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General Common Fixed Point Theorems for Occasionally Weakly Compatible Maps

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Abstract

The aim of this contribution is to establish and prove two general common fixed point theorems for four occasionally weakly compatible maps in a metric space. These results unify and complement several various results, especially the main result of Djoudi [2] and references therein.

Keywords: Occasionally weakly compatible maps, general common fixed point theorems, integral type, metric space.

1 Introduction

Before giving our main results, let us begin by the following historic:

To generalize commuting maps, Sessa [8] introduced the concept of weakly commuting maps. Maps f and g of a metric space $(\mathcal{X}, d)$ into itself are weakly commuting if, for all $x \in \mathcal{X}$

$$d(fgx, gfx) \leq d(gx, fx). \tag{1}$$

Later on, Jungck [3] gave a generalization of commuting and weakly commuting maps by introducing the notion of compatible maps. He defines f and g to be compatible if,

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0 \tag{2}$$

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whenever $(x_n)_{n\in N}$ is a sequence in $\mathcal{X}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in \mathcal{X}$.

In 1993, the same author with Murthy and Cho [5] introduced another generalization of weak commutativity by giving the concept of compatibility of type (A). They define f and g to be compatible of type (A) if, f and g satisfy instead of (2) the two conditions:

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \ \textit{and} \ \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0. \tag{3}$$

Further, in 1995, Pathak and Khan [7] generalized the concept of compatible maps of type (A) by making the notion of compatible maps of type (B). f and g are called compatible of type (B) if, they satisfy instead of (2), the following inequalities:

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right] \tag{4}$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right]. \tag{5}$$

Afterwards, in 1998, Pathak et al. [6] added another extension of compatible maps of type (A) by giving the concept of compatible maps of type (C). The above maps are said to be compatible of type (C) if, we have in place of condition (2) the below inequalities:

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n) \ + \lim_{n\to\infty} d(ft, g^2x_n)\right] \tag{6}$$

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) \ + \lim_{n\to\infty} d(gt, f^2x_n)\right]. \tag{7}$$

In his paper [4], Jungck generalized all the above concepts by giving the notion of weakly compatible maps. He defines f and g to be weakly compatible if, $ft = gt$, $t \in \mathcal{X}$ implies $fgt = gft$.

Recently in 2008, Al-Thagafi and Shahzad [1] gave a proper generalization of non-trivial weakly compatible maps which do have a coincidence point called occasionally weakly compatible maps. Two self-maps f and g of a set $\mathcal{X}$ are occasionally weakly compatible maps (shortly (owc)) if and only if, there is a point t in $\mathcal{X}$ which is a coincidence point of f and g at which f and g commute.

Now, let R^+ be the set of all non-negative real numbers and Φ be the set of all continuous functions $\varphi : (R^+)^6 \to R$ satisfying the conditions:

(φ_1): φ is nondecreasing in variables t_5 and t_6,

(φ_2): there exists $\theta \in (1, \infty)$, such that for every u, $v \geq 0$ with

(φ_a): $\varphi(u, v, u, v, u+v, 0) \geq 0$ or

(φ_b): $\varphi(u, v, v, u, 0, u+v) \geq 0$ we have $u \geq \theta v$,

(φ_3): $\varphi(u, u, 0, 0, u, u) < 0 \ \forall u > 0$.

In his paper [2], Djoudi proved the next common fixed point theorem for four weakly compatible maps by using general implicit relations on a complete metric space.

Theorem 1.1 *Let h, k, f and g be maps from a complete metric space $\mathcal{X}$ into itself having the following conditions:*

(i) h, k are surjective,

(ii) the pairs of maps h, f as well as k, g are weakly compatible,

(iii) the inequality

$$\varphi(d(hx, ky), d(fx, gy), d(hx, fx), d(ky, gy), d(hx, gy), d(ky, fx)) \geq 0 \tag{8}$$

for all x, $y \in \mathcal{X}$, where $\varphi \in \Phi$. Then h, k, f and g have a unique common fixed point.

Our purpose henceforth is to improve and extend the result of [2] by weakening weakly compatibility, dropping the surjectivity and deleting some conditions required on function φ in a metric space which is more general than complete metric space.

2 Main Results

Theorem 2.1 *Let $(\mathcal{X}, d)$ be a metric space and let h, k, f, $g : \mathcal{X} \to \mathcal{X}$ be four maps satisfying the following inequality:*

$$\varphi(d(hx, ky), d(fx, gy), d(hx, fx), d(ky, gy), d(hx, gy), d(ky, fx)) \geq 0 \tag{9}$$

for all x, y in $\mathcal{X}$, where φ is a function from $(R^+)^6$ into R, satisfies property

$$\varphi(u, u, 0, 0, u, u) < 0, \ \forall u > 0, \tag{10}$$

then, h, k, f and g have at most one common fixed point in $\mathcal{X}$.

