



A note on weak amenability of semigroup algebras B. Mohamadzadeh^a, A. Yousofzadeh1^{b,1}

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Abstract

A Banach algebra A is weakly amenable if every continuous derivation from A into A^* is inner. In this paper, we show that the semigroup algebra $M_a(S)$ is weakly amenable for a certain class of locally compact semigroups. **Keywords:** Clifford semigroup; derivation; idempotent; weak amenability.

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1 Introduction

Let S be a locally compact topological semigroup, and let M(S) denote the space of all bounded complex regular measures on S. This space with the convolution product *, and norm $||\mu|| = |\mu|(S)$ is a Banach algebra. The space of all measures $\mu \in M(S)$ for which the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into M(S) are weakly continuous is denoted by $M_a(S)$ (or $\tilde{L}(S)$ as in [1], where δ_x denotes the Dirac measure at x. Note that the measure algebra $M_a(S)$ defines a two-sided closed L-ideal of M(S)(see [1]). Recall that $M_a(S)^*$ can be made into a Banach $M_a(S)$ -module, with module actions defined by

$$\langle f\mu,\nu\rangle = \langle f,\mu*\nu\rangle, \quad \langle \mu f,\nu\rangle = \langle f,\nu*\mu\rangle \quad (\nu \in M_a(S))$$

for all $f \in M_a(S)^*$ and $\mu \in M_a(S)$.

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Recall that a linear map D from $M_a(S)$ in to $M_a(S)^*$ is called a *derivation* if $D(\mu * \nu) = D(\mu)\nu + \mu D(\nu)$ $(\mu, \nu \in M_a(S))$. For example, if $f \in M_a(S)^*$, then $ad_f : \mu \longrightarrow \mu f - f\mu$ is a derivation which is called *inner*. $M_a(S)$ is called *weakly amenable* if each continuous derivation from $M_a(S)$ into $M_a(S)^*$ is inner. It is well-known result that the group algebra $L^1(G)$ is weakly amenable for each locally compact group G. Let S = (N, +), then S is a commutative discrete semigroup with identity which is a subsemigroup of group G = (Z, +). But $\ell^1(S)$ is not weakly amenable. Indeed; since $N^2 = N + N \neq N$, from Proposition 4.2 of [2] it follows that $\ell^1(S)$ is not weakly amenable.

An element e of semigroup S is called *idempotent* if $e^2 = e$. We denote be E_S the set of idempotents in S. We recall that a semigroup S is called *Clifford semigroup* if it is an inverse semigroup for which the element of E_S are central(c.f. [4], 4.2)

In this paper, we show that $M_a(S)$ is weakly amenable for a certain class of Clifford semigroup S.

2 Main Results

Let S be a locally compact semigroup. Denote by $L^{\infty}(S, M_a(S))$ the set of all complexvalued bounded functions g on S that are $M_a(S)$ -measurable. We identify functions in $L^{\infty}(S, M_a(S))$ that agree μ -almost everywhere for all $\mu \in M_a(S)$. Note that in the case where S is discrete (resp. a locally compact group), $L^{\infty}(S, M_a(S))$ is equal to $\ell^{\infty}(S)$ (resp. $L^{\infty}(S)$). Observe that $L^{\infty}(S, M_a(S))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_{\infty}$ is a commutative C^* -algebra. A semigroups S is called a *foundation semigroup*; if $\cup \{\operatorname{supp}(\mu) : \mu \in M_a(S)\}$ is dense in S. Let S be a foundation semigroup with identity. In view of Proposition 3.6 of [5], there is an isometric isomorphism of $L^{\infty}(S; M_a(S))$ onto $M_a(S)^*$. Recall that $L^{\infty}(S; M_a(S))$ can be made into a Banach M(S)-module, with module actions defined by

$$\langle f \circ \mu, \nu \rangle = \langle f, \mu * \nu \rangle, \quad \langle \mu \circ f, \nu \rangle = \langle f, \nu * \mu \rangle \quad (\nu \in M_a(S))$$

B. Mohamadzadeh and A. Yousofzadeh

for all $f \in L^{\infty}(S; M_a(S))$ and $\mu \in M(S)$.

Lemma 2.1 Let S be a foundation semigroup with identity and $D : M_a(S) \longrightarrow L^{\infty}(S, M_a(S))$ be a bounded derivation. Then:

- (a) D has an extension to a bounded derivation $\tilde{D}: M(S) \longrightarrow L^{\infty}(S, M_a(S))$.
- (b) $\tilde{D}(\delta_e) = 0$, whenever e is a central idempotent in S.

Proof. (a) In this case, recall Proposition 5.9 of [5] that $M_a(S)$ has a bounded approximate identity. Let $\mu \in M(S)$, $\nu \in M_a(S)$, and let (e_α) be a bounded approximate identity of $M_a(S)$. By Corollary 5.17 of [6], there exists $\nu_1, \nu_2 \in M_a(S)$ such that $\nu = \nu_1 * \nu_2$. Now

$$\begin{array}{lll} \langle D(\mu \ast e_{\alpha}), \nu \rangle & = & \langle D(\mu \ast e_{\alpha}) \circ \nu_{1}, \nu_{2} \rangle \\ \\ & = & \langle D(\mu \ast e_{\alpha} \ast \nu_{1}, \nu_{2} \rangle - \langle D(\nu_{1}), \nu_{2} \ast \mu \ast e_{\alpha} \rangle \\ \\ & \longrightarrow & \langle D(\mu \ast \nu_{1}), \nu_{2} \rangle - \langle D(\nu_{1}), \nu_{2} \ast \mu \rangle, \end{array}$$

so that the $weak^* - \lim D(\mu * e_\alpha)$ exists in $L^{\infty}(S, M_a(S))$. Define

$$\tilde{D}(\mu) = weak^* - \lim D(\mu * e_\alpha).$$

It follows that

$$\tilde{D}(\mu * \nu_1) = \mu \circ \tilde{D}(\nu_1) + D(\mu) \circ \nu_2,$$

and similar calculations then show that \tilde{D} is a derivation.

