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Tchebychev Multi-wavelet Basis for Solving Boundary Integral Equation

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Abstract

In this paper, using orthogonality of Tchebychev polynomials, we present an orthonormal wavelet basis for $L^2[0,1]$. We use this basis for solving Neumann problems with Galerkin method. The property of this basis is that a variety of integral operators is represented in this basis as sparse matrices, to high precision. Some examples are solved to illustrate the efficiency and accuracy of this method.

Keywords: Galerkin Method, Tchebychev multi-wavelet, boundary integral equations.

1 Introduction

The boundary value problems of classical potential theory are ubiquitous in engineering and physics. Most such problems can be reduced to boundary integral equations which are, from a mathematical point of view, more tractable than the original differential equations. Although the mathematical benefits of such reformulations were realized and exploited in the 19th century, until recently boundary integral equations were rarely used as mathematical tools, since most integral operators upon discretization turn into dense matrices. By using fast algorithms such as wavelet methods, we can greatly

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reduce the cost of applying dense matrices resulting from solving boundary integral equations by other methods. In this paper we present Tchebychev multi-wavelet as an orthonormal basis and numerical implementation of the fast wavelet Galerkin method for solving Neumann problems. one advantage of using Tchebychev multi-wavelet is that it has a close form which provides convenience for computation, and another advantage is that the matrix representation of a large class of integral operators in this basis is sparse. In particular, an integral operator $\mathcal{K}$ whose kernel is smooth except along a finite number of singular bands has a sparse representation (see Alpert[1]). Let D be a bounded open simple connected region in the plane, and let its boundary, S be a simple closed curve and $f \in C(S)$ be a given boundary function. We consider the interior and exterior Neumann problems as follows :

The Interior Neumann Problem. Find $u \in C^1(\overline{D}) \cap C^2(D)$ that satisfies

$$\begin{cases} \Delta u(P) = 0, & P \in D \\ \frac{\partial u(P)}{\partial n_P} = f(P), & P \in S \end{cases} \tag{1}$$

The Exterior Neumann Problem. Find $u \in C^1(\overline{D_e}) \cap C^2(D_e)$ that satisfies

$$\begin{cases} \Delta u(P) = 0, & P \in D_e \\ \frac{\partial u(P)}{\partial n_P} = f(P), & P \in S \end{cases} \tag{2}$$

with $f \in C(S)$ and $D_e \equiv \mathbf{R}^2 \setminus \overline{D}$.

We can rewrite (1) as an integral equation of second kind (see [2]),

$$u(p) + \frac{1}{\pi}\int_S u(Q)\frac{\partial}{\partial n_Q}[\log|P-Q|]ds_Q = \frac{1}{\pi}\int_S f(Q)\log|P-Q|]ds_Q, \quad P \in S \tag{3}$$

where it's kernel has logarithm-like singularity along the diagonal $s = t$ but is continuous on the unit square. The interior Neumann problem is solvable if and only if the boundary function f satisfies the condition $\int_S f(Q)ds = 0$ [2]. The simplest way to lead with the lack of uniqueness in solving (1) is to introduce an additional condition such as $u(P^*) = 0$ for some fixed point $P^* \in S$ [2]. The exterior Neumann problem can be rewritten as following, if u satisfies $u(\infty)=0$, (see [2])

$$u(p)-\frac{1}{\pi}\int_S u(Q)\frac{\partial}{\partial n_Q}[\log|P-Q|]ds_Q=-\frac{1}{\pi}\int_S f(Q)\log|P-Q|]ds_Q, \quad P\in S \quad (4)$$

that is, a second kind integral equation which is uniquely solvable.

We will present Tchebychev multi-wavelet in sec. 2 and sec. 3 first, in sec. 4 we will describe numerical implementation of wavelet Galerkin method for equations (3) and (4). The second problem that we investigate in sec. 5 is the convergence and error analysis of our proposed method. Lastly in sec. 6 we provide numerical examples to demonstrate the efficiency and accuracy of described methods.

