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Pure Azimuthal Shear of an Elastic Dielectric Material

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Abstract

The purpose of this research is to examine the effect of polarization for the problem of pure azimuthal shear of an elastic dielectric material. The present problem is investigated in context of finite deformation theory. In this paper, the author studied the effect of polarization on the stresses for Neoprene rubber and compare the results with elastic material (Mooney-Rivlin material) graphically. Twisting of a rigid cylinder in an infinite elastic medium is considered as a special case in this research.

Keywords: Azimuthal shear, Finite deformation, Dielectric material, Polarization, Isotropic, Electrostatic.

1 Introduction

A dielectric material is a substance that is a poor conductor of electricity, but an efficient supporter of electrostatic fields. If the flow of current between opposite electric charge poles is kept to a minimum while the electrostatic lines of flux are not impeded or interrupted, an electrostatic field can store energy. Due to this property dielectric materials are vital component of capacitors, electronic devices which can store charge.

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Eringen [1] gave modified form of Toupin's theory [2] of elastic dielectric and obtained specific forms of the basic field equations, the boundary conditions and the constitutive equations that must satisfies the stress, electrical and polarization fields by using a variational principle in electroelastostatics. Recently authors [3-5] and Singh and Verma [19] have investigated some basic problems of practical interest for circular cylinders composed of isotropic hyperelastic incompressible materials by using this theory. The problem of circular shearing (azimuthal shear) of a compressible hyperelastic cylinder has been studied by Ertepinar [6], Haughton [11], Jiang and Ogden [12], Polgnone and Horgan [9], Simmonds and Warne [7], Tao et al. [8], Wineman and Waldron [10], Dorfmann and Ogden [20-21]. Shear problems in circular cylinders for incompressible materials with limiting chain extensibility have been investigated by Horgan and Saccomandi [13-14]. The present research is related to examine the effect of polarization to the problem of azimuthal shear of a hollow circular dielectric cylindrical tube. The inner surface of the tube is bonded to a rigid cylinder and uniformly distributed azimuthal shear traction is applied to the outer surface of the tube with zero radial traction maintained at the same surface. The formulation of the problem is based on the theory of finite elastic deformations [15-18]. We show the effect of polarization on the normal stresses and compare the results with elastic material (neo-Hookean) material graphically.

2 Fundamental Equations

The basic equations of an incompressible, homogeneous, isotropic, hyperelastic dielectric can be classified in the following three groups:

(a). Field Equations.

$$t^k_{l;k} + \rho\, f_l = 0, \tag{1}$$

$$_L E_k - \phi_{,k} = 0, \tag{2}$$

$$\epsilon_0 \nabla^2 \phi - div \overrightarrow{P} = -q_f, \quad in \quad V_d, \tag{3}$$

where t^k_l is the Cauchy stress tensor, ρ is the volume density, f_l is the body force per unit mass, ${}_LE_k$ is the local electrostatic field, ϵ_0 is the material constant, ϕ is the electrostatics potential, $\overrightarrow{P}$ is the polarization vector, q_f is the volume free charge, V_d is the volume that the dielectric occupies. Semicolon and comma indicate the covariant and partial derivatives respectively.

(b). Boundary Conditions.

$$\|t^k_l\|n_k = 0, \tag{4}$$

$$\|\epsilon_0\phi^k_, - \overrightarrow{P^k}\|n_k + \omega_f = 0, \quad on \quad S_d, \tag{5}$$

where n_k is the exterior normal to S_d, S_d is the surface enclosing the dielectric volume, ω_f is the free surface charge, the double bracket stands for discontinuity across the surface .The Cauchy stress tensor t^k_l is defined as

$$t^k_l \equiv_L t^k_l +_M t^k_l, \tag{6}$$

$$_Mt^k_l \equiv \epsilon_0(\phi^k_,\phi_{,l} - 1/2\phi^m_,\phi_{,m}\delta^k_l), \tag{7}$$

where ${}_Mt^k_l$ is the Maxwell stress tensor.

