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A Generalization of The Banach Contraction Principle of Presic Type For Three Maps

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Abstract

In this paper we obtain a Presic type unique common fixed point theorem for three maps and obtain the main theorem of Ciric and Presic as corollary.

Keywords: Weakly compatible pair,complete metric space,unique point.

1 Introduction

In 1932, Banach [1] proved the following theorem

Theorem 1.1 *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be satisfying $d(Tx,Ty) \leq \alpha\ d(x,y)$ for all $x, y \in X$, where $0 \leq \alpha < 1$. Then T has a unique fixed point in X .*

Consider the k-th order nonlinear difference equation

(A) $x_{n+k} = f(x_n, ..., x_{n+k-1}), n \in N$

with the initial values $x_0, x_1, ..., x_k \in X$, where (X,d) is a metric space, $k \in N, k \geq 1$ and $f : X^k \longrightarrow X$.

Equation (A) can be studied by means of a fixed point theory in view of the fact that $x^* \in X$ is a solution of (A) if and only if x^* is a fixed point of f,that is, $x^* = f(x^*, ..., x^*)$.

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One of the most important results on this direction has been obtained by S.B.Presic in [3] by generalizing the Banach contraction mapping principle.

Theorem 1.2 *([3]). Let (X, d) be a complete metric space, k a positive integer and $T : X^k \longrightarrow X$ a mapping satisfying the following contractive type condition*

$$(1.2.1) d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + ... + q_k d(x_k, x_{k+1})$$

for every $x_1, x_2, x_3, ..., x_k, x_{k+1}$ in X, where $q_1, q_2, ..., q_k$ are non -negative constants such that $q_1 + q_2 + ... + q_k < 1$.
Then there exists a unique point x in X such that $T(x, x, ..., x) = x$.
Moreover, if $x_1, x_2, ..., x_k$ are arbitrary points in X and for $n \in N$,
$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1})$ then the sequence $\{x_n\}$ is convergent and $lim x_n = T(lim x_n, lim x_n, ..., lim x_n)$.

Ciric and Presic [2] generalized Theorem 1.2 as follows:

Theorem 1.3 *Let (X, d) be a complete metric space , k a positive integer and $T : X^k \longrightarrow X$ a mapping satisfying the following contractive type condition*

$$(1.3.1) d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq \lambda \ max\{d(x_i, x_{i+1})/1 \leq i \leq k\}$$

for every $x_1, x_2, x_3, ..., x_k, x_{k+1}$ in X, where $\lambda \in (0, 1)$ is constant .
Then there exists a point x in X such that $T(x, x, x, ..., x) = x$.
Moreover, if $x_1, x_2, ..., x_k$ are arbitrary points in X and for $n \in N$,
$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1})$ then the sequence $\{x_n\}$ is convergent and $lim x_n = T(lim x_n, lim x_n, ..., lim x_n)$.
If in addition, we suppose that on diagonal $\Delta \subset X^k$,
$(1.3.2) d(T(u, u, ..., u), T(v, v, ..., v)) < d(u, v)$ holds for all $u, v \in X$, with $u \neq v$,
then x is the unique point in X with $T(x, x, x, ..., x) = x$.

Now in this paper we extend and generalize the above theorems for three maps .

Definition 1.4 *Let* X *be a non empty set and* $T : X^{2k} \longrightarrow X, f : X \longrightarrow X$. (f, T) *is said to be* $2k$*- weakly compatible pair, if* $f(T(p, p, ..., p)) = T(fp, fp, ..., fp)$ *whenever* $p \in X$ *such that* $fp = T(p, p, ..., p)$.

2 Main Theorem

Theorem 2.1 *Let* (X, d) *be a metric space,*k *a positive integer and* $S, T : X^{2k} \longrightarrow X, f : X \longrightarrow X$ *be mappings satisfying*

$$(2.1.1) d(S(x_1, x_2, ..., x_{2k-1}, x_{2k}), T(x_2, x_3, ..., x_{2k}, x_{2k+1})) \\ \leq \lambda \ max\{d(fx_i, fx_{i+1})/1 \leq i \leq 2k\}$$

for all $x_1, x_2, x_3, ..., x_{2k}, x_{2k+1}$ *in* X*,*

$$(2.1.2) d(T(y_1, y_2, ..., y_{2k-1}, y_{2k}), S(y_2, y_3, ..., y_{2k}, y_{2k+1})) \\ \leq \lambda \ max\{d(fy_i, fy_{i+1})/1 \leq i \leq 2k\}$$

for all $y_1, y_2, y_3, ..., y_{2k}, y_{2k+1}$ *in* X*, where* $0 \leq \lambda < 1$*.*
(2.1.3) $d(S(u, u, ..., u), T(v, v, ..., v)) < d(fu, fv) \ \forall \, u, v \in X$ *with* $u \neq v$*,*
(2.1.4) *Suppose that* $f(X)$ *is complete and either* (f, S) *or* (f, T) *is* $2k$*-weakly compatible pair.*
Then there exists a unique point $p \in X$ *such that* $fp = p = S(p, p, ..., p) = T(p, p, ..., p)$*.*

