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More On The Commuting Regularity Of Algebraic Structures

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Abstract

In this paper, we study some main properties of the commuting regular ideals and we give a necessary and sufficient condition that a ring (or semigroups) is commuting regular. Some significant results of this investigation will be used for the commutative rings and group rings.

Keywords: commuting regular rings, commutative rings, group rings, loop rings.

1 Introduction

In this paper, R, G, and S denote a ring, a group, and a semigroup, respectively. Also, we use $Z(R)$, $N(R)$, and $J(R)$, to denote the center, the nilradical, and jacobson radical of the ring R, respectively. A non-empty subset of a ring R (or semigroup with zero) is called nilpotent if there exists a positive integer n such that $I^n = 0$. An element s of semigroup S is called cancellable if for every r and t, $sr = st$ implies $r = t$. The semigroup S is called cancellative, if all elements of S are cancellable.

A quasigroup is a set Q with a binary operation, here denoted by " $\cdot$ " , with the property that for all $a, b \in Q$ there are unique elements $x, y \in Q$ such that $x \cdot a = b$ and $a \cdot y = b$. A quasigroup with an identity element is called a loop.

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An element a of a ring R is called regular if there exists an element x in R such that $axa = a$. A ring R is called Von Neumen (or regular) if all its elements are regular ([5]). A ring R is called commuting regular ([8]) if and only if for each $x, y \in R$ there exists an element a of R such that $xy = yxayx$. The commuting regular semigroups have been defined in a similar way in [3]. Also, a two-sided (left, right) ideal I of a ring (or semigroup) is said to be commuting regular two-sided (left, right) ideal if for every $a, b \in I$ there exists an element $c \in I$ such that $ab = bacba$. (see [3])

Let $R[G]$ be the set of all linear combinations of the form $a = \sum_{g\in G} \alpha(g)g$ where $\alpha(g) \in R$ and $\alpha(g) = 0$ except a finite number of coefficient. The sum and product of elements of $R[G]$ are defined by:

$$(\sum_{g\in G} \alpha(g)g) + (\sum_{g\in G} \beta(g)g) = \sum_{g\in G}(\alpha(g) + \beta(g)g),$$

$$(\sum_{g\in G} \alpha(g)g)(\sum_{h\in G} \beta(h)h) = \sum_{g,h\in G} \alpha(g)\beta(h)gh.$$

It is easy to verify that $R[G]$ is a ring, which is called the group ring of G over R.

If we replace the group G in the above definition by a semigroup S (or loop L), we get $R[S]$ (or $R[L]$) the semigroup ring (or loop ring).

Following [7], let $L_n(m) = \{e, 1, 2, \ldots, n\}$ be a set where $n > 3$ is an odd integer and m is a positive integer such that $(m, n) = 1$ and $(m-1, n) = 1$ with $m < n$. Define on $L_n(m)$, a binary operation "." as follows:

(1) $e \cdot i = i \cdot e = i$ for all $i \in L_n(m)$,

(2) $i^2 = e$ for all $i \in L_n(m)$,

(3) $i \cdot j = t$ where $t \equiv (mj - (m-1)i)\ (mod\ n)$ for all $i, j \in L_n(m)$, $i \neq e$ and $j \neq e$.

Then $L_n(m)$ is a loop.

2 Some main results on commuting regular rings

Some new results of the commuting regular rings are as follows. We omit the proofs where they are easy.

Proposition 2.1 *Let R be a commuting regular ring, then for every $a \in R$, $aR = Ra$.*

Proposition 2.2 *Let R is a commuting regular ring, then every left ideal Ra^2 is generated by an idempotent.*

Proof. Let $a \in R$, there exists $b \in R$ such that $a^2 = a^2ba^2$. So, $ba^2 = ba^2ba^2$ and therefore, $e = ba^2$ is an idempotent. We show that $Ra^2 = Re$. Let $y \in Ra^2$, there exists $r \in R$ such that $y = ra^2$ and so, $y = ra^2 = ra^2ba^2 = ra^2e$. Therefore, $Ra^2 \subseteq Re$. Also $e = ba^2 \in Ra^2$ and $Re \subseteq Ra^2$.

