



Variational iteration method for solving tenth-order boundary value problems

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Abstract

In this paper, the tenth-order linear special case boundary value problems are solved using the variational iteration method. The algorithm approximates the solutions, and their higher-order derivatives, of differential equations and it avoids the complexity provided by other numerical approaches. Three examples compared with those considered by Siddiqi, Twizell and Akram [S.S. Siddiqi, E.H. Twizell, Spline solutions of linear tenth order boundary value problems, *Int. J. Comput. Math.* 68 (1998) 345-362; S.S.Siddiqi, G.Akram, Solutions of tenth-order boundary value problems using eleventh degree spline , *Applied Mathematics and Computation* 185 (1)(2007) 115-127] show that the method is simple and valid.

Keywords: Tenth-order boundary value problems; variational iteration method.

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1 Introduction

The variational iteration method (VIM), which was proposed originally by He [1-7], has been proved by many authors to be a powerful mathematical tool for various kinds of linear and nonlinear problems [8-11]. The reliability of the method and the reduction in the size of computation gave this method a wider applications.

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In this paper, we tend to extend the use of VIM to the following class of tenth-order linear special case boundary value problems and compare with other methods

$$\left\{ \begin{array}{l} u^{(10)}(x) + h(x)u(x) = g(x), a \leq x \leq b, \\ u(a) = \alpha_0, u(b) = \alpha_1, \\ u'(a) = \beta_0, u'(b) = \beta_1, \\ u^{(2)}(a) = \gamma_0, u^{(2)}(b) = \gamma_1, \\ u^{(3)}(a) = \xi_0, u^{(3)}(b) = \xi_1, \\ u^{(4)}(a) = \varsigma_0, u^{(4)}(b) = \varsigma_1, \end{array} \right. \quad (1.1)$$

where $\alpha_i, \beta_i, \gamma_i, \xi_i, \varsigma_i, i = 0, 1$ are finite real constants and $h(x), g(x)$ are continuous on $[a, b]$. The basic idea for high order boundary value problems was first proposed by He and his student in [6] and the method is systematically illustrated in [7].

Higher order differential equations arise in many fields e.g. when an infinite horizontal layer of fluid is heated from below and a uniform magnetic field is also applied across the fluid in the same direction as gravity under the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modelled by a tenth-order boundary value problem. However, there are few literature on the numerical solutions of tenth-order boundary value problems and associated eigenvalue problems. Higher order boundary value problems were researched in [12-19]. Wazwaz [13] presented a modified Adomian Decomposition method for tenth-order and twelfth-order boundary value problems. Twizell et al. [16,18] developed numerical methods for eighth, tenth and twelfth-order eigenvalue problems arising in thermal instability and boundary value problems with order $2m$. Siddiqi and Twizell [12,19] gave the solution of sixth-order boundary value problems and tenth-order linear special case boundary value problems using spline technique. Siddiqi and Akram [14,15] gave the solutions of tenth-order linear special case boundary value problems using non-polynomial spline technique and eleventh degree spline.

2 Analysis and application of the variational iteration method

Consider the differential equation

$$Lu + Nu = g(x), \quad (2.1)$$

where L and N are linear and nonlinear operators, respectively, and $g(x)$ is the source inhomogeneous term. In [1-7], the VIM was introduced by He where a correct functional for Eq.(2.1) can be written as:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{Lu_n(t) + N\tilde{u}_n(t) - g(t)\} dt, \quad (2.2)$$

where λ is a general Lagrangian multiplier [2], which can be identified optimally via variational theory, and \tilde{u}_n is a restricted variation which means $\delta\tilde{u}_n = 0$. By this method, it is required first to determine the Lagrangian multiplier λ that will be identified optimally. The successive approximates u_{n+1} , $n \geq 0$, of the solution u will be readily obtained upon using the determined Lagrangian multiplier and any selective function u_0 . Consequently, the solution is given by

$$u = \lim_{n \rightarrow \infty} u_n.$$

The variational iteration method has been shown to solve easily and accurately a large class of problems with approximations converging rapidly to accurate solutions. Generally one iteration leads to high accurate solution by VIM if the initial solution is carefully chosen with some unknown parameters. The convergence of the method is systematically discussed by Tatari and Dehghan [20].