Proof. Suppose $x_1, x_2, ..., x_{2k}$ are arbitrary points in X and for $n \in N$,define
$fx_{2k+2n-1} = S(x_{2n-1}, x_{2n}, x_{2n+1}, ..., x_{2n+2k-2})$ and
$fx_{2k+2n} = T(x_{2n}, x_{2n+1}, x_{2n+2}, ..., x_{2n+2k-1})$.
Let $\alpha_n = d(fx_n, fx_{n+1})$. Let $\theta = \lambda^{1/2k}$ and $K = max\{\alpha_1/\theta^1, \alpha_2/\theta^2, ..., \alpha_{2k}/\theta^{2k}\}$.
Claim: $\alpha_n \leq K\theta^n$ for all $n \in N$ (2.1.5)

By selection of K we have $\alpha_n \leq K\theta^n$ for $n = 1, 2, ..., 2k$.

$$\begin{aligned} Now\ \ \alpha_{2k+1} &= d(fx_{2k+1}, fx_{2k+2}) \\ &= d(S(x_1, x_2, ..., x_{2k-1}, x_{2k}), T(x_2, x_3, ..., x_{2k}, x_{2k+1})) \\ &\leq \lambda max\{d(fx_i, fx_{i+1}) : i = 1, 2, ..., 2k\}\ \ by (2.1.1) \\ &= \lambda max\{\alpha_1, \alpha_2, ..., \alpha_{2k-1}, \alpha_{2k}\} \\ &\leq \lambda max\{K\theta^1, K\theta^2, ..., K\theta^{2k-1}, K\theta^{2k}\} \\ &= \lambda K\theta = \theta^{2k} K\theta\ \ as\ \ \theta = \lambda^{1/2k} \\ &= K\theta^{2k+1} \end{aligned}$$

Thus $\alpha_{2k+1} \leq K\theta^{2k+1}$.

Similarly

$$\begin{aligned} \alpha_{2k+2} &= d(fx_{2k+2}, fx_{2k+3}) \\ &= d(T(x_2, x_3, ..., x_{2k}, x_{2k+1}), S(x_3, x_4, ..., x_{2k+1}, x_{2k+2})) \\ &\leq \lambda max\{d(fx_i, fx_{i+1}) : i = 2, 3, ..., 2k+1\}\ \ by (2.1.2) \\ &= \lambda max\{\alpha_i : i = 2, 3, ..., 2k+1\} \\ &\leq \lambda max\{K\theta^2, K\theta^3, ..., K\theta^{2k}, K\theta^{2k+1}\} \\ &= \lambda K\theta^2 = \theta^{2k} K\theta^2\ \ \ as\ \ \ \theta = \lambda^{1/2k} \\ &= K\theta^{2k+2} \end{aligned}$$

Thus $\alpha_{2k+2} \leq K\theta^{2k+2}$.

Hence the claim is true .

Now, by claim, for any $n, p \in N$ we have

$$\begin{aligned} d(fx_n, fx_{n+p}) &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + ... + d(fx_{n+p-1}, fx_{n+p}) \\ &= \alpha_n + \alpha_{n+1} + ... + \alpha_{n+p-1} \\ &\leq K\theta^n + K\theta^{n+1} + ... + K\theta^{n+p-1} \\ &\leq K[\theta^n + \theta^{n+1} + ... + \theta^{n+p-1} + ...] \\ &= K\theta^n/1 - \theta \longrightarrow 0\ \ \ as\ \ \ n \longrightarrow \infty \end{aligned}$$

Hence $\{fx_n\}$ is a Cauchy sequence . Since $f(X)$ is a complete , there exists z in $f(X)$ such that $z = lim fx_n$.

There exists $p \in X$ such that $z = fp$.