2 Multi-wavelet bases

By reviewing some papers with Alpert [1], Mallat [9], Meyer [10] and Daubechies [5] and using, orthogonality of Tchebychev polynomials we construct an orthonormal multi-wavelet basis for $L^2[0,1]$. This basis comprised of dilates and translates of a finite set of functions $h_1,...,h_k$ and this basis consists of orthonormal systems

$$h^n_{j,m}=2^{m/2}h_j(2^m x-m) \qquad j=1,...,k; \qquad m,n\in Z \quad (5)$$

where the functions $h_1,...,h_k$ are piecewise polynomials, vanishing outside the interval $[0,1]$, and orthogonal to low-order polynomials (have vanishing moments),

$$\int_0^1 h_k x^i dx=0 \qquad i=0,...,k-1. \quad (6)$$

We suppose k is a positive integer and $m=0,1,2,...$, we define a space V^k_m of piecewise polynomial functions,

$$V^k_m=\{f:f(x)=\begin{cases}\text{a polynomial of degree less than k}, & \frac{n}{2^m}<x<\frac{n+1}{2^m}; n=0,\dots,2^{m-1}\\ 0, & \text{otherwise}\end{cases}\}$$

It is apparent that the space V^k_m has dimension $2^m k$ and

$$V^k_0\subset V^k_1\subset ...\subset V^k_m\subset ....i.e. \quad (7)$$

For $m = 0, 1, 2, ...$ we defined $2^m k$ dimension space W_m^k to be orthogonal complement of V_m^k in V_{m+1}^k,

$$V_m^k \oplus W_m^k = V_{m+1}^k, \qquad V_m^k \perp W_m^k. \tag{8}$$

So we inductively obtain the decomposition

$$V_m^k = V_0^k \oplus W_0^k \oplus W_1^k \oplus ... \oplus W_{m-1}^k \tag{9}$$

and we can write, (see [1])

$$L^2[0,1] = \overline{\bigcup_{i=0}^{\infty} V_i^k}. \tag{10}$$

Suppose that the real functions $h_1, ..., h_k$, defined on $\mathbf{R}$, form an orthogonal basis for W_0^k. Since V_0^k is orthogonal to W_0^k, the first k moments of $h_1, ..., h_k$ vanish,

$$\int_0^1 h_k x^i dx = 0, \qquad i = 0, ..., k-1.$$

The $2k$ dimensional space W_1^k is spanned by the $2k$ orthogonal functions $h_1(2x), ..., h_k(2x)$, $h_1(2x-1), ..., h_k(2x-1)$, of which the first k functions have the support $[0, \frac{1}{2}]$ and the second k functions have support $[\frac{1}{2}, 1]$. In general, the space W_m^k is spanned by $2^m k$ functions obtained from $h_1, ..., h_k$ by translation and dilation. There are some freedom in choosing the functions $h_1, ..., h_k$ within the constraint that they be orthogonal, by requiring normality and additional vanishing moments, we specify them uniquely, up to sign. In the following we exploit only the property that $h_1, ..., h_k$ form an orthonormal basis for W_0^k.

In preparation for the definition of $h_1, ..., h_k$, we construct k functions $f_1, ..., f_k$: $\mathbf{R} \to \mathbf{R}$, supported on the interval $[-1, 1]$, with the following properties :

1. The restriction of f_i to interval $(0, 1)$ is a polynomial of degree $k-1$.
2. The function f_i is extended to the interval $(-1, 0)$ as an even or odd function according to the parity of $i + k - 1$.
3. The functions $f_1, ..., f_k$ satisfy the following orthogonality and normality conditions

$$\int_{-1}^1 f_i(x) f_j(x) dx \equiv \langle f_i, f_j \rangle = \delta_{ij}, \qquad i, j = 1, ..., k.$$

4. The function f_j has vanishing moments,

$$\int_{-1}^{1} f_j(x)x^i dx = 0, \qquad i = 0, 1, ..., j + k - 2.$$

Properties 1 and 2 imply that there are k^2 polynomial coefficient that determine the functions $f_1, ..., f_k$, while properties 3 and 4 provide k^2 (non-trivial) constraints. It turns out that the equations uncoupled to give k nonsingular linear systems that may be solved to obtain the coefficients, yielding the functions uniquely.