(c). Constitutive Equations.

$$_Lt^k_l = -p\delta^k_l + 2[^{-1}c^k_l(\frac{\partial\Sigma}{\partial I_1} + I_1\frac{\partial\Sigma}{\partial I_2}) -^{-2} c^k_l\frac{\partial\Sigma}{\partial I_2} +^{-1} c^k_m P^m P_l\frac{\partial\Sigma}{\partial I_4}$$

$$+^{-2}c^k_m P^m P_l\frac{\partial\Sigma}{\partial I_5} + (^{-1}c^k_m)(^{-1}c^l_n)P^m P_l\frac{\partial\Sigma}{\partial I_5}], \tag{8}$$

$$_LE^k = 2[^{-1}c^k_l\frac{\partial\Sigma}{\partial I_4} +^{-2} c^k_l\frac{\partial\Sigma}{\partial I_5} + \delta^k_l\frac{\partial\Sigma}{\partial I_6}]P^l, \tag{9}$$

where p is the arbitrary hydrostatic pressure, δ^k_l is a Kronecker delta, $\Sigma = \Sigma(I_1, I_2, I_4, I_5, I_6)$ and $I's$ are the invariants based on Finger's strain measure ^{-1}c and polarization $\overrightarrow{P}$.These are given by

$$I = I_1 = \delta^{k-1}_l c^k_l, \quad II = I_2 = \frac{1}{2}\delta^{km-1}_{ln} c^{l}_k {}^{-1}c^n_m, \quad III = I_3 = \frac{1}{6}\delta^{kmp-1}_{lnq} c^l_k {}^{-1}c^n_m {}^{-1}c^q_p,$$

$$I_4 = {}^{-1}c_l^k P^l P_k, \qquad I_5 = {}^{-1}c_m^k\, {}^{-1}c_l^m P^l P_k, \qquad I_6 = P^2. \tag{10}$$

The deformation tensors c_l^k and ${}^{-1}c_l^k$ are given by

$${}^{-1}c_l^k = f_{ml} G^{KM} x^k_{,K} x^m_{,M},$$

$$c_l^k = f^{km} G_{ML} X^M_{,m} X^L_{,l}. \tag{11}$$

Where G_{ij}, f_{ij} are the metric tensor, X^i and x^i are the co-ordinates in the undeformed and deformed states respectively.

3 Formulation of The Problem

A thick incompressible circular cylindrical shell, carrying a uniform surface charge at the inner surface, is elongated uniformly in the axial direction. For the cylindrical tube,with inner surface bonded to a rigid cylinder and a uniformly distributed azimuthal shear traction applied to the outer surface, the deformation is that of pure azimuthal shear (no radial deformation) described by

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z, \tag{12}$$

where the material and spatial cylindrical polar coordinates are denoted by (R, Θ, Z) and (r, θ, z) respectively, with $a \leq R \leq b$.
Using (11), we can find the deformation tensors as

$$\|c_{kl}\| = \begin{bmatrix} 1 + R^2 g'^2 & -Rg' & 0 \\ -Rg' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \left\|{}^{-1}c_{kl}\right\| = \begin{bmatrix} 1 & Rg' & 0 \\ Rg' & 1 + R^2 g'^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{13}$$

with the help of (10) and (13), we calculate the principal invariants

$$I_1 = I_2 = 3 + R^2 g'^2, \quad I_3 = 1, \quad I_4 = P^2, \quad I_5 = (1 + R^2 g'^2) P^2, \quad I_6 = P^2. \tag{14}$$

4 Electrostatic and Maxwell Fields

To determine the electrostatic field, we solve equation (3) with the boundary conditions (4)and (5). We assume that electric field and polarization field have single component i.e.,$\overrightarrow{E} = [E(R), 0, 0]$, $\overrightarrow{P} = [P(R), 0, 0]$.
As a final result [18], we find

$$\phi = \alpha_1, \qquad for \quad 0 < R < a$$

$$\epsilon_0 \phi = -a\omega_f \log R + \int^R P(R)dR + \beta_2, \qquad for \quad a < R < b$$

$$\epsilon_0 \phi = -a\omega_f \log R, \qquad for \qquad R > b. \tag{15}$$

The unknown constants α_1 and β_2 are immaterial for electric and stress field. Using $(12), (15)_2, (7)$, we obtain

$$_Mt_1^1 = -_Mt_2^2 = -_Mt_3^3 = \frac{1}{2\epsilon_0}(\frac{a\omega_f}{R} - P)^2. \tag{16}$$

