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A simple iterative method with fifth-order convergence by using Potra and Pták's method

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Abstract

In this paper, we present a new iterative method for solving non-linear equations $f(x) = 0$, by using Potra and Pták's method [1]. It is shown that the order of convergence of this method is five or higher. Several numerical examples are given to illustrate the performance of the presented method.

Keywords: Newton's method; Iterative methods; Non-linear equations; Potra and Pták's method; Root-finding; Fifth-order.

1 Introduction

In this paper, we consider an iterative method to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a non-linear equation $f(x) = 0$. In recent years, some fifth-order iteration methods have been proposed and analyzed for solving non-linear equations that these methods improve some classical methods such as the Newton's method, Chebyshev-Halley's methods, Ostrowski's method. Probably the most well-known and widely used algorithm to find a root of single non-linear equation is Newton's method. This method is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

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and converges quadratically in some neighborhood of α.

There exists a modification of Newton's method with third-order convergence due to Potra and Pták [1], defined by

$$x_{n+1} = x_n - \frac{f(x_n) + f(x_n - \frac{f(x_n)}{f'(x_n)})}{f'(x_n)}. \tag{2}$$

In this paper, we introduce a new iterative method by using Potra and Pták's method, then compare this method with previous methods with fifth-order convergence.

The rest of this paper is organized as follows. In Section 2, we give the suggested method based on Potra and Pták's method. In Section 3, we illustrate the result with some numerical examples and in the last Section, the conclusion is presented.

2 The suggested method

In this section, we consider a simple new iterative method for solving non-linear equations, based on Potra and Pták's method, as follows:

$$x_{n+1} = z_n - \frac{f(z_n) + f(z_n - \frac{f(z_n)}{f'(x_n)})}{f'(x_n)}, \tag{3}$$

$$z_n = x_n - \frac{f(x_n) + f(x_n - \frac{f(x_n)}{f'(x_n)})}{f'(x_n)}. \tag{4}$$

It is clear that per iteration of this method requires four evaluations of the function and one evaluation of its first derivative.

Theorem 2.1 *Assume that the function $f : I \subseteq \Re \to \Re$ for an open interval I, has a simple root $\alpha \in I$. Let $f(x)$ be sufficiently smooth in the neighborhood of the root α, then the order of convergence of the method defined by (3)-(4) is five or higher and it satisfies the following error equation:*

$$e_{n+1} = 8c_2^4 e_n^5 + O(e_n^6), \tag{5}$$

where

$$c_2 = \frac{f''(\alpha)}{2f'(\alpha)}.$$

Proof Let α be a simple zero of f, $e_n = x_n - \alpha$ and $h_n = z_n - \alpha$. Using Taylor expansion around $x = \alpha$ and taking into account $f(\alpha) = 0$, we get

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + \cdots], \tag{6}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \cdots], \tag{7}$$

$$f(z_n) = f'(\alpha)[h_n + c_2h_n^2 + c_3h_n^3 + c_4h_n^4 + c_5h_n^5 + \cdots], \tag{8}$$

$$f'(z_n) = f'(\alpha)[1 + 2c_2h_n + 3c_3h_n^2 + 4c_4h_n^3 + 5c_5h_n^4 + 6c_6h_n^5 + \cdots], \tag{9}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!\ f'(\alpha)}$. We know

$$(1+a)^{-1} = \sum_{k\geq 0} \binom{-1}{k} a^k, \tag{10}$$

then

$$\frac{1}{f'(x_n)} = \frac{1}{f'(\alpha)}[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \cdots]^{-1}$$
$$= \frac{1}{f'(\alpha)}\sum_{k\geq 0} \binom{-1}{k} \left(2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \cdots\right)^k. \tag{11}$$

For simplicity we compute

$(2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \cdots)^k$, for k=2,3,4,5.