Proposition 2.3 *Let R be a commuting regular ring and I be an ideal of R, then R/I is a commuting regular ring.*

Proof. Suppose that $a+I$, $b+I \in R/I$ where $a, b \in R$. By the hypothesis there exists $x \in R$ such that $ab = baxba$. So, $(a+I)(b+I) = (b+I)(a+I)(x+I)(b+I)(a+I)$.

Note that if $\alpha : R \to S$ is a homomorphism of rings and R is a commuting regular ring, then $R/Ker(\alpha)$ and $\alpha(S)^{-1}$ are commuting regular.

Proposition 2.4 *If R is a commuting regular ring, then each homomorphic image of R is a commuting regular ring.*

Proof. Let R' be a ring, $\alpha : R \to R'$ be an epimorphism, and a, $b \in R'$, then there exist $r, s \in R$ such that $a = \alpha(r)$ and $b = \alpha(s)$. Since R is a commuting regular ring, there exists t such that $rs = srtsr$ and

$$ab = \alpha(r)\alpha(s) = \alpha(rs) = \alpha(srtsr) = \alpha(s)\alpha(r)\alpha(t)\alpha(s)\alpha(r) = bacba,$$

where $c = \alpha(t)$.

Lemma 2.5 *The center of a commuting regular ring R is a commuting regular ideal.*

Proof. Let $a, b \in Z(R)$, there exists $x \in R$ such that $ab = baxba = (ba)^2x = x(ba)^2$. So, $abx = (ba)^2x^2 = x^2(ba)^2$. Therefore, $ab = baxba = (ba)^2x^2ba = ba(bax^2)ba$. We show that $bax^2 \in Z(R)$. Note that $bax \in Z(R)$ because if $y \in R$, then

$$(bax)y = ba(xy) = (xy)ba = xy(baxba) = (xba)y(xba) = x(ba)^2yx = bayx = y(bax)$$

and so, $bax^2y = (bax)(xy) = xy(bax) = (xba)yx = y(xba)x = y(bax^2)$. If we consider $bax^2 = t$, then $ab = ba(bax^2)ba = batba$ and $Z(R)$ is a commuting regular ideal.

Lemma 2.6 *If R is a commuting regular ring such that $R = R^2$, then every maximal ideal M in R is prime.*

Proof. Suppose $ab \in M$ but $a \notin M$ and $b \notin M$. Then each of the ideals $M + (a)$ and $M + (b)$ properly contains M. By maximality, $M + (a) = R = M + (b)$. Since R is commuting regular and $a, b \in R$, so, $(a)(b) \subseteq (ab) \subseteq M$. Therefore,

$$R = R^2 = (M + (a))(M + (b)) \subseteq M^2 + (a)M + M(b) + (a)(b) \subseteq M.$$

This contradicts the fact that $M \neq R$. Therefore, $a \in M$ or $b \in M$.

Proposition 2.7 *If R is a commuting regular ring, then $J(R)$ is a nilpotent ideal.*

Proof. The proof is easy by combining the assertions (5) and (6) of the Theorem I of [8].

Not that $I^n = 0$ means that $a_1a_2...a_n = 0$ for any set elements $a_1, a_2, ..., a_n \in I$. This condition is much stronger than I being nil. For instance, in the commutative ring $R = Z[x_1, x_2, ...]/(x_1^2, x_2^3, ...)$ the ideal I generated by $\bar{x}_1, \bar{x}_2, ...$ is nil, where $\bar{x}_i = x_i + (x_1^2, x_2^2, \ldots)$, but easily shown to be not nilpotent.

