have $(Au)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. Therefore, $\{\psi_{ij}(x)\}_{(1,1)}^{(\infty,n)}$ is the complete system of $\bigoplus_n W_2^2[0,1]$. So the proof of the Theorem 3.1 is complete. □

Theorem 3.2. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *and the solution of Eq.*(1.2) *is unique, then the solution of Eq.*(1.2) *satisfies the form*

$$u(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{n}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}F_k(x_l, u_1(x_l), u_2(x_l), \cdots, u_n(x_l))\overline{\psi}_{ij}(x), \tag{3.3}$$

where $F(x, u_1(x), u_2(x), \cdots, u_n(x)) = f(x) + g(u_1(x), u_2(x), \cdots, u_n(x)) = (F_1, F_2, \cdots, F_n)^{\top}$.

Proof. Applying Theorem 3.1, it is easy to know that $\{\overline{\psi}_{ij}(x)\}_{(1,1)}^{(\infty,n)}$ is the complete normal orthogonal basis of $\bigoplus_n W_2^2[0,1]$.
Note that $(v(x), \varphi_{ij}(x)) = v_j(x_i)$ for each $v(x) \in \bigoplus_n W_2^1[0,1]$, hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}\sum_{j=1}^{n}(u(x), \overline{\psi}_{ij}(x))\overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{n}(u(x), \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}\psi_{lk}(x))\overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{n}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}(u(x), A^*\varphi_{lk}(x))\overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{n}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}(Au(x), \varphi_{lk}(x))\overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{n}\sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}F_k(x_l, u_1(x_l), u_2(x_l), ..., u_n(x_l))\overline{\psi}_{ij}(x) \end{aligned} \tag{3.4}$$

So, the proof of the theorem is complete. □

Remark:

Case(i): $Eq.(1.2)$ is linear, that is, $g(u_1(x), u_2(x), ..., u_n(x)) = 0$. Then (3.4) is the required analytical solution to $Eq.(1.2)$.

Case(ii): $Eq.(1.2)$ is nonlinear. In this case, we will obtain analytical solution to $Eq.(1.2)$ using following method.

The implementation method

Write $F(x, u(x)) = F(x, u_1(x), u_2(x), ..., u_n(x))$ simply. (3.3) can be denoted by

$$u(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{n}A_{ij}\overline{\psi}_{ij}(x), \tag{3.5}$$

where $A_{ij} = \sum_{l=1}^{i}\sum_{k=1}^{j}\beta_{lk}^{ij}F_k(x_l, u(x_l))$. Let $x_1 = 0$, it follows that $u(x_1)$ is known from the initial conditions of $Eq.(1.2)$. So $F(x_1, u(x_1))$ is known. Considering the numerical computation, we put $u_0(x_1) = u(x_1)$ and define the m-term approximation to $u(x)$ by

$$u_m(x) = \sum_{i=1}^{m}\sum_{j=1}^{n}B_{ij}\overline{\psi}_{ij}(x), \tag{3.6}$$

where

$$
\begin{aligned}
&B_{1j} = \sum_{k=1}^{j} \beta_{1k}^{1,j} F_k(x_1, u_0(x_1)), j = 1, 2, \cdots, n \\
&u_1(x) = \sum_{j=1}^{n} B_{1j}\overline{\psi}_{1j}(x), \\
&B_{2j} = \sum_{l=1}^{2} \sum_{k=1}^{j} \beta_{lk}^{ij} F_k(x_l, u_1(x_l)), j = 1, 2, \cdots, n \\
&u_2(x) = \sum_{i=1}^{2} \sum_{j=1}^{n} B_{ij}\overline{\psi}_{ij}(x), \\
&\cdots\cdots \\
&u_{m-1}(x) = \sum_{i=1}^{m-1} \sum_{j=1}^{n} B_{ij}\overline{\psi}_{ij}(x), \\
&B_{mj} = \sum_{l=1}^{m} \sum_{k=1}^{j} \beta_{lk}^{ij} F_k(x_l, u_{m-1}(x_l)), j = 1, 2, \cdots, n
\end{aligned} \tag{3.7}
$$

Next, the convergence of $u_m(x)$ will be proved.