$$\left(2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \cdots\right)^2 = 4c_2^2e_n^2 + 12c_2c_3e_n^3 +$$
$$(16c_2c_4 + 9c_3^2)e_n^4 + (20c_2c_5 + 24c_3c_4)e_n^5 + \cdots,$$
$$\left(2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \cdots\right)^3 = 8c_2^3e_n^3 + 36c_2^2c_3e_n^4 +$$
$$(48c_2^2c_4 + 54c_2c_3^2)e_n^5 + \cdots,$$
$$\left(2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \cdots\right)^4 = 16c_2^4e_n^4 + 96c_2^3c_3e_n^5$$

$$+\cdots,$$

$$(2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+\cdots)^5=32c_2^5e_n^5+\cdots.$$

By substitution of above expansions in (11), we have

$$\frac{1}{f'(x_n)}=\frac{1}{f'(\alpha)}[1-2c_2e_n+(-3c_3+4c_2^2)e_n^2+(-4c_4+12c_2c_3-8c_2^3)e_n^3+$$
$$(-5c_5+16c_2c_4+9c_3^2-36c_2^2c_3+16c_2^4)e_n^4+(-6c_6+20c_2c_5+24c_3c_4-48c_2^2c_4$$
$$-54c_2c_3^2+96c_2^3c_3-32c_2^5)e_n^5+\cdots], \tag{12}$$

and from (6) and (12), we compute

$$\frac{f(x_n)}{f'(x_n)}=e_n-c_2e_n^2+(2c_2^2-2c_3)e_n^3+(-3c_4+7c_2c_3-4c_2^3)e_n^4+(-4c_5+10c_2c_4$$
$$+6c_3^2-20c_2^2c_3+8c_2^4)e_n^5+\cdots. \tag{13}$$

Furthermore, we have

$$f(x)=f'(\alpha)[(x-\alpha)+c_2(x-\alpha)^2+c_3(x-\alpha)^3+c_4(x-\alpha)^4+c_5(x-\alpha)^5+\cdots],$$

then

$$f(x_n-\frac{f(x_n)}{f'(x_n)})=f'(\alpha)[(e_n-\frac{f(x_n)}{f'(x_n)})+c_2(e_n-\frac{f(x_n)}{f'(x_n)})^2+c_3(e_n-\frac{f(x_n)}{f'(x_n)})^3$$
$$+c_4(e_n-\frac{f(x_n)}{f'(x_n)})^4+\cdots], \tag{14}$$

wherein

$$(e_n-\frac{f(x_n)}{f'(x_n)})=c_2e_n^2+(2c_3-2c_2^2)e_n^3+(3c_4-7c_2c_3+4c_2^3)e_n^4+(4c_5-10c_2c_4$$
$$-6c_3^2+20c_2^2c_3-8c_2^4)e_n^5+\cdots,$$

$$(e_n - \frac{f(x_n)}{f'(x_n)})^2 = c_2^2 e_n^4 + (4c_2c_3 - 4c_2^3)e_n^5 + \cdots,$$
$$(e_n - \frac{f(x_n)}{f'(x_n)})^3 = O(e_n^6),$$

thus according to (14), we get

$$f(x_n - \frac{f(x_n)}{f'(x_n)}) = f'(\alpha)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + (4c_5 - 10c_2c_4 - 6c_3^2 + 24c_2^2c_3 - 12c_2^4)e_n^5 + \cdots], \quad (15)$$

also

$$f(x_n) + f(x_n - \frac{f(x_n)}{f'(x_n)}) = f'(\alpha)[e_n + 2c_2e_n^2 + (3c_3 - 2c_2^2)e_n^3 + (4c_4 - 7c_2c_3 + 5{c_2}^3)e_n^4 + (5c_5 - 10c_2c_4 - 6c_3^2 + 24c_2^2c_3 - 12c_2^4)e_n^5 + \cdots. \quad (16)$$