If the sequence $\{f_j\}$ satisfy the above properties, defining $h_1, ..., h_k : \mathbf{R} \to \mathbf{R}$, by the formula

$$h_i(x) = \sqrt{2} f_i(2x - 1), \qquad i = 1, .., k, \tag{11}$$

then we have

$$W_0^k = \text{Linear span}\{h_i(x) : \quad i = 1, ..., k\} \tag{12}$$

and more generally,

$$W_m^k = \text{Linear span}\{h_{j,m}^n(x) : h_{j,m}^n(x) = \sqrt{2^m} h_j(2^m x - n),\ j = 1, .., k; n = 0, ..., 2^m - 1\}. \tag{13}$$

In [1] it is shown that dilates and translates of the piecewise polynomial functions $h_1, ..., h_k$ form an orthogonal basis for $L^2(R)$. Furthermore, a subset of these dilates and translates, combined with a basis for V_0^k, form a basis for $L^2[0, 1]$.

3 Construction of Tchebychev Multi-wavelet basis

It is well known that Tchebychev polynomials of the first kind $T_m : [-1, 1] \to R \quad (m \geq 0)$ defined by the formula

$$P_m(x) = \cos(m \arccos(x)),$$

and are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1 - x^2}} dx$$

and satisfy the following formula

$$P_0(x) = 1, \quad P_1(x) = x,$$

$$2xP_m(x) = P_{m+1}(x) + P_{m-1}(x), \quad m \in N. \tag{14}$$

We let normalized Tchebychev polynomials in $[0,1]$ as

$$T_m(x) = \frac{\sqrt{2}P_m(2x-1)}{\sqrt[4]{\pi^2(x-x^2)}},$$

which are orthonormal on $[0,1]$ with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

Now we can obtain functions $h_1, ..., h_k$ by the above discussion. In this work we consider normalized Tchebychev polynomials in $[0,1]$ as an orthonormal basis for V_0^k and by dilates and translates of the obtained functions and we introduce a new basis for $L^2(\mathbf{R})$ which we call ***Tchebychev Multi-Wavelet Basis*** using 1- 4 introduced in section 2. For example for **Case** $(k = 3)$:

$$h_1(x) = \frac{1}{\sqrt[4]{x-x^2}} \begin{cases} -10.05425 + 30.0671x - 20.7404x^2 & 0 < x < \frac{1}{2} \\ 0.72761 - 11.4138x + 20.7404x^2 & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_2(x) = \frac{1}{\sqrt[4]{x-x^2}} \begin{cases} 17.048672 - 44.62946x + 28.085650x^2 & 0 < x < \frac{1}{2} \\ 0.504859 - 11.54184x + 28.085650x^2 & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_3(x) = \frac{1}{\sqrt[4]{x-x^2}} \begin{cases} -13.779968 + 33.55531x - 19.984565x^2, & 0 < x < \frac{1}{2} \\ 0.209227 - 6.413824x + 19.984565x^2, & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$W_0^3 = \text{Linear Space}\{h_1(x), h_2(x), h_3(x)\}.$$

Case $(k = 4)$:

$$h_1(x) = \frac{1}{\sqrt[4]{x-x^2}} \begin{cases} 30.612 - 147.80x + 224.4x^2 - 108.00x^3 & 0 < x < \frac{1}{2} \\ -0.761 + 22.95x - 99.6x^2 + 108.00x^3 & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_2(x) = \frac{1}{\sqrt[4]{x-x^2}} \begin{cases} -76.632 + 324.82x - 443.22x^2 + 195.64x^3 & 0 < x < \frac{1}{2} \\ -0.610 + 25.30x - 143.70x^2 + 195.64x^3 & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_3(x) = \frac{1}{\sqrt[4]{x-x^2}} \begin{cases} 93.20601 - 370.2467x + 478.1984x^2 - 201.5097x^3, & 0 < x < \frac{1}{2} \\ -0.35206 + 18.3791x - 126.3308x^2 + 201.5097x^3, & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_4(x) = \frac{1}{\sqrt[4]{x-x^2}} \begin{cases} -61.25014 + 230.75890x - 284.9064x^2 + 115.51472x^3, & 0 < x < \frac{1}{2} \\ -0.11705 + 7.49022x - 61.63774x^2 + 115.51472x^3, & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$W_0^4 = \text{Linear Space}\{h_1(x), h_2(x), h_3(x), h_4(x)\}.$$

