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A Generalization of The Banach Contraction Principle of Presic Type For Three Maps

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Abstract

In this paper we obtain a Presic type unique common fixed point theorem for three maps and obtain the main theorem of Ciric and Presic as corollary.
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1 Introduction

In 1932, Banach [1] proved the following theorem

Theorem 1.1 Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be satisfying $d(Tx, Ty) \leq \alpha \ d(x, y)$ for all $x, y \in X$, where $0 \leq \alpha < 1$. Then T has a unique fixed point in X.

Consider the k-th order nonlinear difference equation

(A) $x_{n+k} = f(x_n, ..., x_{n+k-1}), n \in N$

with the initial values $x_0, x_1, ..., x_k \in X$, where (X, d) is a metric space, $k \in N, k \ge 1$ and $f: X^k \longrightarrow X$.

Equation (A) can be studied by means of a fixed point theory in view of the fact that $x^* \in X$ is a solution of (A) if and only if x^* is a fixed point of f, that is, $x^* = f(x^*, ..., x^*)$.

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One of the most important results on this direction has been obtained by S.B.Presic in [3] by generalizing the Banach contraction mapping principle.

Theorem 1.2 ([3]). Let (X, d) be a complete metric space, k a positive integer and $T: X^k \longrightarrow X$ a mapping satisfying the following contractive type condition

$$(1.2.1)d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1}))$$

$$\leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \dots + q_k d(x_k, x_{k+1})$$

for every $x_1, x_2, x_3, ..., x_k, x_{k+1}$ in X, where $q_1, q_2, ..., q_k$ are non -negative constants such that $q_1 + q_2 + ... + q_k < 1$.

Then there exists a unique point x in X such that T(x, x, ..., x) = x.

Moreover, if $x_1, x_2, ..., x_k$ are arbitrary points in X and for $n \in N$,

 $x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1})$ then the sequence $\{x_n\}$ is convergent and $limx_n = T(limx_n, limx_n, ..., limx_n)$.

Ciric and Presic [2] generalized Theorem 1.2 as follows:

Theorem 1.3 Let (X, d) be a complete metric space, k a positive integer and T: $X^k \longrightarrow X$ a mapping satisfying the following contractive type condition

$$(1.3.1)d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1}))$$

$$\leq \lambda \ max\{d(x_i, x_{i+1})/1 \leq i \leq k\}$$

for every $x_1, x_2, x_3, ..., x_k, x_{k+1}$ in X, where $\lambda \in (0, 1)$ is constant.

Then there exists a point x in X such that T(x, x, x, ..., x) = x.

Moreover, if $x_1, x_2, ..., x_k$ are arbitrary points in X and for $n \in N$,

 $x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1})$ then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, ..., \lim x_n)$.

If in addition, we suppose that on diagonal $\Delta \subset X^k$,

(1.3.2)d(T(u, u, ..., u), T(v, v, ..., v)) < d(u, v) holds for all $u, v \in X$, with $u \neq v$,

then x is the unique point in X with T(x, x, x, ..., x) = x.

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Now in this paper we extend and generalize the above theorems for three maps .

Definition 1.4 Let X be a non empty set and $T: X^{2k} \longrightarrow X, f: X \longrightarrow X$. (f,T) is said to be 2k- weakly compatible pair, if f(T(p, p, ..., p)) = T(fp, fp, ..., fp) whenever $p \in X$ such that fp = T(p, p, ..., p).

2 Main Theorem

Theorem 2.1 Let (X, d) be a metric space, k a positive integer and $S, T : X^{2k} \longrightarrow X, f : X \longrightarrow X$ be mappings satisfying

$$(2.1.1)d(S(x_1, x_2, \dots, x_{2k-1}, x_{2k}), T(x_2, x_3, \dots, x_{2k}, x_{2k+1}))$$

$$\leq \lambda \ max\{d(fx_i, fx_{i+1})/1 \leq i \leq 2k\}$$

for all $x_1, x_2, x_3, ..., x_{2k}, x_{2k+1}$ in X,

$$(2.1.2)d(T(y_1, y_2, \dots, y_{2k-1}, y_{2k}), S(y_2, y_3, \dots, y_{2k}, y_{2k+1}))$$

$$\leq \lambda \ max\{d(fy_i, fy_{i+1})/1 \leq i \leq 2k\}$$

for all $y_1, y_2, y_3, ..., y_{2k}, y_{2k+1}$ in X, where $0 \le \lambda < 1$.