Theorem 3.3. *Suppose that $\| u_m \|$ is bounded in (3.6) and Eq.(1.2) has a unique solution. If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the n-term approximate solution $u_m(x)$ derived from the above method converges to the exact solution $u(x)$ of Eq.(1.2) and*

$$u(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{n} B_{ij}\overline{\psi}_{ij}(x), \tag{3.8}$$

where B_{ij} is given by (3.7).

Proof. First, we will prove the convergence of $u_n(x)$.
From (3.6), we infer that

$$u_{m+1}(x) = u_m(x) + B_{m+1,j}\overline{\psi}_{m+1,j}(x). \tag{3.9}$$

The orthonormality of $\{\overline{\psi}_{ij}\}_{(1,1)}^{(\infty,n)}$ yields that

$$\| u_{m+1} \|^2 = \| u_m \|^2 + \sum_{j=1}^{n} (B_{m+1,j})^2 = \cdots\cdots = \sum_{i=1}^{m+1} \sum_{j=1}^{n} (B_{ij})^2 \tag{3.10}$$

In terms of (4.6), it holds that $\| u_{m+1} \| \geq \| u_m \|$. Due to the condition that $\| u_m \|$ is bounded, $\| u_m \|$ is convergent and there exists a constant c such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{n} (B_{ij})^2 = c.$$

This implies that

$$\sum_{j=1}^{n} B_{ij}^2 \in l^2, i = 1, 2, \cdots.$$

If $m > n$, then

$$\begin{aligned} \| u_m - u_n \| &= \| u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n \| \\ &\leq \| u_m - u_{m-1} \| + \| u_{m-1} - u_{m-2} \| + \cdots + \| u_{n+1} - u_n \| \end{aligned} \quad (3.11)$$

In view of

$$\| u_m - u_{m-1} \|^2 = \sum_{j=1}^{n} (B_{mj})^2.$$

Consequently,

$$\| u_m - u_n \| = \sum_{l=n+1}^{m} (\sum_{j=1}^{n} (B_{lj})^2)^{\frac{1}{2}} \to 0 \quad as\ n \to \infty.$$

The completeness of $\bigoplus_n W_2^2[0,1]$ shows that $u_m \to \overline{u} \quad as\ m \to \infty$ in the sense of $\| \cdot \|_{\bigoplus_n W_2^2[0,1]}$.

Second, we will prove that $\overline{u}$ is the solution of $Eq.(1.2)$.

Taking limits in (3.6) and using the continuity of $F(x, u(x))$, we get

$$\begin{aligned} \overline{u}(x) &= \sum_{i=1}^{\infty} \sum_{j=1}^{n} B_{ij} \overline{\psi}_{ij}(x) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{n} \sum_{l=1}^{i} \sum_{k=1}^{j} \beta_{lk}^{ij} F_k(x_l, \overline{u}(x_l)) \overline{\psi}_{ij}(x). \end{aligned} \quad (3.12)$$

From $Theorem(3.2)$, it follows that $\overline{u}(x)$ satisfies $Eq.(1.2)$.

Since $\overline{\psi}_{ij}(x) \in \bigoplus_n W_2^2[0,1]$, clearly, $\overline{u}(x)$ satisfies the initial conditions of $Eq.(1.2)$.

That is, $\overline{u}(x)$ is the solution of $Eq.(1.2)$. The application of the uniqueness of solution to $Eq.(1.2)$ then yields that

$$u(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{n} B_{ij} \overline{\psi}_{ij}(x). \quad (3.13)$$

The proof is complete. □

4 Numerical example

In this section, some numerical examples are studied to demonstrate the accuracy of the present method. In the process of computation, all the symbolic and numerical computations performed by using Mathematica 5.0. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 1

Considering equation

$$\begin{cases} u'(x) + xu(x) + 2xv(x) = f_1(x), \\ v'(x) + v(x) + x^2u(x) = f_2(x), \\ u(0) = 0, v(0) = 0, \end{cases}$$

where $0 \leq x \leq 1$, $f_1(x) = 1 + x^2 + 2x\sin(x)$, and $f_2(x) = x^3 + \sin(x) + \cos(x)$. The true solutions $u(x), v(x)$ are x and $\sin(x)$ respectively. Using our method, we take $x_i = \frac{i-1}{N-1}, i = 1, 2, \cdots, N,\ N = 51$ on $[0, 1]$ and obtain approximate solutions $u_{51}(x)$ and $v_{51}(x)$ on $[0, 1]$. The numerical results are given in the following Table 1,2.