The equation $_M\overrightarrow{E} = -grad\phi$ gives the Maxwell electric field as follows

$$_ME^1 = \frac{1}{\epsilon_0}(\frac{a\omega_f}{R} - P)^2, \quad _ME^2 = 0, \quad _ME^3 = 0. \tag{17}$$

5 Local Electric and Stress Fields

Using equation (13), we obtain the local electric field from (9)

$$_LE^1 = 2[\frac{\partial\Sigma}{\partial I_4} + (1 + R^2 g'^2)\frac{\partial\Sigma}{\partial I_5} + \frac{\partial\Sigma}{\partial I_6}]P,$$

$$_LE^2 = 2[Rg'\frac{\partial\Sigma}{\partial I_4} + (2Rg' + R^3 g'^3)\frac{\partial\Sigma}{\partial I_5}]P, \qquad _LE^3 = 0. \tag{18}$$

The local stress tensor from (8) for incompressible dielectric is obtained as

$$_Lt_1^1 =_L t^{11} = -p + 2[(\frac{\partial\Sigma}{\partial I_1} + 2\frac{\partial\Sigma}{\partial I_2}) + \frac{\partial\Sigma}{\partial I_4} + (2 + R^2 g'^2)\frac{\partial\Sigma}{\partial I_5}P^2],$$

$$_Lt_2^2 =_L t^{22} = -p + 2[(1 + R^2 g'^2)\frac{\partial \Sigma}{\partial I_1} + (2 + R^2 g'^2)\frac{\partial \Sigma}{\partial I_2} + R^2 g'^2 P^2 \frac{\partial \Sigma}{\partial I_5}],$$

$$_Lt_3^3 =_L t^{33} = -p + 2[\frac{\partial \Sigma}{\partial I_1} + (2 + R^2 g'^2)\frac{\partial \Sigma}{\partial I_2}],$$

$$_Lt_1^2 =_L t^{12} = 2[\frac{\partial \Sigma}{\partial I_1} + \frac{\partial \Sigma}{\partial I_2} + R^2 g'^2 P^2 \frac{\partial \Sigma}{\partial I_5}], \quad _Lt_1^3 = 0, \quad _Lt_2^3 = 0. \tag{19}$$

By using (6),(18)and (19),the components of Cauchy stress tensor can be written in the form

$$t_1^1 = t^{11} = -p + 2[(\frac{\partial \Sigma}{\partial I_1} + 2\frac{\partial \Sigma}{\partial I_2}) + \frac{\partial \Sigma}{\partial I_4} + (2 + R^2 g'^2)\frac{\partial \Sigma}{\partial I_5} P^2] - \frac{1}{2\epsilon_0}(\frac{a\omega_f}{R} - P)^2,$$

$$t_2^2 = t^{22} = -p + 2[(1 + R^2 g'^2)\frac{\partial \Sigma}{\partial I_1} + (2 + R^2 g'^2)\frac{\partial \Sigma}{\partial I_2} + R^2 g'^2 P^2 \frac{\partial \Sigma}{\partial I_5}] + \frac{1}{2\epsilon_0}(\frac{a\omega_f}{R} - P)^2,$$

$$t_3^3 = t^{33} = -p + 2[\frac{\partial \Sigma}{\partial I_1} + (2 + R^2 g'^2)\frac{\partial \Sigma}{\partial I_2}] + \frac{1}{2\epsilon_0}(\frac{a\omega_f}{R} - P)^2,$$

$$_Lt_2^1 =_L t^{12} = 2[\frac{\partial \Sigma}{\partial I_1} + \frac{\partial \Sigma}{\partial I_2} + R^2 g'^2 P^2 \frac{\partial \Sigma}{\partial I_5}], \quad _Lt_1^3 = 0, \quad _Lt_2^3 = 0. \tag{20}$$