Proposition 2.8 *If I is a nil ideal of commuting regular ring R, then I is a nilpotent ideal.*

Proof. The nil ideal I is contained in $J(R)$ by the Lemma 1.2.2 of [4]. By the Proposition 2.7, we conclude that the ideal I is a nilpotent ideal.

Theorem 2.9 *Let R be a commuting regular ring and $I \neq (0)$ is a non nilpotent ideal of R, then there exists nonzero idempotent e such that $e \in I$.*

Proof. By the Proposition 2.8, there exists $x \in I$ such that $x^n \neq 0$ for all positive integers n. So, there exists $a \in R$ such that $x^2 = x^2ax^2$ for $ax^2 \neq 0$. Therefore, $ax^2 = ax^2ax^2$ and $e = ax^2$ is a nonzero idempotent in I.

Lemma 2.10 *Suppose that S is a commuting regular ring then:*
(i) Every idempotent element is central, i.e., $Id(S) \subseteq Z(S)$.
(ii) For each $x,\ y \in S$, there exist $s,\ t \in S$, such that $xy = sx = yt$.

Proof. See [8].

Theorem 2.11 *Let R has a no zero divisor and I be an ideal of R such that non zero idempotent e belong to I. Then R is a commuting regular ring if and only if I is a commuting regular.*

Proof. Let R be a commuting regular ring and $a, b \in I$, there exists $c \in R$ such that $ab = bacba$. By the Lemma 2.10, $abe = bacebae$ and so, $ab = ba(ce)ba$. Therefore, I is a commuting regular ring. Conversely, let $a, b \in R$, then $ae, be \in I$ and there exists $c \in R$ such that $aebe = beaecbeae$. So, $abe = bacbae$ and $ab = bacba$. Therefore, R is a commuting regular ring.

When a prime ideal is a maximal ideal? This is a well-known question answered in [1] and [7] by considering the Artinian ring. We submit the following result which proposes a new class of rings with this property.

Proposition 2.12 *Let R be a commutative ring with identity. If R is a commuting regular ring, then every prime ideal of R is a maximal ideal.*

Proof. Let P be a prime ideal of R, then R/P is a commuting regular ring by the Proposition 2.3. If $0 \neq a \in R/P$, there exists $b \in R/P$ such that $a^2 = a^2ba^2$ and so, $a^2(1 - ba^2) = 0$. Therefore, $1 - ba^2 = 0$ or $ba^2 = 1$ and $ba = a^{-1}$. So, R/P is a field and P is a maximal ideal of R.

Corollary 2.13 *Let R be a commutative ring with identity. If R is commuting regular, then*

(1) $dim R = 0$,

(2) $J(R) = N(R)$,

(3) If R is a Noetherian ring, then R is an Artinian ring .

Proof. (1) and (2) by the Proposition 2.12. The proof (3) by the Proposition 2.12 and the Proposition 8.38 of [7].

3 Commuting regularity of group rings and group loops

Proposition 3.1 *Let R be a ring with identity, then the commuting regular ring $R[G]$ is a Von Neumann ring.*

Proof. If $a \in R$, then there exists $b \in R$ such that $a = a.1 = (1.a)b(1.a) = aba$. So, R is a Von Neuman ring.

Following [6], a group G is called locally finite whenever every finite subset of G generates a finite subgroup.

Corollary 3.2 *Let R be a ring with identity. If $R[G]$ is commuting regular ring, then*

(1) G is locally finite,

(2) the order of every element of G is invertible in R,

(3) $J(R[G]) = 0$.

Proof. By the Proposition 3.1 and the Theorem III.18 of [6].