For variational iteration method, the key is the identification of Lagrangian multiplier. For linear problems, their exact solutions can be obtained by only one iteration step due to the fact that the Lagrangian multiplier can be identified exactly. For non-linear problems, the lagrange multiplier is difficult to be identified exactly. To overcome

the difficulty, we apply restricted variations to nonlinear term. Due to the approximate identification of the Lagrangian multiplier, the approximate solutions converge to their exact solutions relatively slowly. It should be specially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to their exact solutions.

For Eq.(1.1), according to the VIM, we derive a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_a^x \lambda(t)\{u_n^{(10)}(t) + h(t)\tilde{u}_n(t) - g(t)\}dt, \quad (2.3)$$

where \tilde{u} is a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

Making the above correct functional stationary with respect to u_n , noticing that $\delta u_n = 0$,

$$\begin{aligned} \delta u_{n+1}(x) &= \delta u_n(x) + \delta \int_0^x \lambda(t)\{u_n^{(10)}(t) + h(t)\tilde{u}_n(t) - g(t)\}dt \\ &= \delta u_n(x) + \delta \int_0^x \lambda(t)\{u_n^{(10)}(t)\}dt \\ &= [1 - \lambda^{(9)}(x)]\delta u_n(x) + \sum_{i=0}^8 \lambda^{(i)}(x)\delta u_n^{(9-i)}(x) + \int_0^x \lambda^{(10)}(t)\delta u_n(x)dt \\ &= 0. \end{aligned}$$

We, therefore, have the following stationary conditions:

$$\begin{cases} \lambda^{(10)}(t) = 0, \\ 1 - \lambda^{(9)}(x) = 0, \\ \lambda^{(i)}(x) = 0, \quad i = 0, 1, 2, \dots, 8. \end{cases} \quad (2.4)$$

The Lagrangian multiplier can be easily identified as:

$$\lambda = \frac{t^9}{362880} - \frac{t^8 x}{40320} + \frac{t^7 x^2}{10080} - \frac{t^6 x^3}{4320} + \frac{t^5 x^4}{2880} - \frac{t^4 x^5}{2880} + \frac{t^3 x^6}{4320} - \frac{t^2 x^7}{10080} + \frac{t x^8}{40320} - \frac{x^9}{362880}.$$

Therefore, we have the following iteration formula of Eq.(1.1):

$$u_{n+1}(x) = u_n(x) + \int_a^x \lambda\{u_n^{(10)}(t) + h(t)u_n(t) - g(t)\}dt, \quad (2.5)$$

where

$$\lambda = \frac{t^9}{362880} - \frac{t^8 x}{40320} + \frac{t^7 x^2}{10080} - \frac{t^6 x^3}{4320} + \frac{t^5 x^4}{2880} - \frac{t^4 x^5}{2880} + \frac{t^3 x^6}{4320} - \frac{t^2 x^7}{10080} + \frac{t x^8}{40320} - \frac{x^9}{362880}.$$

3 Numerical examples

Now we apply the variational iteration method to solve some tenth-order boundary value problems. Results obtained by the method are compared with other methods and demonstrate that the present method is more effective.

Example 3.1 Consider the following tenth-order boundary value problem[12,15]:

$$\left\{ \begin{array}{l} u^{(10)}(x) - xu(x) = -(89 + 21x + x^2 - x^3)e^x, -1 \leq x \leq 1, \\ u(-1) = 0, u(1) = 0, \\ u'(-1) = \frac{2}{e}, u'(1) = -2e, \\ u^{(2)}(-1) = \frac{2}{e}, u^{(2)}(1) = -6e, \\ u^{(3)}(-1) = 0, u^{(3)}(1) = -12e, \\ u^{(4)}(-1) = -\frac{4}{e}, u^{(4)}(1) = -20e, \end{array} \right.$$

whose exact solution is $u(x) = (1 - x^2)e^x$.

From (2.4), we obtain the following iteration formulation:

$$u_{n+1}(x) = u_n(x) + \int_{-1}^x \lambda \{u_n^{(10)}(t) - tu_n(t) + ((89 + 21t + t^2 - t^3)e^t)\} dt, \quad (3.1)$$

where

$$\lambda = \frac{t^9}{362880} - \frac{t^8 x}{40320} + \frac{t^7 x^2}{10080} - \frac{t^6 x^3}{4320} + \frac{t^5 x^4}{2880} - \frac{t^4 x^5}{2880} + \frac{t^3 x^6}{4320} - \frac{t^2 x^7}{10080} + \frac{t x^8}{40320} - \frac{x^9}{362880}.$$

Now, we assume that an initial approximation has the form

$$u_0(x) = \sum_{i=0}^9 c_i x^i,$$

where $c_i, i = 0, 1, \dots, 9$ are unknown constants to be further determined. In terms of (3.1), one can obtain the first-order approximation $u_1(x)$.