Proof Suppose that h, k, f and g have two common fixed points t and t' such that $t \neq t'$, then condition (9) gives

$$\varphi(d(ht, kt'), d(ft, gt'), d(ht, ft), d(kt', gt'), d(ht, gt'), d(kt', ft))$$

$$= \varphi(d(t, t'), d(t, t'), 0, 0, d(t, t'), d(t', t)) \geq 0$$

contradicts (10), therefore $t' = t$.

Theorem 2.2 *Let h, k, f and g be maps from a metric space $(\mathcal{X}, d)$ into itself having the following conditions:*
(1) *h and f as well as k and g are owc,*
(2) *inequality (9) holds for all x, $y \in \mathcal{X}$, where φ satisfies property (10), then, h, k, f and g have a unique common fixed point in $\mathcal{X}$.*

Proof Since the pairs $\{h, f\}$ and $\{k, g\}$ are each owc, then, there exist two elements u and v in $\mathcal{X}$ such that $hu = fu$, $kv = gv$ and $hfu = fhu$, $kgv = gkv$.
First, we prove that $hu = kv$. If it is not the case, then, by inequality (9) we get

$$\varphi(d(hu, kv), d(fu, gv), d(hu, fu), d(kv, gv), d(hu, gv), d(kv, fu))$$

$$= \varphi(d(hu, kv), d(hu, kv), 0, 0, d(hu, kv), d(kv, hu)) \geq 0$$

contradicts (10), therefore $hu = kv$.

Now, suppose that $h^2u \neq hu$, then, from condition (9),

$$\varphi(d(h^2u, kv), d(fhu, gv), d(h^2u, fhu), d(kv, gv), d(h^2u, gv), d(kv, fhu))$$

$$= \varphi(d(h^2u, hu), d(h^2u, hu), 0, 0, d(h^2u, hu), d(hu, h^2u)) \geq 0$$

contradicts property (9), hence $h^2u = hu$.

Also, if $k^2v \neq kv$, then, using inequality (9) we obtain

$$\varphi(d(hu, k^2v), d(fu, gkv), d(hu, fu), d(k^2v, gkv), d(hu, gkv), d(k^2v, fu))$$

$$= \varphi(d(kv, k^2v), d(kv, k^2v), 0, 0, d(kv, k^2v), d(k^2v, kv)) \geq 0$$

which contradicts (10), therefore $k^2v = kv$.

Putting $hu = kv = t$ we have $ht = ft = kt = gt = t$ and from Theorem 2.1, the common fixed point is unique.

Corollary 2.3 *Let $(\mathcal{X}, d)$ be a metric space and let h, $f : (\mathcal{X}, d) \to (\mathcal{X}, d)$ be two maps such that*

(1) *the pair $\{h, f\}$ is owc,*

(2) *the inequality*

$$\varphi(d(hx, hy), d(fx, fy), d(hx, fx), d(hy, fy), d(hx, fy), d(hy, fx)) \geq 0 \qquad (11)$$

holds for all x, y in $\mathcal{X}$, where φ satisfies property (10), then, h and f have a unique common fixed point in $\mathcal{X}$.

Corollary 2.4 *Let h, k and f be three self-maps of a metric space $(\mathcal{X}, d)$ satisfying the following conditions:*

(1) *the pairs $\{h, f\}$ and $\{k, f\}$ are owc,*

(2) *the inequality*

$$\varphi(d(hx, ky), d(fx, fy), d(hx, fx), d(ky, fy), d(hx, fy), d(ky, fx)) \geq 0 \qquad (12)$$

holds for all x, y *in* $\mathcal{X}$, *where* φ *satisfies property (10), then,* h, k *and* f *have a unique common fixed point in* $\mathcal{X}$.

Corollary 2.5 *Let* $(\mathcal{X}, d)$ *be a metric space and let* h, k, f *and* g *be self-maps such that the pairs* $\{h, f\}$ *and* $\{g, k\}$ *are owc and satisfy one of the following inequalities:*

$$d^p(hx, ky) \geq \frac{1}{a}[bd^p(fx, gy) - cd^{p-1}(hx, gy)d(ky, fx) - dd^{p-1}(ky, fx)d(hx, gy)], \quad (13)$$

where $a > 0$, b, c, $d \geq 0$, $b > a + c + d$ *and* p *is an integer such that* $p \geq 2$,

$$d^p(hx, ky) \geq \alpha d^p(fx, gy) + \beta d^p(hx, gy) + \gamma d^p(ky, fx), \quad (14)$$

where $\alpha > 1$, β, $\gamma \geq 0$, $\alpha + \beta + \gamma > 1$ *and* p *is an integer such that* $p \geq 1$,
for all x, y *in* $\mathcal{X}$, *then,* h, k, f *and* g *have a unique common fixed point in* $\mathcal{X}$.