 $(2.1.3) \ d(S(u, u, ..., u), T(v, v, ..., v)) < d(fu, fv) \ \forall u, v \in X \ with \ u \neq v,$

(2.1.4) Suppose that f(X) is complete and either (f, S) or (f, T) is 2k-weakly compatible pair.

Then there exists a unique point $p \in X$ such that fp = p = S(p, p, ..., p) = T(p, p, ..., p).

Proof. Suppose $x_1, x_2, ..., x_{2k}$ are arbitrary points in X and for $n \in N$, define $fx_{2k+2n-1} = S(x_{2n-1}, x_{2n}, x_{2n+1}, ..., x_{2n+2k-2})$ and $fx_{2k+2n} = T(x_{2n}, x_{2n+1}, x_{2n+2}, ..., x_{2n+2k-1})$. Let $\alpha_n = d(fx_n, fx_{n+1})$. Let $\theta = \lambda^{1/2k}$ and $K = max\{\alpha_1/\theta^1, \alpha_2/\theta^2, ..., \alpha_{2k}/\theta^{2k}\}$. Claim: $\alpha_n \leq K\theta^n$ for all $n \in N$ (2.1.5)

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By selection of K we have $\alpha_n \leq K\theta^n$ for n = 1, 2, ..., 2k.

$$Now \ \alpha_{2k+1} = d(fx_{2k+1}, fx_{2k+2}) \\ = d(S(x_1, x_2, ..., x_{2k-1}, x_{2k}), T(x_2, x_3, ..., x_{2k}, x_{2k+1})) \\ \le \lambda max\{d(fx_i, fx_{i+1}) : i = 1, 2, ..., 2k\} \ by(2.1.1) \\ = \lambda max\{\alpha_1, \alpha_2, ..., \alpha_{2k-1}, \alpha_{2k}\} \\ \le \lambda max\{K\theta^1, K\theta^2, ..., K\theta^{2k-1}, K\theta^{2k}\} \\ = \lambda K\theta = \theta^{2k} K\theta \ as \ \theta = \lambda^{1/2k} \\ = K\theta^{2k+1}$$

Thus $\alpha_{2k+1} \leq K\theta^{2k+1}$. Similarly

$$\begin{aligned} \alpha_{2k+2} &= d(fx_{2k+2}, fx_{2k+3}) \\ &= d(T(x_2, x_3, \dots, x_{2k}, x_{2k+1}), S(x_3, x_4, \dots, x_{2k+1}, x_{2k+2})) \\ &\leq \lambda max\{d(fx_i, fx_{i+1}) : i = 2, 3, \dots, 2k+1\} \ by(2.1.2) \\ &= \lambda max\{\alpha_i : i = 2, 3, \dots, 2k+1\} \\ &\leq \lambda max\{K\theta^2, K\theta^3, \dots, K\theta^{2k}, K\theta^{2k+1}\} \\ &= \lambda K\theta^2 = \theta^{2k}K\theta^2 \ as \ \theta = \lambda^{1/2k} \\ &= K\theta^{2k+2} \end{aligned}$$

Thus $\alpha_{2k+2} \leq K\theta^{2k+2}$.

Hence the claim is true .

Now, by claim, for any $n, p \in N$ we have

$$\begin{aligned} d(fx_n, fx_{n+p}) &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \dots + d(fx_{n+p-1}, fx_{n+p}) \\ &= \alpha_n + \alpha_{n+1} + \dots + \alpha_{n+p-1} \\ &\leq K\theta^n + K\theta^{n+1} + \dots + K\theta^{n+p-1} \\ &\leq K[\theta^n + \theta^{n+1} + \dots + \theta^{n+p-1} + \dots] \\ &= K\theta^n/1 - \theta \longrightarrow 0 \quad as \quad n \longrightarrow \infty \end{aligned}$$

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Hence $\{fx_n\}$ is a Cauchy sequence. Since f(X) is a complete, there exists z in f(X) such that $z = \lim fx_n$.

There exists $p \in X$ such that z = fp.