Generally we let

$$W_m^k = \text{Linear Space}\{h_{i,m}^n(t) | h_{i,m}^n(x) = \sqrt{2^m} h_i(2^m x - n), i = 1, 2, \ldots, k, n = 0, \ldots, 2^m - 1\}$$

$$V_0^k = \text{Linear Space}\{T_0(x), T_1(x), \ldots, T_{k-1}(x)\}$$

and

$$V_m^k = V_0^k \bigoplus_{j=0}^{m-1} W_j^k.$$

The orthonormal system

$$\mathcal{B}_k = \{T_j(x) : j = 0, ..., k-1\} \cup \{h_{j,m}^k : j = 1, ..., k;\ m = 0, 1, 2, ...; n = 0, ..., 2^m - 1\} \tag{15}$$

span $L^2[0,1]$; we refer to $\mathcal{B}_k$ as the Tchebychev multi-wavelet basis of order k for $L^2[0,1]$. The figures 1. and 2. show the graph of functions $h_1, ..., h_k$ for $k = 3$ and $k = 4$.

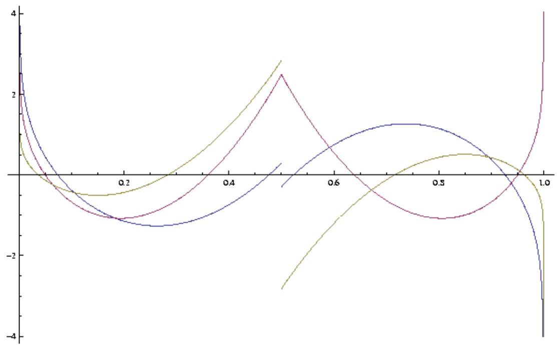

Figure 1: Functions $h_1, \ldots, h_3$

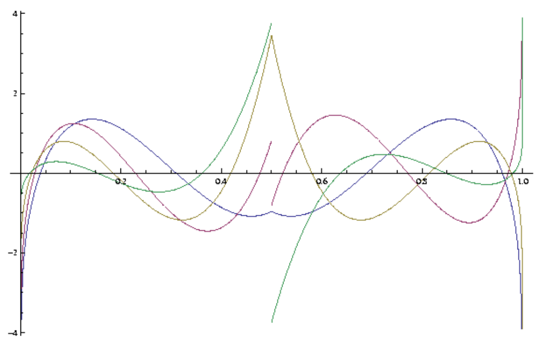

Figure 2: Functions $h_1, \ldots, h_4$

4 Wavelet Galerkin Method for Neumann Problems

In this section, we apply Galerkin method with using Tchebychev multi-wavelet basis as an orthonormal basis to solving equations (3) and (4).

Let D be a bounded open simply connected region in the plane, and let its boundary S be a simple closed curve with parametrization

$$\mathbf{r}(t) = (\xi(t), \eta(t)), \qquad 0 \le t \le L \tag{16}$$

with $\mathbf{r} \in C^2[0, L]$ and $|\mathbf{r}'(t)| \neq 0$ for $0 \le t \le L$.

Using above parametrization for S, we can rewrite (3) and (4) as following equations

$$u(t) + \frac{1}{\pi}\int_0^L K(t,s)u(s)ds = \frac{1}{\pi}g(t), \qquad 0 \le t \le L \tag{17}$$

$$u(t) - \frac{1}{\pi}\int_0^L K(t,s)u(s)ds = -\frac{1}{\pi}g(t), \qquad 0 \le t \le L \tag{18}$$

such that,

$$K(t,s) = \frac{\eta'(s)[\xi(t)-\xi(s)] - \xi'(s)[\eta(t)-\eta(s)]}{[\xi(t)-\xi(s)]^2 + [\eta(t)-\eta(s)]^2}, \qquad s \neq t \tag{19}$$

$$K(t,t) = \frac{\eta'(t)\xi''(t) - \xi'(t)\eta''(t)}{2[\xi'(t)^2 + \eta'(t)^2]}, \tag{20}$$

$$g(t) = \int_0^L f(\mathbf{r}(s))\sqrt{\xi'(s)^2 + \eta'(s)^2}\log|\mathbf{r}(t) - \mathbf{r}(s)|ds. \tag{21}$$

