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# A note on weak amenability of semigroup algebras

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**Abstract**

A Banach algebra $A$ is weakly amenable if every continuous derivation from $A$ into $A^*$ is inner. In this paper, we show that the semigroup algebra $M_a(S)$ is weakly amenable for a certain class of locally compact semigroups.

**Keywords:** Clifford semigroup; derivation; idempotent; weak amenability.



## 1 Introduction

Let $S$ be a locally compact topological semigroup, and let $M(S)$ denote the space of all bounded complex regular measures on $S$. This space with the convolution product $*$, and norm $\|\mu\| = |\mu|(S)$ is a Banach algebra. The space of all measures $\mu \in M(S)$ for which the mappings $x \mapsto \delta_x * \mid \mu \mid$ and $x \mapsto \mid \mu \mid * \delta_x$ from $S$ into $M(S)$ are weakly continuous is denoted by $M_a(S)$ (or $\widetilde{L}(S)$ as in [1], where $\delta_x$ denotes the Dirac measure at $x$. Note that the measure algebra $M_a(S)$ defines a two-sided closed $L$-ideal of $M(S)$ (see [1]). Recall that $M_a(S)^*$ can be made into a Banach $M_a(S)$-module, with module actions defined by

$$\langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle, \quad \langle \mu f, \nu\rangle = \langle f, \nu * \mu\rangle \quad (\nu \in M_a(S))$$

for all $f \in M_a(S)^*$ and $\mu \in M_a(S)$.

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Recall that a linear map $D$ from $M_a(S)$ in to $M_a(S)^*$ is called a *derivation* if $D(\mu * \nu) = D(\mu)\nu + \mu D(\nu) \quad (\mu, \nu \in M_a(S))$. For example, if $f \in M_a(S)^*$, then $ad_f : \mu \longrightarrow \mu f - f\mu$ is a derivation which is called *inner*. $M_a(S)$ is called *weakly amenable* if each continuous derivation from $M_a(S)$ into $M_a(S)^*$ is inner. It is well-known result that the group algebra $L^1(G)$ is weakly amenable for each locally compact group $G$. Let $S = (N, +)$, then $S$ is a commutative discrete semigroup with identity which is a subsemigroup of group $G = (Z, +)$. But $\ell^1(S)$ is not weakly amenable. Indeed; since $N^2 = N + N \neq N$, from Proposition 4.2 of [2] it follows that $\ell^1(S)$ is not weakly amenable.

An element $e$ of semigroup $S$ is called *idempotent* if $e^2 = e$. We denote be $E_S$ the set of idempotents in $S$. We recall that a semigroup $S$ is called *Clifford semigroup* if it is an inverse semigroup for which the element of $E_S$ are central(c.f. [4], 4.2)

In this paper, we show that $M_a(S)$ is weakly amenable for a certain class of Clifford semigroup $S$.

## 2 Main Results

Let $S$ be a locally compact semigroup. Denote by $L^\infty(S, M_a(S))$ the set of all complex-valued bounded functions $g$ on $S$ that are $M_a(S)$-measurable. We identify functions in $L^\infty(S, M_a(S))$ that agree $\mu$-almost everywhere for all $\mu \in M_a(S)$. Note that in the case where $S$ is discrete (resp. a locally compact group), $L^\infty(S, M_a(S))$ is equal to $\ell^\infty(S)$ (resp. $L^\infty(S)$). Observe that $L^\infty(S, M_a(S))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative $C^*$-algebra. A semigroups $S$ is called a *foundation semigroup*; if $\cup\{\mathrm{supp}(\mu) : \mu \in M_a(S)\}$ is dense in $S$. Let $S$ be a foundation semigroup with identity. In view of Proposition 3.6 of [5], there is an isometric isomorphism of $L^\infty(S; M_a(S))$ onto $M_a(S)^*$. Recall that $L^\infty(S; M_a(S))$ can be made into a Banach $M(S)$-module, with module actions defined by

$$\langle f \circ \mu, \nu \rangle = \langle f, \mu * \nu \rangle, \quad \langle \mu \circ f, \nu \rangle = \langle f, \nu * \mu \rangle \quad (\nu \in M_a(S))$$

for all $f \in L^\infty(S; M_a(S))$ and $\mu \in M(S)$.