Then for any integer n, using (2.1.1) and (2.1.2), we have

$$\begin{split} &d(S(p, p, ..., p), fx_{2n+2k-1}) \\ &= d(S(p, p, ..., p), S(x_{2n-1}, x_{2n}, ..., x_{2n+2k-2})) \\ &\leq d(S(p, p, ..., p), T(p, p, ..., x_{2n-1})) + d(T(p, p, ..., x_{2n-1}), S(p, p, ..., p, x_{2n-1}, x_{2n})) \\ &+ d(S(p, p, ..., x_{2n-1}, x_{2n}), T(p, p, ..., p, x_{2n-1}, x_{2n}, x_{2n+1})) \\ &+ d(T(p, p, ..., p, x_{2n-1}, x_{2n}, x_{2n+1}), S(p, p, ..., p, x_{2n-1}, x_{2n}, x_{2n+1}, x_{2n+2})) \\ &+ ... + d(S(p, p, x_{2n-1}, x_{2n}, ..., x_{2n+2k-4}), T(p, x_{2n-1}, x_{2n}, ..., x_{2n+2k-3}, x_{2n+2k-3})) \\ &+ d(T(p, x_{2n-1}, x_{2n}, ..., x_{2n+2k-4}, x_{2n+2k-3}), S(x_{2n-1}, x_{2n}, ..., x_{2n+2k-3}, x_{2n+2k-2})) \\ &\leq \lambda d(fp, fx_{2n-1}) + \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}) \} \\ &+ \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), d(fx_{2n}, fx_{2n+1}), d(fx_{2n+1}, fx_{2n+2}) \} \\ &+ ... \\ &+ \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-3}, fx_{2n+2k-3}) \} \\ &+ \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-3}, fx_{2n+2k-3}) \} \\ &+ \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-3}, fx_{2n+2k-3}) \} \\ &+ \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-3}, fx_{2n+2k-3}) \} \\ &+ \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-3}, fx_{2n+2k-3}) \} \\ &+ \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-3}, fx_{2n+2k-3}) \} \\ &+ \lambda max \{ d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-3}, fx_{2n+2k-2}) \} \\ \end{array}$$

$$\begin{split} S(p,p,...,p) &= fp & (i).\\ \text{Consider } d(fp,T(p,p,...,p)) &= d(S(p,p,...,p),T(p,p,...,p)) \leq \lambda(0) = 0\\ \text{Thus } T(p,p,...,p) &= fp & (ii). \end{split}$$

Now suppose that (f, S) is 2k-weakly compatible pair. Then we have f(S(p, p, ..., p)) = S(fp, fp, ..., fp). $f^2p = f(fp) = f(S(p, p, ..., p)) = S(fp, fp, ..., fp)$. Suppose $fp \neq p$. Then from (2.1.3) ,we have $d(f^2p, fp) = d(S(fp, fp, ..., fp), T(p, p, ..., p)) < d(f^2p, fp)$. It is a contradiction. Therefore fp = p. Now from (i) and (ii), we have fp = p = S(p, p, ..., p) = T(p, p, ..., p). Uniqueness of p: Suppose there exists a point $q \neq p$ in X such that 278

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$$fq = q = S(q, q, q, ..., q) = T(q, q, q, ..., q).$$

Consider $d(fp, fq) = d(S(p, p, ..., p), d(T(q, q, ..., q)) < d(fp, fq)$ from (2.1.3)
It is a contradiction. Therefore $q = p$.

When S = T and 2k is replaced by k in Theorem 2.1, we get the following .

Corollary 2.2 Let (X,d) be a metric space, k a positive integer and $T: X^k \longrightarrow X, f: X \longrightarrow X$ be mappings satisfying

$$(2.2.1)d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1}))$$

$$\leq \lambda \ max\{d(fx_i, fx_{i+1})/1 \leq i \leq k\}$$

for every $x_1, x_2, x_3, ..., x_k, x_{k+1}$ in X, where $\lambda \in (0, 1)$

 $(2.2.2)d(T(u, u, ..., u), T(v, v, ..., v)) < d(fu, fv) \; \forall \; u, v \in X \; \textit{ with } u \neq v,$

(2.2.3) Suppose that f(X) is complete and (f,T) is k-weakly compatible pair.

Then there exists a unique point $p \in X$ such that fp = p = T(p, p, ..., p, p).

Remark : If f = I (Identity map) in Corollary (2.2), we get the main theorem of Ciric and Presic [2].

3 Conclusion

In this paper, we can obtain an iterative method for solution of simultaneous nonlinear difference equations f(x) = S(x, x, ..., x) = x and f(x) = T(x, x, ..., x) = x using Theorem 2.1. Also we obtain the main theorem of Ciric and Presic [2] as a corollary.

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