For notation simplicity we restrict our attention to the interval $[0, L] = [0, 1]$ and assume that (17) and (18) as the following form

$$u(t) - \int_0^1 K(t,s)u(s)ds = g(t), \qquad 0 \le t \le 1. \tag{22}$$

By the Galerkin method, we seek an approximate solution $u_n \in V_n^k$ to (22) by requiring that the residual

$$e_n = u_n(t) - \int_0^1 K(t,s)u_n(s)ds - g(t)$$

be orthogonal to V_n^k. We choose a basis for V_n^k denoted by $\{w_l : \quad l = 1, ..., 2^n k\}$, then we seek coefficients $c_l^n, l = 1, ..., 2^n k$ for

$$u_n(t) = \sum_{l=1}^{2^n k} c_l^n w_l(t), \qquad t \in [0, 1] \tag{23}$$

such that

$$\langle e_n, w_l \rangle = \langle u_n(t) - \int_0^1 K(t,s)u_n(s)ds - g(t), w_l \rangle = 0, \quad l = 1, ..., 2^n k. \tag{24}$$

That is, we need to solve the following linear algebraic system for the unknown coefficient vector $\mathbf{C}_n = (c_1^n, ..., c_{2^n k}^n)^T$

$$(\mathbf{I} - \mathbf{A}_n)\mathbf{C}_n = \mathbf{G}_n \tag{25}$$

where $\mathbf{I}$ is the $2^n k$-identity matrix, $\mathbf{A}_n = (a_{l,l'})$, and $\mathbf{G}_n = (g_1, ..., g_{2^n k})$ with

$$a_{l,l'} = \int_0^1 \left(\int_0^1 K(t,s) w_{l'}(s) ds \right) w_l(t) dt \quad \text{and} \quad g_l = \int_0^1 g(t) w_l(t) dt. \tag{26}$$

Entries $a_{l,l'}$ for $l, l' = 1, ..., 2^n k$ will be computed numerically. It is well-known [2] that if $u \in H^2$ (Sobolev space) and kernel function $K(t,s)$ is such that the operator $\int_0^1 K(t,s)u(s)ds$ is compact on L^2, such an approximate solution u_n has the error estimate

$$\|u_n - u\|_{L^2} = O(2^{-2n}). \tag{27}$$

The multi-resolution nature of the wavelet approximations is clearly displayed in this linear system. From level n to level $n+1$, the coefficient matrix $\mathbf{A}_n$ and right-hand-side vector $\mathbf{G}_n$ are expanded to $\mathbf{A}_{n+1}$ and $\mathbf{G}_{n+1}$ by adding new blocks to $\mathbf{A}_n$ and $\mathbf{G}_n$ respectively

$$\mathbf{A}_{n+1} = \begin{pmatrix} \mathbf{A}_n & * \\ * & * \end{pmatrix}, \qquad \mathbf{G}_{n+1} = \begin{pmatrix} \mathbf{G}_n \\ * \end{pmatrix}$$

where the new blocks (marked with $*$) all contain information from the new $(n+1)$-level resolution and are of the same size as their corresponding parts $\mathbf{A}_n$ and $\mathbf{G}_n$ respectively. It should be noted though that the solution vector C_{n+1} usually does not have the same structure as $\mathbf{G}_{n+1}$ above.