The equations of force equilibrium with the vanishing body force are

$$\frac{\partial t_2^1}{\partial R} + \frac{1}{R}(t_1^1 - t_2^2) = 0. \tag{21}$$

$$\frac{\partial t_1^1}{\partial R} + 2\frac{t_2^1}{R} = 0. \tag{22}$$

From (22), we have

$$\frac{d}{dR}(R^2 t_2^1) = 0. \tag{23}$$

we find, on integration (23), that

$$t_2^1 = \frac{b^2}{R^2} T_0. \tag{24}$$

where T_0 is the prescribed azimuthal shear stress on the outer boundary. On using (20), we obtain a first order differential equation for $g(R)$, namely

$$2Rg'[\frac{\partial\Sigma}{\partial I_1}+\frac{\partial\Sigma}{\partial I_2}+R^2g'^2P^2\frac{\partial\Sigma}{\partial I_5}]=\frac{b^2}{R^2}T_0 \tag{25}$$

also

$$\frac{\partial t_1^1}{\partial R}-\frac{2}{R}[R^2g'^2(\frac{\partial\Sigma}{\partial I_1}+\frac{\partial\Sigma}{\partial I_2})-(\frac{\partial\Sigma}{\partial I_4}+2\frac{\partial\Sigma}{\partial I_5})P^2-\frac{1}{2\epsilon_0}(\frac{a\omega_f}{R}-P)^2]=0 \tag{26}$$

Obtain $g(R)$ from (25), subject to the boundary condition

$$g(a)=0 \tag{27}$$

Integrating (27) from R to b and use of the boundary condition

$$t_1^1=0 \tag{28}$$

Yields

$$t_1^1(R)=\int_R^b\frac{-2}{s}[s^2g'^2(\frac{\partial\Sigma}{\partial I_1}+\frac{\partial\Sigma}{\partial I_2})-(\frac{\partial\Sigma}{\partial I_4}+2\frac{\partial\Sigma}{\partial I_5})P^2-\frac{1}{2\epsilon_0}(\frac{a\omega_f}{s}-P)^2]ds. \tag{29}$$

6 Special Material

We consider a special type of dielectric material characterized by the strain energy function Σ of the form

$$\Sigma=\alpha_1(I_1-3)+\alpha_2(I_2-3)+\alpha_4I_4+\alpha_5I_5+\alpha_6I_6, \tag{30}$$

where $\alpha's$ are constants. Now (2), (17), (18) gives polarization of the form

$$P=-K_1\frac{a\omega_f}{R}, \tag{31}$$

where

$$K_1=[2\epsilon_0(\alpha_6-\alpha_5)-1]^{-1}. \tag{32}$$

and

$$2Rg'[\alpha_4+(2+R^2g'^2)\alpha_5]P=0. \tag{33}$$

Now as $R, P, g' > 0$, therefore

$$[\alpha_4 + (2 + R^2 g'^2)\alpha_5] = 0. \tag{34}$$

From equation (25), (30) and (31), we have

$$2K_1^2 a^2 \omega_f^2 \alpha_5 g'^2 + 2Rg'(\alpha_1 + \alpha_2) - \frac{b^2}{R^2} T_0 = 0. \tag{35}$$

The quadratic equation (35) for Rg' yields

$$Rg' = -R^2 M_3 + \frac{1}{R}\sqrt{R^4 M_3^2 + \frac{b^2}{M_2} T_0}. \tag{36}$$

where $M_3 = \frac{M_1}{M_2}$, $M_1 = \alpha_1 + \alpha_2$ and $M_2 = 2K_1^2 a^2 \omega_f^2 \alpha_5$ .The positive sign is taken here to ensure that $g' > 0$. On integration of (36) and using the boundary condition $g(a) = 0$, we obtain

$$g(R) = \frac{bT_0^{\frac{1}{2}}}{2M_2^{\frac{1}{2}}}\left[\frac{M_1}{bT_0^{\frac{1}{2}} M_2^{\frac{1}{2}}}(a^2 - R^2) + W(R) - W(a) - \ln\{(\frac{a}{R})^2(\frac{W(R)+1}{W(a)+1})\}\right], \tag{37}$$

where

$$W(R) = \sqrt{1 + \frac{M_1^2 R^4}{b^2 T_0 M_2}} \tag{38}$$

Now we can write (29) by using (30), (31), (32) and (34) in the form

$$t_1^1(R) = -\int_R^b \left[\frac{b^2 T_0}{s^2} g' - (2s^2 g'^3 + 2sg'^2)\frac{a^2 K_1^2 \omega_f^2}{s^2}\alpha_5 + \frac{1}{s^3 \epsilon_0}(1 + K_1)^2 a^2 \omega_f^2\right] ds. \tag{39}$$