Then for any integer n, using (2.1.1) and (2.1.2),we have

$$\begin{aligned}
& d(S(p,p,...,p), fx_{2n+2k-1}) \\
& = d(S(p,p,...,p), S(x_{2n-1}, x_{2n}, ..., x_{2n+2k-2})) \\
& \leq d(S(p,p,...,p), T(p,p,...,x_{2n-1})) + d(T(p,p,...,x_{2n-1}), S(p,p,...,p,x_{2n-1},x_{2n})) \\
& \quad + d(S(p,p,...,x_{2n-1},x_{2n}), T(p,p,...,p,x_{2n-1},x_{2n},x_{2n+1})) \\
& \quad + d(T(p,p,...,p,x_{2n-1},x_{2n},x_{2n+1}), S(p,p,...,p,x_{2n-1},x_{2n},x_{2n+1},x_{2n+2})) \\
& \quad + ... + d(S(p,p,x_{2n-1},x_{2n},...,x_{2n+2k-4}), T(p,x_{2n-1},x_{2n},...,x_{2n+2k-4},x_{2n+2k-3})) \\
& \quad + d(T(p,x_{2n-1},x_{2n},...,x_{2n+2k-4},x_{2n+2k-3}), S(x_{2n-1},x_{2n},...,x_{2n+2k-3},x_{2n+2k-2})) \\
& \leq \lambda d(fp, fx_{2n-1}) + \lambda max\{d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n})\} \\
& \quad + \lambda max\{d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), d(fx_{2n}, fx_{2n+1})\} \\
& \quad + \lambda max\{d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), d(fx_{2n}, fx_{2n+1}), d(fx_{2n+1}, fx_{2n+2})\} \\
& \quad + ... \\
& \quad + \lambda max\{d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-4}, fx_{2n+2k-3})\} \\
& \quad + \lambda max\{d(fp, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{2n+2k-3}, fx_{2n+2k-2})\}
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we get $d(S(p,p,...,p), fp) \leq 0$ so that

$S(p,p,...,p) = fp$ (i).

Consider $d(fp, T(p,p,...,p)) = d(S(p,p,...,p), T(p,p,...,p)) \leq \lambda(0) = 0$

Thus $T(p,p,...,p) = fp$ (ii).

Now suppose that (f, S) is $2k$-weakly compatible pair. Then we have $f(S(p,p,...,p)) = S(fp, fp, ..., fp)$.

$f^2p = f(fp) = f(S(p,p,...,p)) = S(fp, fp, ..., fp)$.

Suppose $fp \neq p$. Then from (2.1.3) ,we have

$d(f^2p, fp) = d(S(fp, fp, ..., fp), T(p,p,...,p)) < d(f^2p, fp)$. It is a contradiction.

Therefore $fp = p$. Now from (i) and (ii), we have $fp = p = S(p,p,...,p) = T(p,p,...,p)$.

Uniqueness of p : Suppose there exists a point $q \neq p$ in X such that

$fq = q = S(q, q, q, ..., q) = T(q, q, q, ..., q)$.

Consider $d(fp, fq) = d(S(p, p, ..., p), d(T(q, q, ..., q)) < d(fp, fq)$ from (2.1.3)

It is a contradiction. Therefore $q = p$.

When $S = T$ and $2k$ is replaced by k in Theorem 2.1, we get the following .

Corollary 2.2 *Let (X, d) be a metric space, k a positive integer and $T : X^k \longrightarrow X, f : X \longrightarrow X$ be mappings satisfying*

$$(2.2.1) d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq \lambda \ max\{d(fx_i, fx_{i+1})/1 \leq i \leq k\}$$

for every $x_1, x_2, x_3, ..., x_k, x_{k+1}$ in X, where $\lambda \in (0, 1)$

$(2.2.2) d(T(u, u, ..., u), T(v, v, ..., v)) < d(fu, fv) \ \forall \ u, v \in X$ *with* $u \neq v$,

(2.2.3) *Suppose that $f(X)$ is complete and (f, T) is k-weakly compatible pair.*

Then there exists a unique point $p \in X$ such that $fp = p = T(p, p, ..., p, p)$.

Remark : If $f = I$ (Identity map) in Corollary (2.2), we get the main theorem of Ciric and Presic [2].

3 Conclusion

In this paper , we can obtain an iterative method for solution of simultaneous nonlinear difference equations $f(x) = S(x, x, ..., x) = x$ and $f(x) = T(x, x, ..., x) = x$ using Theorem 2.1. Also we obtain the main theorem of Ciric and Presic [2] as a corollary.

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