5 Convergence and Error Analysis

5.1 Second Kind Integral Equations

A linear Fredholm integral equation of the second kind is an expression of the form

$$f(x) - \int_a^b K(x,t)f(t)dt = g(x), \tag{28}$$

where we assume that the kernel K is in $L^2[a,b]^2$ and the unknown f and right- hand-side g are in $L^2[a,b]$. For notational simplicity, we restrict our attention to the interval $[a,b]=[0,1]$. We use the symbol $\mathcal{K}$ to denote the integral operator of Eq. (28), given by the formula

$$(\mathcal{K}f)(x) = \int_0^1 K(x,t)f(t)dt,$$

for all $f \in L^2[0,1]$ and $x \in [0,1]$. Suppose that $\{b_1, b_2, \ldots\}$ is a complete orthonormal basis for $L^2[0,1]$, the expansion of K in this basis is given by the formula

$$K(x,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K_{ij} b_i(x) b_j(t), \tag{29}$$

where the coefficient K_{ij} is given by the expression

$$K_{ij} = \int_0^1 \int_0^1 K(x,t) b_i(x) b_j(t) dx dt, \quad i,j = 1,2,\ldots. \tag{30}$$

Similarly, the functions f and g have expansions

$$f(x) = \sum_{i=1}^{\infty} f_i b_i(x), \qquad g(x) = \sum_{i=1}^{\infty} g_i b_i(x),$$

where the coefficients f_i and g_i are given by

$$f_i = \int_0^1 f(x) b_i(x) dx, \qquad g_i = \int_0^1 g(x) b_i(x) dx, \quad i = 1,2,\ldots.$$

The integral equation (28) then corresponds to the infinite system of linear equations

$$f_i - \sum_{j=1}^{\infty} K_{ij} f_j = g_i, \quad i = 1,2,\ldots.$$

The expansion for K may be truncated at a finite number of terms, yielding the integral operator $\mathcal{R}$ defined by the formula

$$(\mathcal{R}f)(x) = \int_0^1 \sum_{i=1}^{n} \sum_{j=1}^{n} (K_{ij} b_i(x) b_j(t)) f(t) dt, \quad f \in L^2[0,1], x \in [0,1],$$

which approximates $\mathcal{K}$. Integral equation (28) is thereby approximated by the system

$$f_i - \sum_{j=1}^{n} K_{ij} f_j = g_i, \quad i = 1,2,\ldots n, \tag{31}$$

which is a system of n equations in n unknowns. system (31) may be solved numerically to yield an approximate solution to Eq. (28), given by the expression

$$f_R(x) = \sum_{i=1}^{n} f_i b_i(x).$$

How large is the error $e_R = f - f_R$ of the approximate solution?

We follow the derivation by Delves and Mohamed in [1]. Defining g_R by the formula

$$g_R(x) = \sum_{i=1}^{n} g_i b_i(x),$$

we rewrite Eqs. (28) and (31) in terms of operators $\mathcal{K}$ and $\mathcal{R}$ to obtain

$$(\mathcal{I}-\mathcal{K})f = g$$

$$(\mathcal{I}-\mathcal{R})f_R = g_R.$$

Combining the latter equations yields

$$(\mathcal{I}-\mathcal{K})e_R = (\mathcal{K}-\mathcal{R})f_R + (g-g_R).$$

Provided that $(\mathcal{I}-\mathcal{K})^{-1}$ exists, we obtain the error bound

$$\|e_R\| \leq \|(\mathcal{I}-\mathcal{K})^{-1}\|.\|(\mathcal{K}-\mathcal{R})f_R + (g-g_R)\|. \tag{32}$$

The error depends, therefore, on the conditioning of the original integral equation, as is apparent from the term $\|(\mathcal{I}-\mathcal{K})^{-1}\|$, on the fidelity of the finite-dimensional operator $\mathcal{R}$ to the integral operator $\mathcal{K}$, and on the approximation of g_R to g.

5.2 Convergence of the Multi-wavelet basis

For a function $u \in L^2[0,1]$, a positive integer k, and $n = 0,1,2,...$, we define the orthogonal projection $Q_n^k u$ of u onto V_n^k by the formula

$$\left(Q_n^k u\right)(x) = \sum_{l=1}^{2^n k} \langle u, w_{l,n}\rangle . w_{l,n}(x) \tag{33}$$

where $\{w_{l,n}\}$ is an orthonormal basis (in this work we use Tchebychev multi-wavelet basis) for V_n^k. The projection $Q_n^k u$ converges (in the mean) to u as $n \to \infty$. If the function u is several times differentiable, we can bound the error, as established by the following lemma.