Incorporating the boundary condition of Example 3.1 into $u_1(x)$, the unknown constants in $u_1(x)$ can be obtained $c_0 = -1.27224 \times 10^{-6}, c_1 = -1.00001, c_2 = 0.99994, c_3 = 0.166504, c_4 = -0.166961, c_5 = -0.00870105, c_6 = 0.00801073, c_7 + 8.935 \times 10^{-7}, c_8 =$

Table 1: Comparison of maximum absolute errors of the present method with other methods for Example 3.1

$u_n^{(\mu)}$	$[9]x \in [x_5, x_{k-5}]$	$x \notin [x_5, x_{k-5}]$	[15]	Present method(u_1)
$\mu = 0$	2.65×10^{-4}	$4.16 \times 10^{+13}$	3.28×10^{-6}	9.08×10^{-12}
$\mu = 2$	6.55×10^{-4}	$2.41 \times 10^{+16}$	1.40×10^{-3}	9.02×10^{-11}
$\mu = 4$	1.02×10^{-3}	$7.30 \times 10^{+17}$	$7.76 \times 10^{+2}$	2.57×10^{-9}
$\mu = 6$	4.04×10^{-3}	$3.83 \times 10^{+14}$	1.97×10^{-1}	1.71×10^{-6}
$\mu = 8$	1.10×10^{-2}	$3.17 \times 10^{+17}$	$2.73 \times 10^{+4}$	1.83×10^{-4}

$-0.000280092, c_9 = -0.0000238477$. And therefore, the first-order approximation $u_1(x)$ is obtained.

The maximum error in absolute value $|u^{(\mu)}(x) - u_1^{(\mu)}(x)|, \mu = 0, 2, 4, 6, 8$ compared with $|u^{(\mu)}(x) - u_n^{(\mu)}(x)|, \mu = 0, 2, 4, 6, 8$ considered by Siddiqi and Twizell [12], Siddiqi and Akram [15] are shown in Table 1. It is evident from Table 1 that the maximum absolute errors are less than those presented by Siddiqi and Twizell [12], Siddiqi and Akram [15].

Example 3.2 Consider the following tenth-order boundary value problem [12,15]:

$$\begin{cases} u^{(10)}(x) + u(x) = -10(2x \sin(x) - 9 \cos(x)), -1 \leq x \leq 1, \\ u(-1) = 2 \sin(1), u(1) = 0, \\ u'(-1) = -2 \cos(1), u'(1) = 2 \cos(1), \\ u^{(2)}(-1) = 2 \cos(1) - 4 \sin(1), u^{(2)}(1) = 2 \cos(1) - 4 \cos(1), \\ u^{(3)}(-1) = 6 \cos(1) + 6 \sin(1), u^{(3)}(1) = -6 \cos(1) - 6 \sin(1), \\ u^{(4)}(-1) = -12 \cos(1) + 8 \sin(1), u^{(4)}(1) = -12 \cos(1) + 8 \sin(1), \end{cases}$$

whose exact solution is $u(x) = (x^2 - 1) \cos(x)$.

From (2.4), we obtain the following iteration formulation:

$$u_{n+1}(x) = u_n(x) + \int_{-1}^x \lambda \{u_n^{(10)}(t) + u_n(t) + (10(2t \sin(t) - 9 \cos(t)))\} dt, \quad (3.2)$$

where

$$\lambda = \frac{t^9}{362880} - \frac{t^8 x}{40320} + \frac{t^7 x^2}{10080} - \frac{t^6 x^3}{4320} + \frac{t^5 x^4}{2880} - \frac{t^4 x^5}{2880} + \frac{t^3 x^6}{4320} - \frac{t^2 x^7}{10080} + \frac{t x^8}{40320} - \frac{x^9}{362880}.$$

Table 2: Comparison of maximum absolute errors of the present method with other methods for Example 3.2