Proof For proof of (13) and (14), we use Theorems 2.1 and 2.2 with the following functions φ which satisfy, for every case, hypothesis (10)
for (i):

$$\varphi(d(hx, ky), d(fx, gy), d(hx, fx), d(ky, gy), d(hx, gy), d(ky, fx))$$
$$= ad^p(hx, ky) - bd^p(fx, gy) + cd^{p-1}(hx, gy)d(ky, fx) + dd^{p-1}d(ky, fx)d(hx, gy),$$

for (ii):

$$\varphi(d(hx, ky), d(fx, gy), d(hx, fx), d(ky, gy), d(hx, gy), d(ky, fx))$$
$$= d^p(hx, ky) - \alpha d^p(fx, gy) - \beta d^p(hx, gy) - \gamma d^p(ky, fx).$$

Corollary 2.6 *Let* h, k, f *and* g *be four self-maps of a metric space* $(\mathcal{X}, d)$ *satisfying the following conditions:*
(1) *the pairs* $\{h, f\}$ *and* $\{k, g\}$ *are owc,*
(2) *the inequality*

$$d^p(hx, ky) \geq \alpha d^p(fx, gy) \quad (15)$$

for all x, y *in* $\mathcal{X}$, *where* $\alpha > 1$ *and* p *is an integer such that* $p \geq 1$, *then,* h, k, f *and* g *have a unique common fixed point in* $\mathcal{X}$.

Proof This corollary is an interesting particular case of the previous corollary. We obtain the result by using (ii) in the above corollary with $\beta=\gamma=0$.

Now, another common fixed point result of integral type is established as follows:

Corollary 2.7 *Let h, k, f and g be maps from a metric space $(\mathcal{X},d)$ into itself. Suppose there exists a function $\Phi:(R^+)^6\to R^+$ such that*

$$\int_0^{\Phi(d(hx,ky),d(fx,gy),d(hx,fx),d(ky,gy),d(hx,gy),d(ky,fx))}\varphi(t)dt\geq 0, \tag{16}$$

for all x, $y\in\mathcal{X}$, where Φ satisfies property

$$\int_0^{\Phi(u,u,0,0,u,u)}\varphi(t)dt<0 \text{ implies } u=0 \tag{17}$$

with $\varphi:R^+\to R$ is a Lebesgue-integrable map which is summable. Suppose that h and f as well as k and g are owc, then, h, k, f and g have a unique common fixed point in $\mathcal{X}$.

Proof Since h and f as well as k and g are owc, there exist two points u and v in $\mathcal{X}$ such that $hu=fu$ and $hfu=fhu$; $kv=gv$ and $kgv=gkv$.
Suppose that $d(hu,kv)>0$. By inequality (16), we get

$$\int_0^{\Phi(d(hu,kv),d(fu,gv),d(hu,fu),d(kv,gv),d(hu,gv),d(kv,fu))}\varphi(t)dt$$

$$=\int_0^{\Phi(d(hu,kv),d(hu,kv),0,0,d(hu,kv),d(kv,hu))}\varphi(t)dt\geq 0,$$

which contradicts property (17), therefore, $d(hu,kv)=0$ which implies that $fu=hu=kv=gv$.
Now, suppose that $d(h^2u,hu)>0$. Using condition (16), we obtain

$$\int_0^{\Phi(d(hhu,kv),d(fhu,gv),d(hhu,fhu),d(kv,gv),d(hhu,gv),d(kv,fhu))}\varphi(t)dt$$

$$=\int_0^{\Phi(d(h^2u,hu),d(h^2u,hu),0,0,d(h^2u,hu),d(hu,h^2u))}\varphi(t)dt\geq 0,$$

which contradicts property (17), hence, $d(h^2u, hu) = 0$ which implies that $fhu = hhu = hu$.
Similarly, $khu = ghu = hu$. Thus, $hu = kv$ is a common fixed point of h, k, f and g. The uniqueness of the common fixed point follows easily from inequality (16) and property (17).

finally using the recurrence on n, we get the next result

Corollary 2.8 *Let f, g, $\{h_n\}_{n\in N^*}$, where $N^* = \{1, 2, \ldots\}$ be self-maps of a metric space $(\mathcal{X}, d)$. Suppose there exists a function $\Phi : (R^+)^6 \to R^+$ such that*

$$\int_0^{\Phi(d(h_nx,h_{n+1}y),d(fx,gy),d(h_nx,fx),d(h_{n+1}y,gy),d(h_nx,gy),d(h_{n+1}y,fx))} \varphi(t)dt \geq 0, \qquad (18)$$

for all x, $y \in \mathcal{X}$, where Φ satisfies property (17) and φ is as in the above corollary. If the pairs (h_n, f) and (h_{n+1}, g) are owc, then, f, g and $\{h_n\}_{n\in N^*}$ have a unique common fixed point in $\mathcal{X}$.

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