Lemma 1: [1] Suppose that the function $u : [0,1] \to \mathbf{R}$ is k times continuously differentiable, $u \in C^k[0,1]$. Then $Q_n^k u$ approximates u with mean error bounded as follows:

$$\|Q_n^k u - u\| \leq 2^{-nk} \frac{2}{4^k k!} \sup_{x\in[0,1]} |u^{(k)}(x)| \tag{34}$$

5.3 Error Analysis of the Quadrature Rule

Since the entries of the coefficient matrix $\mathbf{A}_n$ are obtained numerically, we are actually solving a perturbed version of the linear system (24), say,

$$(\mathbf{I}-\widetilde{\mathbf{A}}_n)\widetilde{\mathbf{C}}_n=\widetilde{\mathbf{G}}_n \tag{35}$$

where

$$\widetilde{\mathbf{A}}_n=(\tilde{a}_{l,l'}) \qquad \text{with} \qquad \widetilde{a}_{l,l'}=a_{l,l'}+\varepsilon_{l,l'}$$

$$\widetilde{\mathbf{G}}_n=(\tilde{g}_l) \qquad \text{with} \qquad \widetilde{g}_l=g_l+\varepsilon_l$$

and $\varepsilon_{l,l'}$ and ε_l denote the quadrature error for computing the double integral $a_{l,l'}$ and single integral g_l, respectively. in the following we analyze the effect of such quadrature error on approximation solution $\widetilde{u}_n$. By (24) and (34) we can write

$$(\mathbf{I}-\mathbf{A}_n)(\mathbf{C}_n-\widetilde{\mathbf{C}}_n)=(\mathbf{G}_n-\widetilde{\mathbf{G}})+(\mathbf{A}_n-\widetilde{\mathbf{A}}_n)\widetilde{\mathbf{C}}_n$$

$$\|(\mathbf{C}_n-\widetilde{\mathbf{C}}_n)\|_2=\|(\mathbf{I}-\mathbf{A}_n)^{-1}\|_2 \;\; \|(\mathbf{G}_n-\widetilde{\mathbf{G}})+(\mathbf{A}_n-\widetilde{\mathbf{A}}_n)\widetilde{\mathbf{C}}_n\|_2$$

therefor we have

$$\|(\mathbf{C}_n-\widetilde{\mathbf{C}}_n)\|_2=O(\|(\mathbf{G}_n-\widetilde{\mathbf{G}}_n)\|_2)+O(\|(\mathbf{A}_n-\widetilde{\mathbf{A}}_n)\|_2) \tag{36}$$

If we let,

$$\varepsilon_n^2=\max_{l,l'}\{|\varepsilon_l|^2,|\varepsilon_{l,l'}|^2\}$$

we can write

$$\|\mathbf{G}_n-\widetilde{\mathbf{G}}_n\|_2{}^2=\sum_{l=1}^{2^n k}|\varepsilon_l|^2\le\varepsilon_n^2 2^n k\le 2^{2n}k^2\varepsilon_n^2 \tag{37}$$

and

$$\|\mathbf{A}_n-\widetilde{\mathbf{A}}_n\|_2{}^2=\sum_{l=1}^{2^n k}\sum_{l'=1}^{2^n k}|\varepsilon_{l,l'}|^2\le 2^{2n}k^2\varepsilon_n^2 \tag{38}$$

On the other hand since we use the Tchebychev multi-wavelet basis, we have

$$\|u_n-\widetilde{u}_n\|_{L^2}=\|(\mathbf{C}_n-\widetilde{\mathbf{C}}_n)\|_2. \tag{39}$$

Now by (35) – (38) we have

$$\|u_n - \widetilde{u}_n\|_{L^2} = O(2^n \varepsilon_n). \tag{40}$$

In order for $\widetilde{u}_n$ to be an approximation to u_n in order as in (27), we need that quadrature error ε_n be as

$$\varepsilon_n \sim 2^{-3n}$$

as n increases. Note that this estimate provides only the rate of decrease for quadrature errors as resolution level increases , and does not specify on the absolute errors of the quadrature for a given k.