(b) We note that

$$\delta_e \circ f = f \circ \delta_e \qquad (f \in L^{\infty}(S, M_a(S))).$$

Thus

$$\tilde{D}(\delta_e) = \tilde{D}(\delta_e * \delta_e) = 2\delta_e \circ \tilde{D}(\delta_e).$$

Mathematical Sciences Vol. 3, No. 3 (2009)

This implies that

$$\delta_e \circ \tilde{D}(\delta_e) = 2\delta_e \circ \tilde{D}(\delta_e),$$

and so $\tilde{D}(\delta_e) = 2\delta_e \circ \tilde{D}(\delta_e) = 0.$

Theorem 2.2 Let S be a Clifford foundation semigroup with identity. Then the semigroup algebra $M_a(S)$ is weakly amenable.

Proof. Let S be a Clifford semigroup. By Theorem 4.2.1 of [4], it follows that $S = \bigcup \{G_e : e \in E_S\}$ where the group G_e is the union of the algebraic subgroups of S having e as their identity element and $G_e.G_f \subseteq G_{ef}$ for all $e, f \in E_S$. Let \dot{S} be the collection of all $x \in S$ for which every neighbourhood X of x the set $X^{-1}x \cap xX^{-1}$ is a neighbourhood of e. We note that \dot{S} is an ideal of S. From Theorem 11.5 of [6], it follows that for any $e \in E_S \cap \dot{S}$, G_e is a closed subgroup of S and

$$M_a(S) = \bigoplus_1 \{ L^1(G_e) : e \in E_S \cap \dot{S} \}.$$
 (*)

Now, let $D: M_a(S) \mapsto L^{\infty}(S; M_a(S))$ be a bounded derivation. For $e \in E_S \cap \dot{S}$ define derivation $D_e: L^1(G_e) \mapsto L^{\infty}(G_e)$ by

$$D_e(f_e)(g_e) = D(\tilde{f}_e)(\tilde{g}_e),$$

where $\tilde{f}_e = (f_u)_{u \in E_S \cap \dot{S}} \in \oplus L^1(G_e)$ with $f_u = f_e$ if u = e and $f_u = 0$ if $u \neq e$. In fact, we note that $M_a(G_e)$ is a subalgebra of $M_a(S)$. Since G_e is weakly amenable, then there exist $\psi_e \in L^{\infty}(G_e)$ such that $D_e = ad_{\psi_e}$ and $\|\psi_e\| \leq \|D_e\| \leq \|D\|$. Since $G_e \cap G_f = \emptyset$ for all $e, f \in E_S \cap \dot{S}$, for each $x \in S$ there is a unique $e \in E_S \cap \dot{S}$ that $x \in S$ and so we may define $\psi \in L^{\infty}(S; M_a(S))$ by $\psi(x) = \psi_e(x)$. Let \tilde{D} be as in Lemma 2.1. For any $f_u \in L^1(G_u)$ and $g_v \in L^1(G_v)$, when $u, v \in E_S \cap \dot{S}$ we have

$$(ad_{\psi}\tilde{g_{v}})(\tilde{f_{u}}) = (\tilde{g_{v}}.\psi - \psi.\tilde{g_{v}})(\tilde{f_{u}})$$
$$= \psi(\tilde{f_{u}} * \tilde{g_{v}} - \tilde{g_{v}} * \tilde{f_{u}})$$
$$= \psi_{uv}(f_{u} * g_{v} - g_{v} * f_{u})$$

B. Mohamadzadeh and A. Yousofzadeh

$$= \psi_{uv}(f_u * \delta_v * \delta_u * g_v - g_v * \delta_u * f_u * \delta_v)$$

$$= ad_{\psi_{uv}}(g_v * \delta_u)(f_u * \delta_v)$$

$$= D_{uv}(g_v * \delta_u)(f_u * \delta_v)$$

$$= [\delta_v \circ D(\tilde{g_v} * \delta_u)](\tilde{f_u})$$

$$= (D(\delta_v * \tilde{g_v} * \delta_u) - \tilde{D}(\delta_v) \circ (\tilde{g_v} * \delta_u))(\tilde{f_u})$$

$$= D(\tilde{g_v} * \delta_u)(\tilde{f_u})$$

$$= (D(\tilde{g_v}) \circ \delta_u + \tilde{g_v} \circ \tilde{D}(\delta_u))(\tilde{f_u})$$

$$= D(\tilde{g_v})(\delta_u * \tilde{f_u}) = D(\tilde{g_v})(\tilde{f_u}).$$

So we have

$$D(\tilde{g_v}) = ad_{\psi}(\tilde{g_v}) \qquad (v \in E_S \cap \dot{S}, g_v \in L^1(G_v)).$$

Now, linearity of D together with (*) implies that $D = ad_{\psi}$ and the proof is complete.

Remark. Let T = [0,1]. Then T with the semigroup structure defined by $x.y = \max\{x,y\}$ is a commutative semigroup with identity. Let G be any locally compact group. Then $S = T \times G$ with the product topology and coordinatewise multiplication defines a foundation semigroup with identity (see [3], Page 43). Let $G_t = \{t\} \times G$ for $t \in T$. It is clear that G_t is a subgroup of S with identity (t, e_G) . It is also clear that $S = \bigcup_{t \in T} G_t$ is a Clifford semigroup with $E_S = \{(t, e_G) : t \in T\}$ and satisfy in the Theorem 2.2 which is not a subset of any group.

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246