**Lemma 2.1** *Let $S$ be a foundation semigroup with identity and $D : M_a(S) \longrightarrow L^\infty(S, M_a(S))$ be a bounded derivation. Then:*

(a) *$D$ has an extension to a bounded derivation $\tilde{D} : M(S) \longrightarrow L^\infty(S, M_a(S))$.*

(b) *$\tilde{D}(\delta_e) = 0$, whenever $e$ is a central idempotent in $S$.*

*Proof.* (a) In this case, recall Proposition 5.9 of [5] that $M_a(S)$ has a bounded approximate identity. Let $\mu \in M(S)$, $\nu \in M_a(S)$, and let $(e_\alpha)$ be a bounded approximate identity of $M_a(S)$. By Corollary 5.17 of [6], there exists $\nu_1, \nu_2 \in M_a(S)$ such that $\nu = \nu_1 * \nu_2$. Now

$$\begin{aligned}
\langle D(\mu * e_\alpha), \nu \rangle &= \langle D(\mu * e_\alpha) \circ \nu_1, \nu_2 \rangle \\
&= \langle D(\mu * e_\alpha * \nu_1, \nu_2 \rangle - \langle D(\nu_1), \nu_2 * \mu * e_\alpha \rangle \\
&\longrightarrow \langle D(\mu * \nu_1), \nu_2 \rangle - \langle D(\nu_1), \nu_2 * \mu \rangle,
\end{aligned}$$

so that the $weak^* - \lim D(\mu * e_\alpha)$ exists in $L^\infty(S, M_a(S))$. Define

$$\tilde{D}(\mu) = weak^* - \lim D(\mu * e_\alpha).$$

It follows that

$$\tilde{D}(\mu * \nu_1) = \mu \circ \tilde{D}(\nu_1) + D(\mu) \circ \nu_2,$$

and similar calculations then show that $\tilde{D}$ is a derivation.

(b) We note that

$$\delta_e \circ f = f \circ \delta_e \qquad (f \in L^\infty(S, M_a(S))).$$

Thus

$$\tilde{D}(\delta_e) = \tilde{D}(\delta_e * \delta_e) = 2\delta_e \circ \tilde{D}(\delta_e).$$

This implies that

$$\delta_e \circ \tilde{D}(\delta_e) = 2\delta_e \circ \tilde{D}(\delta_e),$$

and so $\tilde{D}(\delta_e) = 2\delta_e \circ \tilde{D}(\delta_e) = 0$.

**Theorem 2.2** *Let $S$ be a Clifford foundation semigroup with identity. Then the semigroup algebra $M_a(S)$ is weakly amenable.*

*Proof.* Let $S$ be a Clifford semigroup. By Theorem 4.2.1 of [4], it follows that $S = \cup\{G_e : e \in E_S\}$ where the group $G_e$ is the union of the algebraic subgroups of $S$ having $e$ as their identity element and $G_e.G_f \subseteq G_{ef}$ for all $e, f \in E_S$. Let $\dot{S}$ be the collection of all $x \in S$ for which every neighbourhood $X$ of $x$ the set $X^{-1}x \cap xX^{-1}$ is a neighbourhood of $e$. We note that $\dot{S}$ is an ideal of $S$. From Theorem 11.5 of [6], it follows that for any $e \in E_S \cap \dot{S}$, $G_e$ is a closed subgroup of $S$ and

$$M_a(S) = \oplus_1\{L^1(G_e) : e \in E_S \cap \dot{S}\}. \quad (*)$$

Now, let $D : M_a(S) \mapsto L^\infty(S; M_a(S))$ be a bounded derivation. For $e \in E_S \cap \dot{S}$ define derivation $D_e : L^1(G_e) \mapsto L^\infty(G_e)$ by