$u_n^{(\mu)}$	$[9]x \in [x_5, x_{k-5}]$	$x \notin [x_5, x_{k-5}]$	[15]	Present method(u_1)
$\mu = 0$	2.65×10^{-4}	$4.16 \times 10^{+13}$	8.85×10^{-8}	2.89×10^{-11}
$\mu = 2$	6.55×10^{-4}	$2.48 \times 10^{+16}$	3.65×10^{-6}	2.85×10^{-10}
$\mu = 4$	1.62×10^{-3}	$5.75 \times 10^{+17}$	5.92×10^{-0}	7.97×10^{-9}
$\mu = 6$	4.04×10^{-3}	$1.65 \times 10^{+16}$	1.78×10^{-2}	4.40×10^{-6}
$\mu = 8$	1.10×10^{-2}	$3.20 \times 10^{+19}$	$2.08 \times 10^{+3}$	3.74×10^{-4}

Now, we assume that an initial approximation has the form

$$u_0(x) = \sum_{i=0}^9 c_i x^i,$$

where $c_i, i = 0, 1, \dots, 9$ are unknown constants to be further determined. In terms of (3.2), one can obtain the first-order approximation $u_1(x)$.

Incorporating the boundary condition of Example 3.2 into $u_1(x)$, the unknown constants in $u_1(x)$ can be obtained $c_0 = 1.00001, c_1 = 1.00008, c_2 = -0.49963, c_3 = -0.832327, c_4 = -0.456528, c_5 = -0.156093, c_6 = -0.0383208, c_7 = -0.00693432, c_8 = -0.000857655, c_9 = -0.000054744$. And therefore, the first-order approximation $u_1(x)$ is obtained.

The maximum error in absolute value $|u^{(\mu)}(x) - u_1^{(\mu)}(x)|, \mu = 0, 2, 4, 6, 8$ compared with $|u^{(\mu)}(x) - u_n^{(\mu)}(x)|, \mu = 0, 2, 4, 6, 8$ considered by Siddiqi and Twizell [12], Siddiqi and Akram [15] are shown in Table 2. It is evident from Table 2 that the maximum absolute errors are less than those presented by Siddiqi and Twizell [12], Siddiqi and Akram [15].

Example 3.3 Consider the following tenth-order boundary value problem [15]:

$$\left\{ \begin{array}{l} u^{(10)}(x) + (x^2 - 2x)u(x) = 10 \cos(x) - (x - 1)^3 \sin(x), -1 \leq x \leq 1, \\ u(-1) = 2 \sin(1), u(1) = 0, \\ u'(-1) = 2 \cos(1) - \sin(1), u'(1) = \sin(1), \\ u^{(2)}(-1) = 2 \cos(1) - 2 \sin(1), u^{(2)}(1) = 2 \cos(1), \\ u^{(3)}(-1) = 2 \cos(1) + 3 \sin(1), u^{(3)}(1) = -3 \sin(1), \\ u^{(4)}(-1) = -4 \cos(1) + 2 \sin(1), u^{(4)}(1) = -4 \cos(1), \end{array} \right.$$

whose exact solution is $u(x) = (x - 1) \sin(x)$.

From (2.4), we obtain the following iteration formulation

$$u_{n+1}(x) = u_n(x) + \int_{-1}^x \lambda \{u_n^{(10)}(t) + (t^2 - 2t)u_n(t) - (10 \cos(t) - (t - 1)^3 \sin(t))\} dt \quad (3.3)$$

where

$$\lambda = \frac{t^9}{362880} - \frac{t^8 x}{40320} + \frac{t^7 x^2}{10080} - \frac{t^6 x^3}{4320} + \frac{t^5 x^4}{2880} - \frac{t^4 x^5}{2880} + \frac{t^3 x^6}{4320} - \frac{t^2 x^7}{10080} + \frac{t x^8}{40320} - \frac{x^9}{362880}.$$

Now, we assume that an initial approximation has the form

$$u_0(x) = \sum_{i=0}^9 c_i x^i,$$

where $c_i, i = 0, 1, \dots, 9$ are unknown constants to be further determined. In terms of (3.3), one can obtain the first-order approximation $u_1(x)$.

Incorporating the boundary condition of Example 3.3 into $u_1(x)$, the unknown constants in $u_1(x)$ can be obtained $c_0 = -1.00001, c_1 = -0.000114839, c_2 = 1.49947, c_3 = -0.00145348, c_4 = -0.544302, c_5 = -0.00330464, c_6 = 0.0401419, c_7 = -0.00179426, c_8 = -0.00216034, c_9 = -0.000193759$. And therefore, the first-order approximation $u_1(x)$ is obtained.