On using (36) in (39) we find that

$$t_1^1(R) = -\int_R^b \Big[\frac{b^2 T_0}{s^2}(-sM_3 + \frac{1}{s}\sqrt{s^4 M_3^2 + \frac{b^2 T_0}{M_2}}) - \{2s^2(-sM_3 + \frac{1}{s}\sqrt{s^4 M_3^2 + \frac{b^2 T_0}{M_2}})^3$$

$$+2s(-sM_3 + \frac{1}{s}\sqrt{s^4 M_3^2 + \frac{b^2 T_0}{M_2}})\}\frac{a^2 K_1^2 \omega_f^2}{s^2}\alpha_5 + \frac{1}{s^3 \epsilon_0}(1 + K_1)^2 a^2 \omega_f^2\Big] ds, \tag{40}$$

which on integration gives

$$t_1^1 = b^2 T_0 M_3 \ln(\frac{b}{R}) + \frac{M_3}{2}\ln\{(\frac{R}{b})^2(\frac{J(R)+1}{J(b)+1})\} + \frac{1}{2}\{J(b) - J(R)\}$$

$$+\xi a^2K_1^2\omega_f^2\alpha_5+\frac{1}{\epsilon_0}(1+K_1)^2\omega_f^2\{(\frac{a}{b})^2-(\frac{a}{R})^2\} \tag{41}$$

where

$$\xi=\frac{M_3^3}{4}(R^4-b^4)-\frac{1}{2}\frac{b^2T_0M_3}{M_2}[J(b)-J(R)-2\ln\{(\frac{R}{b})^2(\frac{J(R)+1}{J(b)+1}\}]$$

$$-\frac{1}{2}\frac{M^2M_3^2}{b^2T_0}[J(b)b^4-J(R)R^4]+M_3^2(b^2-R^2)-\frac{b^2T_0}{M_2}(\frac{1}{b^2}-\frac{1}{R^2})$$

$$+\frac{3}{4}\frac{b^2T_0}{M_2}[\frac{M_3M_2^{\frac{1}{2}}}{bT_0^{\frac{1}{2}}}(b^2W(b)-R^2W(R))+\cosh^{-1}W(b)-\cosh^{-1}W(R)]$$

$$+\frac{5}{4}\frac{bT_0^{\frac{1}{2}}M_3}{M_2^{\frac{1}{2}}}[\frac{M_1}{bT_0^{\frac{1}{2}}M_2^{\frac{1}{2}}}(b^2-R^2)+W(R)-W(b)-\ln\{(\frac{R}{b})^2(\frac{W(R)+1}{W(b)+1}\}], \tag{42}$$

where

$$J(R)=\sqrt{1+\frac{b^2T_0M_2}{M_1^2R^4}}. \tag{43}$$

The remaining non-zero stresses are

$$t_2^2=t_1^1+b^2T_0M_3(J(R)-1)+\frac{1}{\epsilon_0}(1+K_1)^2\omega_f^2(\frac{a}{R})^2$$

$$+2R^2M_3^2[J(R)-1]^2[1-R^2M_3(J(R)-1)]a^2K_1^2\omega_f^2\alpha_5,$$

$$t_3^3=t_1^1+R^2M_3^2[J(R)-1]^22k_1^2a^2\omega_f^2\alpha_5+\frac{1}{\epsilon_0}(1+K_1)^2\omega_f^2(\frac{a}{R})^2,$$

$$t_2^1=\frac{b^2}{R^2}T_0. \tag{44}$$

7 Special Case

If $\frac{\partial\Sigma}{\partial I_5} = \alpha_5 = 0$, then from (34)$\frac{\partial\Sigma}{\partial I_4} = \alpha_4 = 0$ By using (30), we an write (25)as

$$2Rg'[\alpha_1 + \alpha_2] = \frac{b^2}{R^2}T_0 \tag{45}$$

So that, on integrating (39) and use of the boundary condition $g(a) = 0$, we obtain the well known result

$$g(R) = \frac{T_0}{4(\alpha_1 + \alpha_2)}[(\frac{b}{a})^2 - (\frac{b}{R})^2]. \tag{46}$$