$$D_e(f_e)(g_e) = D(\tilde{f}_e)(\tilde{g}_e),$$

where $\tilde{f}_e = (f_u)_{u \in E_S \cap \dot{S}} \in \oplus L^1(G_e)$ with $f_u = f_e$ if $u = e$ and $f_u = 0$ if $u \neq e$. In fact, we note that $M_a(G_e)$ is a subalgebra of $M_a(S)$. Since $G_e$ is weakly amenable, then there exist $\psi_e \in L^\infty(G_e)$ such that $D_e = ad_{\psi_e}$ and $\|\psi_e\| \leq \|D_e\| \leq \|D\|$. Since $G_e \cap G_f = \emptyset$ for all $e, f \in E_S \cap \dot{S}$, for each $x \in S$ there is a unique $e \in E_S \cap \dot{S}$ that $x \in S$ and so we may define $\psi \in L^\infty(S; M_a(S))$ by $\psi(x) = \psi_e(x)$. Let $\tilde{D}$ be as in Lemma 2.1. For any $f_u \in L^1(G_u)$ and $g_v \in L^1(G_v)$, when $u, v \in E_S \cap \dot{S}$ we have

$$\begin{aligned} (ad_\psi \tilde{g_v})(\tilde{f_u}) &= (\tilde{g_v}.\psi - \psi.\tilde{g_v})(\tilde{f_u}) \\ &= \psi(\tilde{f_u} * \tilde{g_v} - \tilde{g_v} * \tilde{f_u}) \\ &= \psi_{uv}(f_u * g_v - g_v * f_u) \end{aligned}$$

$$
\begin{aligned}
&= \psi_{uv}(f_u * \delta_v * \delta_u * g_v - g_v * \delta_u * f_u * \delta_v) \\
&= ad_{\psi_{uv}}(g_v * \delta_u)(f_u * \delta_v) \\
&= D_{uv}(g_v * \delta_u)(f_u * \delta_v) \\
&= D(\tilde{g_v} * \delta_u)(\tilde{f}_u * \delta_v) \\
&= [\delta_v \circ D(\tilde{g_v} * \delta_u)](\tilde{f}_u) \\
&= (D(\delta_v * \tilde{g_v} * \delta_u) - \tilde{D}(\delta_v) \circ (\tilde{g_v} * \delta_u))(\tilde{f}_u) \\
&= D(\tilde{g_v} * \delta_u)(\tilde{f}_u) \\
&= (D(\tilde{g_v}) \circ \delta_u + \tilde{g_v} \circ \tilde{D}(\delta_u))(\tilde{f}_u) \\
&= (D(\tilde{g_v}) \circ \delta_u)(\tilde{f}_u) \\
&= D(\tilde{g_v})(\delta_u * \tilde{f}_u) = D(\tilde{g_v})(\tilde{f}_u).
\end{aligned}
$$

So we have

$$
D(\tilde{g_v}) = ad_\psi(\tilde{g_v}) \qquad (v \in E_S \cap \dot{S}, g_v \in L^1(G_v)).
$$

Now, linearity of $D$ together with $(*)$ implies that $D = ad_\psi$ and the proof is complete.

**Remark**. *Let $T = [0,1]$. Then $T$ with the semigroup structure defined by $x.y = \max\{x,y\}$ is a commutative semigroup with identity. Let $G$ be any locally compact group. Then $S = T \times G$ with the product topology and coordinatewise multiplication defines a foundation semigroup with identity (see [3], Page 43). Let $G_t = \{t\} \times G$ for $t \in T$. It is clear that $G_t$ is a subgroup of $S$ with identity $(t, e_G)$. It is also clear that $S = \cup_{t\in T} G_t$ is a Clifford semigroup with $E_S = \{(t, e_G) : t \in T\}$ and satisfy in the Theorem 2.2 which is not a subset of any group.*

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