6 Numerical results

In this section we apply the method discussed in the pervious sections to the solutions of the following problems, using Tchebychev Multi-Wavelet :

6.1 Interior Neumann Problem

Consider the interior Neumann problem (1) such that

$$D = \{P = (p_1, p_2) : p_1^2 + p_2^2 < 1\}$$

$$S = \partial D = \{(p_1, p_2) : p_1^2 + p_2^2 = 1\}.$$

Consider the following parametrization on S,

$$P = (p_1, p_2) = (\cos(2\pi t), \sin(2\pi t)), \qquad 0 \le t \le 1$$

$$Q = (q_1, q_2) = (\cos(2\pi s), \sin(2\pi s)), \qquad 0 \le s \le 1$$

and we let,

$$f(P) = p_1 = \cos(2\pi t)$$

we can rewrite (1) as the second kind integral equation (22) for the unknown

$$u(t) = u(\cos(2\pi t), \sin(2\pi t)), \qquad 0 \leq t \leq 1$$

and the right-hand-side

$$g(t) = 2\cos(2\pi s)\log(4\sin^2(\pi(s-t))).$$

Note that since in this example we have,

$$\int_S f(Q)ds = 0$$

therefore the interior Neumann problem (1) with above assumptions, has the unique exact solution

$$u(P) = \cos(2\pi t), \qquad P \in D.$$

We then solve (22) by using our method and numerical results are shown in Table.1.

m	k	$\|u - u_n\|_{L^2}$ in V_m^k
1	6	0.01443675
2	5	0.00357842
3	4	0.001543250
4	3	0.00037692
5	2	0.00020961
6	1	0.0000812495

6.2 Exterior Neumann Problem

Consider the exterior Neumann problem (2) by the following assumptions,

$$D = \{P = (p_1, p_2) : p_1^2 + \frac{p_2^2}{4} < 1\}$$

$$S = \partial D = \{(p_1, p_2) : p_1^2 + \frac{p_2^2}{4} = 1\}$$

$$P = (p_1, p_2) = (\cos(2\pi t), 2\sin(2\pi t)), \qquad 0 \leq t \leq 1$$

$$Q = (q_1, q_2) = (\cos(2\pi s), 2\sin(2\pi s)), \qquad 0 \le s \le 1$$

$$f(P) = -\frac{2\sqrt{2}\,(\cos(2\pi t) + 3\cos(6\pi t))}{(5 - 3\cos(4\pi t))^2\,\sqrt{3\cos(4\pi t) + 5}}.$$

Now we can rewrite (2) as a second kind integral equation (22) for the unknown

$$u(t) = u(\cos(2\pi t), 2\sin(2\pi t)), \qquad 0 \le t \le 1$$

with the kernel

$$K(t,s) = -\frac{4}{3\cos(2\pi(s+t)) + 5}$$

and the right-hand-side

$$g(t) = \frac{2(\cos(2\pi s) + 3\cos(6\pi s))\log\left(2(3\cos(2\pi(s+t)) + 5)\sin^2(\pi(s-t))\right)}{(5 - 3\cos(4\pi s))^2}.$$

This problem has the unique exact solution

$$u(P) = \frac{p_1}{p_1^2 + p_2^2} = \frac{\cos(2\pi t)}{1 + 3\sin^2(2\pi t)}.$$

We use our method for current problem and numerical results are shown in Table.2.

m	k	$\|u - u_n\|_{L^2}$ in V_m^k
1	6	0.10076491
2	5	0.04522286
3	4	0.00352981
4	3	0.00122951
5	2	0.000374764
6	1	0.00010232

7 Conclusion

In this work, we constructed Tchebychev multi-wavelet basis as an orthonormal basis for $L^2[0,1]$ and presented a fast Galerkin method for solving boundary integral equations. We solved interior and exterior Neumann problems by this method . The most important advantage of using Tchebychev multi-wavelet basis for solving boundary integral equations is that the representation of the resulting matrices by this basis is sparse. Other advantage of using this basis is that it has a closed form which provides simple computing and produce good results by solving a small linear system. We can have better solutions by increasing the resolution of this basis (by increase k and m in V_m^k), easily.

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