Thus, we find the non-zero normal stresses as

$$t_1^1 = \frac{1}{8}\frac{T_0^2}{(\alpha_1 + \alpha_2)}[1 - (\frac{b}{R})^4] + \frac{1}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2[(\frac{a}{b})^2 - (\frac{a}{R})^2],$$

$$t_2^2 = \frac{1}{8}\frac{T_0^2}{(\alpha_1 + \alpha_2)}[1 - 3(\frac{b}{R})^4] + \frac{1}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2[(\frac{a}{b})^2 + (\frac{a}{R})^2],$$

$$t_3^3 = \frac{1}{8}\frac{T_0^2}{(\alpha_1 + \alpha_2)}[1 - (\frac{b}{R})^4] + \frac{1}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2[(\frac{a}{b})^2 + (\frac{a}{R})^2], \tag{47}$$

It is clearly seen from (47) that polarization effects normal stresses. To get numerical results of the present problem the authors used the numerical for Neoprene rubber ($\epsilon_0 = 6.6$) with $\alpha_1 = .4, \alpha_2 = .1, \alpha_6 = .05, b = 2, a = .9, R = 1$. It is observed from Figure 7.1 and Figure 7.2, that the radial stress is less compressive and hoop stress is more tensile in case of dielectric material than elastic material (neo-Hookean material) but most important is the effect of polarization on axial stress (Figure 7.3), due to polarization axial stress becomes compressive in the region where azimuthal shear stress (T_0) $< .2777$, and tensile for remaining region.

To compare the predictions of the dielectric material, we first consider the dependence of the relative angle of twist, defined by

$$\psi = g(b), \tag{48}$$

on the prescribed azimuthal shear T_0 , we have

$$\psi = \frac{1}{4}\frac{T_0}{(\alpha_1+\alpha_2)}[(\frac{b}{a})^2 - 1], \tag{49}$$

which we write in the non-dimensional form

$$\psi = \frac{\overline{T}}{4}\{\eta^2 - 1\}, \tag{50}$$

where

$$\eta = \frac{b}{a}(> 1), \tag{51}$$

and

$$\overline{T} = \frac{T_0}{(\alpha_1+\alpha_2)}, \tag{52}$$

Thus, ψ depends on in a linear fashion. The result (52) also hold good for Mooney-Rivlin material.

Another global stress measure worth comparing is the hoop stress at the outer boundary

$$\overline{\tau} = \frac{t_h}{\alpha_1+\alpha_2} = \frac{t_2^2(b)}{\alpha_1+\alpha_2}, \tag{53}$$

For the Mooney-Rivlin material, (47) yields

$$\overline{\tau} = \frac{1}{2}\frac{T_0^2}{(\alpha_1+\alpha_2)} = \frac{\overline{T}^2}{2}, \tag{54}$$

where the non-dimensional prescribed azimuthal shear stress $\overline{T}$ was defined in (52). For dielectric material, we find from (47) that

$$\overline{\tau} = \frac{1}{2}\frac{T_0^2}{(\alpha_1+\alpha_2)} + \frac{1}{\epsilon_0}(1+K_1)^2\omega_f^2(\frac{a}{b})^2 = \frac{\overline{T}^2}{2} + \frac{1}{(\alpha_1+\alpha_2)}\frac{1}{\epsilon_0}(1+K_1)^2\omega_f^2(\frac{a}{b})^2, \tag{55}$$

Thus (55) shows hoop stress increases in the case of dielectric material due to polarization.