By using (12) and (16), we give the following equation

$$\frac{f(x_n) + f(x_n - \frac{f(x_n)}{f'(x_n)})}{f'(x_n)} = e_n - 2c_2^2e_n^3 + (-7c_2c_3 + 9c_2^3)e_n^4 + (-10c_2c_4 - 6c_3^2 + 44c_2^2c_3 - 30c_2^4)e_n^5 + \cdots, \quad (17)$$

this equation implies that

$$h_n = e_n - \frac{f(x_n) + f(x_n - \frac{f(x_n)}{f'(x_n)})}{f'(x_n)} = 2c_2^2e_n^3 + (7c_2c_3 - 9c_2^3)e_n^4 + (10c_2c_4 + 6c_3^2 - 44c_2^2c_3 + 30c_2^4)e_n^5 + \cdots, \quad (18)$$

$$h_n^2 = O(e_n^6), \quad (19)$$

by using above expansions and (8), we can write

$$f(z_n) = f'(\alpha)[2c_2^2e_n^3 + (7c_2c_3 - 9c_2^3)e_n^4 + (10c_2c_4 + 6c_3^2 - 44c_2^2c_3 + 30c_2^4$$

$$)e_n{}^5 + \cdots]. \tag{20}$$

Now for computing $\frac{f(z_n)}{f'(x_n)}$, we use (8) and (12) and we have

$$\frac{f(z_n)}{f'(x_n)} = 2c_2^2 e_n^3 + (7c_2c_3 - 13c_2^3)e_n^4 + (10c_2c_4 + 6c_3^2 - 64c_2^2c_3 + 56c_2^4)e_n^5 + \cdots, \tag{21}$$

also

$$f(z_n - \frac{f(z_n)}{f'(x_n)}) = f'(\alpha)[(h_n - \frac{f(z_n)}{f'(x_n)}) + c_2(h_n - \frac{f(z_n)}{f'(x_n)})^2 + \cdots], \tag{22}$$

wherein

$$(h_n - \frac{f(z_n)}{f'(x_n)}) = 4c_2^3 e_n^4 + (20c_2^2c_3 - 26c_2^4)e_n^5 + \cdots,$$

$$(h_n - \frac{f(z_n)}{f'(x_n)})^2 = O(e_n^8),$$

thus

$$f(z_n - \frac{f(z_n)}{f'(x_n)}) = f'(\alpha)[4c_2^3 e_n^4 + (20c_2^2c_3 - 26c_2^4)e_n^5 + \cdots], \tag{23}$$

also

$$f(z_n) + f(z_n - \frac{f(z_n)}{f'(x_n)}) = f'(\alpha)[2c_2^2 e_n^3 + (7c_2c_3 - 5c_2^3)e_n^4 + (-24c_2^2c_3 + 4c_2^4 +$$

$$10c_2c_4 + 6c_3^2)e_n^5 + \cdots]. \tag{24}$$

Now, by using (12) and (24), we can write

$$\frac{f(z_n) + f(z_n - \frac{f(z_n)}{f'(x_n)})}{f'(x_n)} = 2c_2^2 e_n^3 + (7c_2c_3 - 9c_2^3)e_n^4 + (10c_2c_4 + 6c_3^2 - 44c_2^2c_3 +$$

$$22c_2^4)e_n^5 + \cdots. \tag{25}$$

The proof of theorem will be complete if we substitute (18) and (25) in (3), i.e.,

$$x_{n+1} = z_n - \frac{f(z_n) + f(z_n - \frac{f(z_n)}{f'(x_n)})}{f'(x_n)},$$

then

$$e_{n+1} = h_n - \frac{f(z_n) + f(z_n - \frac{f(z_n)}{f'(x_n)})}{f'(x_n)} = 8c_2^4 e_n^5 + \cdots,$$

and hence

$$e_{n+1} = 8c_2^4 e_n^5 + O(e_n{}^6).$$

3 Numerical Examples

All computations were done **MATHEMATICA** using 120 digit floating point arithmetic (Digits:=120). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs:

$$(i)|x_{n+1} - x_n| < \epsilon, \qquad (ii)|f(x_{n+1})| < \epsilon,$$

and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations, we used the fixed stopping criterion $\epsilon = 10^{-15}$. We present some numerical test results to illustrate the efficiency of the new iterative method in Table 1.