Example 2

Considering equation

$$\begin{cases} u'(x) + v(x) + 2w(x) + \sin(u) = f_1(x), \\ v'(x) + v(x) + xu(x) + w(x) + v^2(x) = f_2(x), \\ 2u(x) + xv(x) + w'(x) + w(x) + w^2(x) = f_3(x), \\ u(0) = 0, v(0) = 0, w(0) = 0, \end{cases}$$

where $0 \leq x \leq 1$, $f_1(x) = 1 + 4x + \sin(x), f_2(x) = 2x + x^2 + \sin(x) + \cos(x)$, and $f_3(x) = 2 + 4x + x\sin(x)$. The true solutions $u(x), v(x), w(x)$ are x, $\sin(x)$ and $2x$ respectively. Using our method, we take $x_i = \frac{i-1}{N-1}, i = 1, 2, \cdots, N,\ N = 21$ on $[0, 1]$ and obtain approximate solutions $u_{21}(x)$, $v_{21}(x)$ and $w_{21}(x)$ on $[0, 1]$. The numerical results are given in the following Table 3,4,5.

Table 1: The numerical results of Example 1(u_{51}(x))

x	True solution u(x)	Approximate solution u_{51}	Absolute error
0.04	0.04	0.0399941	5.8E-6
0.12	0.12	0.119983	1.7E-5
0.24	0.24	0.239966	3.4E-5
0.40	0.40	0.399946	5.3E-5
0.56	0.56	0.55993	6.9E-5
0.72	0.72	0.719918	8.1E-5
0.88	0.88	0.87991	8.9E-5
1.00	1.00	0.999905	9.4E-5

Table 2: The numerical results of Example 1(v_{51}(x))

x	True solution u(x)	Approximate solution v_{51}	Absolute error
0.04	0.0399893	0.0399874	1.8E-6
0.12	0.119712	0.119707	5.6E-6
0.24	0.237703	0.237691	1.1E-5
0.40	0.389418	0.3894	1.8E-5
0.56	0.531186	0.531162	2.3E-5
0.72	0.659385	0.659357	2.8E-5
0.88	0.770739	0.770708	3.0E-5
1.00	0.841471	0.841439	3.1E-5

Table 3: The numerical results of Example 2(u_{21}(x))

x	True solution u(x)	Approximate solution u_{21}	Absolute error
0.08	0.08	0.0801445	1.4E-4
0.24	0.24	0.24041	4.0E-4
0.40	0.40	0.400637	6.3E-4
0.56	0.56	0.560829	8.2E-4
0.72	0.72	0.720975	9.7E-4
0.88	0.88	0.881063	1.0E-3
0.96	0.96	0.961093	1.0E-3

Table 4: The numerical results of Example 2(v_{21}(x))

x	True solution u(x)	Approximate solution v_{21}	Absolute error
0.08	0.0799147	0.0798621	5.2E-5
0.24	0.237703	0.2376	1.0E-4
0.40	0.389418	0.389314	1.0E-4
0.56	0.531186	0.531091	9.5E-5
0.72	0.659385	0.659281	1.0E-4
0.88	0.770739	0.770595	1.4E-4
0.96	0.819192	0.819019	1.7E-4

Table 5: The numerical results of Example 2(w_{21}(x))

x	True solution u(x)	Approximate solution w_{21}	Absolute error
0.08	0.16	0.159863	1.3E-4
0.24	0.48	0.479605	3.9E-4
0.40	0.80	0.799406	5.9E-4
0.56	1.12	1.11927	7.2E-4
0.72	1.44	1.4392	8.0E-4
0.88	1.76	1.75916	8.3E-4
0.96	1.92	1.91917	8.3E-4

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