8 Displacement Boundary Conditions

It is of interest to consider a different boundary value from the preceding where the traction boundary conditions on the outer surface are replaced by prescribed the angular displacement. Thus, we now have the boundary conditions

$$g(a) = 0, \quad g(b) = g_0, \tag{56}$$

where $g_0 > 0$ is the prescribed angular displacement. The governing differential equation for $g(R)$ is still given by (23), which on integration gives

$$R^2 t_2^1 \equiv 2R^3 g'[\alpha_1 + \alpha_2] \equiv K, \tag{57}$$

where K is an unknown constant, to be determined in terms of g_0 .
For dielectric material, it follows from (56) and (57) that

$$g(R) = g_0 \frac{\eta^2 - (\frac{b}{R})^2}{\eta^2 - 1}. \tag{58}$$

And the normal stresses are

$$t_1^1 = -2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2-1)^2}(\frac{b}{R})^4 + \frac{1}{2}\frac{1}{\epsilon_0}(1+K_1)^2\omega_f^2(\frac{a}{R})^2,$$

$$t_2^2 = 2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2-1)^2}[3(\frac{b}{R})^4] + \frac{3}{2}\frac{1}{\epsilon_0}(1+K_1)^2\omega_f^2(\frac{a}{R})^2,$$

$$t_3^3 = -2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2-1)^2}(\frac{b}{R})^4 + \frac{3}{2}\frac{1}{\epsilon_0}(1+K_1)^2\omega_f^2(\frac{a}{R})^2, \tag{59}$$

As from (59) only normal stresses depends upon in a quadratic fashion and have an effect of dielectric and polarization. To get numerical results of the present problem the authors used the numerical values for Neoprene rubber ($\epsilon_0 = 6.6$) with $\alpha_1 = .4, \alpha_2 = .1, \alpha_6 = .05, b = 2, a = 1, R = 1.5$. From Figure 8.1 and 8.2, shows the similar results for different values of angle of twist. From Figure 8.3 axial stress is compressive when the angle of twist $(g_0) < 19.945$ degree and becomes tensile for the remaining region.In the absence of polarization, we obtain results similar to the results obtained by Horgan and Saccomandi [14] for isotropic hyperelastic materials with limiting chain extensibility. The boundary conditions (56) have been used by Jiang and Ogden [12] for analysis of compressible materials.

9 Twisting of a Rigid Cylinder in an Infinite Elastic Medium

Another interesting set of boundary conditions, for a hollow tube surrounded by a rigid casing, is

$$g(a) = g_0, \quad g(b) = 0, \tag{60}$$

In this case, integration of (57) for dielectric material gives

$$g(\overline{R}) = \frac{g_0(\overline{R}^2 - \eta^2)}{\overline{R}^2(\eta^2 - 1)}, \tag{61}$$

And the normal stresses are

$$t_1^1 = -2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2 - 1)^2}(\frac{\eta}{\overline{\overline{R}}})^4 + \frac{1}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2(\frac{1}{\overline{\overline{R}}})^2,$$

$$t_2^2 = 2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2 - 1)^2}[3(\frac{\eta}{\overline{\overline{R}}})^4] + \frac{3}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2(\frac{1}{\overline{\overline{R}}})^2,$$

$$t_3^3 = -2(\alpha_1 + \alpha_2)\frac{g_0^2}{(\eta^2 - 1)^2}(\frac{b}{\overline{\overline{R}}})^4 + \frac{3}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2(\frac{1}{\overline{\overline{R}}})^2. \tag{62}$$

These solutions may be simplified on considering the limit as $\eta \to \infty$ i.e. the boundary-value problem corresponding to the twisting of a rigid cylinder of radius bonded to an infinite elastic medium. In this case, we have from (61)

$$g(\overline{R}) = \frac{g_0}{\overline{\overline{R}}^2} \tag{63}$$

From (62) and (63), we get

$$t_1^1 = -2(\alpha_1 + \alpha_2)g^2(\overline{R}) + \frac{1}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2(\frac{1}{\overline{\overline{R}}})^2,$$

$$t_2^2 = 6(\alpha_1 + \alpha_2)g^2(\overline{R}) + \frac{3}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2(\frac{1}{\overline{\overline{R}}})^2,$$

$$t_3^3 = -2(\alpha_1 + \alpha_2)g^2(\overline{R}) + \frac{3}{2}\frac{1}{\epsilon_0}(1 + K_1)^2\omega_f^2(\frac{1}{\overline{\overline{R}}})^2. \quad (64)$$

From (64), shows the dependence of normal stresses on polarization. In the absence of polarization, we obtain results similar to the results obtained by Horgan and Saccomandi [14] for isotropic hyperelastic materials with limiting chain extensibility also from (64) axial and radial stresses are equal in the absence of radial electric field.

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