We compare the Newton's method (NM), the Grau and Diaz-Barrero's method [3] (GM) defined by

$$x_{n+1} = x_n - \left(1 + \frac{f''(x_n)(f(x_n) + f(z_n))}{2f'^2(x_n)}\right)\frac{f(x_n) + f(z_n)}{f'(x_n)}, \tag{26}$$

$$z_n = x_n - \left(1 + \frac{1}{2}\frac{f''(x_n)f(x_n)}{f'^2(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{27}$$

Noor and Noor's method [4] (NNM) defined by

$$x_{n+1} = x_n - \frac{2[f(x_n) + h(z_n)]f'(x_n)}{2f'^2(x_n) - [f(x_n) + h(z_n)]f''(x_n)}, \tag{28}$$

$$h(x) = f(x) - f(x_n) - (x - x_n)f'(x_n) - \frac{1}{2}(x - x_n)^2 f''(x_n), \tag{29}$$

$$z_n = x_n - \left(1 + \frac{t(x_n)}{2 - t(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{30}$$

$$t(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}, \tag{31}$$

the method of Kou and Li [2] (KM) defined by

$$x_{n+1} = z_n - \left(1 + \frac{M(x_n)}{1 + M(x_n)}\right)\frac{f(z_n)}{f'(x_n)}, \tag{32}$$

$$z_n = x_n - (1 + \frac{1}{2}\ \frac{t(x_n)}{1 - t(x_n)})\frac{f(x_n)}{f'(x_n)}, \tag{33}$$

$$t(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}, \tag{34}$$

$$M(x_n) = \frac{f''(x_n)(f(x_n) - f(z_n))}{f'^2(x_n)}, \tag{35}$$

Yoon Mee Ham and Changbum Chun's method [5] with $D = -1$, $A = 1$, $B = 3$, $C = 5$ (YCM) defined by

$$x_{n+1} = y_n - \frac{f'(y_n) + 3f'(x_n)}{5f'(y_n) - f'(x_n)}\ \frac{f(y_n)}{f'(x_n)}, \tag{36}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{37}$$

and (EAM) method defined by (3)-(4), introduced in the present contribution. We used the following test functions and display the approximate zero x_* found up to the 26th decimal places.

$$f_1(x) = x^3 + 4x^2 - 10, \quad x_* = 1.36523001341409684576080682,$$

$$f_2(x) = x^2 - e^x - 3x + 2, \quad x_* = 0.25753028543986076045536730,$$

$$f_3(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, x_* = -1.20764782713091892700941675,$$

$$f_4(x) = \sin(x)e^x + \ln(x^2 + 1), \quad x_* = 0,$$

$$f_5(x) = (x - 1)^3 - 2, \quad x_* = 2.25992104989487316476721060,$$

$$f_6(x) = (x + 2)e^x - 1, \quad x_* = -0.44285440100238858314132800,$$

$$f_7(x) = \sin^2(x) - x^2 + 1, \quad x_* = 1.40449164821534122603508681.$$

Table 1. Comparison of the number of iterations (NIT) in (NM), (GM), (NNM), (KM), (YCM) and (EAM) methods

	NIT					
$f(x)$	NM	GM	NNM	KM	YCM	EAM
$f_1(x), x_0 = 1$	6	4	6	4	3	4
$f_2(x), x_0 = 1$	5	4	5	4	3	3
$f_3(x), x_0 = -1$	6	4	6	Failed	4	4
$f_4(x), x_0 = 2$	7	Failed	7	4	4	4
$f_5(x), x_0 = 3$	7	4	7	4	4	4
$f_6(x), x_0 = 2$	9	5	9	Failed	5	5
$f_7(x), x_0 = 1$	7	Failed	7	5	4	6

4 Conclusion

In this paper, we defined and analyzed a simple new iterative method for solving nonlinear equations and proved that the order of convergence of this method is at least five.

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