The numerical results are summarized in Table 3, Figures 1,2. Table 3 is the results obtained using the method in [15] and Figure 1,2 are the results obtained using the present method. Comparing them, we can find that the method in this paper is more

Table 3: Maximum absolute errors for Example 3.3 using method in [15]

$u_n^{(\mu)}$	n=14	n=28	n=42	n=56
$\mu = 0$	5.96×10^{-6}	7.99×10^{-7}	1.72×10^{-7}	3.73×10^{-8}
$\mu = 2$	1.79×10^{-6}	7.10×10^{-6}	1.45×10^{-6}	4.83×10^{-7}
$\mu = 4$	$4.31 \times 10^{+1}$	2.35×10^{-0}	4.28×10^{-1}	1.93×10^{-1}
$\mu = 6$	4.42×10^{-1}	2.84×10^{-2}	6.70×10^{-3}	7.30×10^{-3}
$\mu = 8$	$5.55 \times 10^{+2}$	$3.52 \times 10^{+2}$	$4.46 \times 10^{+2}$	$6.36 \times 10^{+2}$

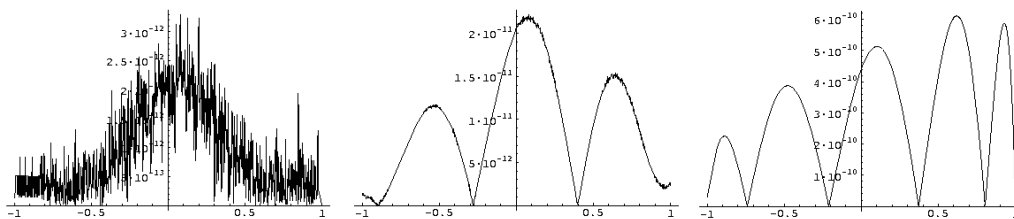


Figure 1: The figure of absolute error $|u - u_1|, |u'' - u_1''|, |u^{(4)} - u_1^{(4)}|$ for Example 3.3

efficient.

Remark: This paper is not a repetition of the paper [13]. The method used in [13] is modified Adomian method. The comparison of the VIM with Adomian method was conducted by many authors via illustrative examples, especially Wazwaz gave a completely comparison between the two method [21], revealing the VIM has many merits over the Adomian method; it can completely overcome the difficulty arising in the calculation of the Adomian polynomial.

4 Conclusion

In this paper, the extended variational iteration method is used to solve a class of linear tenth-order boundary value problems. Comparing with other methods, the results of three numerical examples demonstrate that this method are more accurate than the stated existing methods, and one iteration is enough to obtain accurate solu-

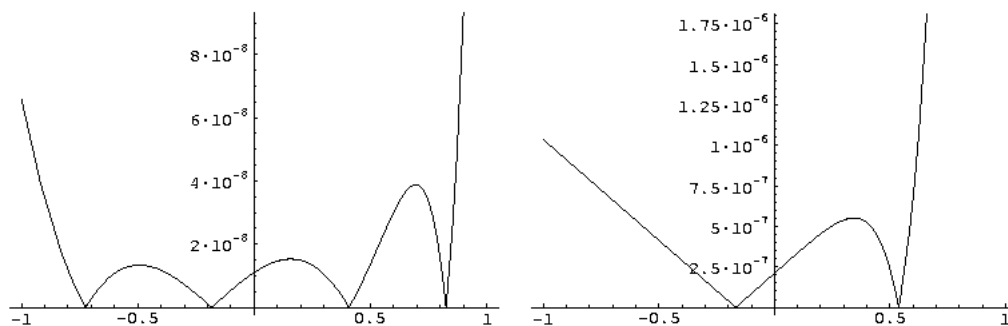


Figure 2: The figure of absolute error $|u^{(6)} - u_1^{(6)}|, |u^{(8)} - u_1^{(8)}|$ for Example 3.3

tions. And this method avoids the complexity provided by other numerical approaches. Moreover, the higher-order derivatives of approximate solutions can also approximate the higher-order derivatives of exact solutions well. Therefore, our conclusion is that the variational iteration method is a satisfactory method for solving linear tenth-order boundary value problems.

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