Lemma 3.3 *If $R[G]$ is a commuting regular ring, then R is a commuting regular ring.*

Proof. Let I be the ideal of $R[G]$ generated by $\{e-g|g \in G\}$, where $e \in G$ is the unit element of the group G. Let $\alpha : G \to \{e\}$ is defined by $\alpha(g) = e$ for all $g \in G$, then it may be naturally extended to a ring epimorphism $\tilde{\alpha} : R[G] \to R[e] \cong R$. Therefore, $R \cong R[G]/I$. Using the notations of [6], we have the following properties: If H is a subgroup of G, and $\alpha : R[G] \to R[G/H]$ is defined by $\alpha(g) = gH$ for all $g \in G$, then $Ker(\alpha) = \omega(H)$ where $\omega(H)$ is defined the left ideal of $R[G]$ generated by $\{e-h|g \in H\}$. Therefore, R is a commuting regular ring by the Proposition 2.3.

Referring to lemma 2.5, if $R[G]$ is a commuting regular ring, its center is a commuting regular ideal.

Proposition 3.4 *Let F be a field of characteristic zero and G be any finite group. Then the group ring $F[G]$ contains commuting regular elements a and b such that $a \neq b$.*

Proof. Let $p \mid o(G)$ where p is a prime. Let H be a subgroup of G, such that $o(H) = p$. Let $g \in H$, $a = \frac{1}{p}(1 + g + \ldots + g^{p-1})$ and $b = \frac{-1}{p}(1 + g + \ldots + g^{p-1}) \in F[G]$. Then $a^2 = a$, $b^2 = b$, and $ab = ba = b$. Therefore, $ab = (ba)b(ba)$.

Example 3.5 *Let $F = Z_3$ and $G = \langle g|g^2 = 1\rangle$. Clearly in the group ring*

$$Z_3[G] = \{0,\ 1,\ 2,\ g,\ 1+g,\ 2+g,\ 2g,\ 1+2g,\ 2+2g\},$$

$a = 2 + 2g$ and $b = 1 + g$ are commuting regular elements.

Example 3.6 *Let $G = \langle g|g^2 = 1\rangle$ and Q field of rational numbers. In the group ring $Q[G]$, suppose that $a = \frac{1}{2}(1+g)$ and $b = \frac{-1}{2}(1+g)$. So, $ab = ba = -\frac{1}{4}(2+2g) = b$, $a^2 = b^2 = \frac{1}{4}(1+2g+g^2) = \frac{1}{4}(2+2g) = a$, and therefore, $ab = (ba)b(ba)$. Then a and b are commuting regular elements.*

Proposition 3.7 *Let $F = Z_p$ where p be prime $(p > 2)$. Then the loop ring $F[L_p(m)]$ contain commuting regular elements a and b such that $a \neq b$.*

Proof. Assume that $a = (p-1)\cdot e + (p-1)\cdot 1 + \ldots + (p-1)\cdot p$ and $b = 1\cdot e + 1\cdot 1 + 1\cdot 2 + \ldots + 1\cdot p$ in $F[L_p(m)]$. Then $a^2 = a$, $b^2 = b$ and $ab = ba = b$. Also, for $a = \frac{p+1}{2} + \frac{p+1}{2}\cdot g$ and $b = \frac{p-1}{2} + \frac{p-1}{2}\cdot g$ in $F[L_p(m)]$ where $g \in L_p(m)$, we have $a^2 = b^2 = a$ and $ab = ba = b$. Therefore, $ab = (ba)b(ba)$.

Example 3.8 *Let $L_5(3)$ be the loop given by the following table:*

.	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	4	2	5	3
2	2	4	e	5	3	1
3	3	2	5	e	1	4
4	4	5	3	1	e	2
5	5	3	1	4	2	e

Then $Z_5[L_5(3)]$ have commuting regular elements $a = 1\cdot e + 1\cdot 1 + 1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 1\cdot 5$ and $b = 4\cdot e + 4\cdot 1 + 4\cdot 2 + 4\cdot 3 + 4\cdot 4 + 4\cdot 5$. Also, $a = 3\cdot e + 3\cdot 1$ and $b = 2\cdot e + 2\cdot 1 \in Z_5[L_5(3)]$ are commuting regular elements.

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