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Characterization of Uniform Fréchet Algebras in which every Element is the Square of Another

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Abstract

Let X be a hemicompact k-space and A be a uniform Fréchet algebra on X. In this note we first show that if each element of a dense subset of A has square root in A then $A = C(X)$ under certain condition. Then we show that $G(C(X))$, the group of invertible elements of $C(X)$, is dense in $C(X)$ if and only if $\dim X$, the covering dimension of X, does not exceed 1. Using this result we give a necessary and sufficient condition under which each continuous function on X is the square of another.

Keywords: Hemicompact space, k-space, Uniform algebra, Uniform Fréchet algebra, Projective limit, Topological dimension, Invertible group.

1 Introduction

Let X be a compact Hausdorff space. By a uniform (Banach) algebra on X we mean a complete subalgebra of $C(X)$ which contains the constants and separates the points of X, where $C(X)$ is endowed with the supremum norm. In 1966 Čirka [6, Theorem 13.15] proved the following theorem.

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Theorem 1.1. *Let X be a locally connected, compact Hausdorff space, and let A be a uniform algebra on X. If the condition $(*)$ below holds for A then $A = C(X)$.*

$$for\ each\ f \in A,\ there\ exists\ g \in A\ with\ g^2 = f. \quad (*)$$

In fact, it is possible to weaken slightly the hypothesis of the theorem: The theorem is still true if we assume only that for a dense subset of f's in A, $f = g^2$, for some $g \in A$. This is the content of the next lemma which is again due to Čirka.

Lemma 1.2. *[6, Lemma 13.16] Let X be a locally connected compact Hausdorff space, and let F and G be subsets of $C(X)$. If every $f \in F$ is of the form $f = g^2$ for some $g \in G$, then every $f \in \overline{F}$ is of the form $f = g^2$ for some $g \in \overline{G}$.*

For a locally connected, compact, Hausdorff space, O. Hatori and T. Miura gave a complete characterization of condition $(*)$ for $C(X)$:

Theorem 1.3. *[3, Theorem 2.2] Let X be a locally connected compact Hausdorff space. Then the following are equivalent.*

$(*)$ *For every $f \in C(X)$ there exists a $g \in C(X)$ such that $f = g^2$.*

$(**)$ *The first Čech cohomology group of X with integer coefficients $\check{H}^1(X, \mathbb{Z})$ is trivial and $dimX \leq 1$.*

In this work we first extend some results of Čirka to uniform Fréchet algebras and then we characterize those algebras of continuous functions on a hemicompact k-space X, which have a dense subset of invertible groups. Using this result we give a necessary and sufficient condition under which each element of $C(X)$ has square root.

We now present some definitions and known results. For further details one can refer, for example, to [1] or [2]

Definition 1.4. *A Hausdorff space X is hemicompact if there exists a sequence (K_n) of compact subsets of X such that $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$, and each compact subset*

K of X is contained in some K_n. We call such sequence (K_n) an admissible exhaustion of X. A Hausdorff space is called a k-space if every subset intersecting each compact subset in a closed set is itself closed.

Note that a complex-valued function f on a k-space X is continuous if and only if it is continuous on each compact subset of X.

Definition 1.5. *A Fréchet algebra A is an LMC-algebra (locally multiplicatively convex algebra) which is also a complete metrizable space, so that the topology of a Fréchet algebra A can be defined by an increasing sequence (p_n) of submultiplicative seminorms. Without loss of generality, we may assume that for each $n \in \mathbb{N}$, $p_n(1) = 1$, if A has unit. A uniform Fréchet algebra (uF-algebra) is a Fréchet algebra A with a defining sequence (p_n) of seminorms such that $p_n(f^2) = (p_n(f))^2$ for all $f \in A$ and $n \in \mathbb{N}$ [2].*

Theorem 1.6. *[2, Theorem 4.1.3]*
The following statements are equivalent for a unital commutative algebra A:

i) A is a uF-algebra.

ii) A is the projective limit of a dense projective sequence of uniform (Banach) algebras.

iii) There is a hemicompact space X such that A is topologically and algebraically isomorphic to a point separating and complete subalgebra of $C(X)$ (endowed with the compact - open topology) which contains the constants.

By the above theorem we can consider each uF-algebra A, as a point separating and complete subalgebra of $C(X)$ (endowed with the compact - open topology) which contains the constants, where X is a hemicompact space. Also, if (X_n) is an admissible exhaustion of X, it is known that A is the projective limit of the dense projective system $\{A_{X_n}; r_n\}$ of uniform Banach algebras where, for each $n \in \mathbb{N}$, A_{X_n} is the completion of the algebra $A|_{X_n}$ with respect to the norm $\|.\|_{X_n}$, and $r_n : A_{X_{n+1}} \longrightarrow A_{X_n}$, $f \mapsto f|_{X_n}$ is the restriction mapping. Also it is known that when X is a hemicompact k-space, $C(X)$ is a Fréchet algebra with respect to the compact - open topology and if (X_n) is an admissible exhaustion of X then we have $C(X) = \varprojlim C(X_n)$.

For a topological space X, let $\exp(C(X))$ contain those elements of $G(C(X))$ which are of exponential type.

Theorem 1.7. *[4, Theorem 2.3] Let X be a hemicompact, k-space with an admissible exhaustion (X_n) such that each X_n has finitely many components. If $f \in C(X)$ and $f|_{X_n} \in \exp(C(X_n))$, for all $n \in \mathbb{N}$, then $f \in \exp(C(X))$.*

Let X be a topological space, and let S^1 be the unit circle in the plane. The homotopy classes of maps $X \longrightarrow S^1$, which is denoted by $\pi^1(X)$, is called the first cohomotopy group of X.

Theorem 1.8. *[4, Theorem 3.1] Let X be a hemicompact, k-space with an admissible exhaustion (X_n) such that each X_n has finitely many components. Then $\pi^1(X)$ is isomorphic with $H^1(C(X)) = G(C(X))/\exp(C(X))$.*

Definition 1.9. *An open covering of a topological space X is a family $\Sigma = \{A_\lambda\}_{\lambda\in\Lambda}$ of open subsets of X such that $\bigcup_{\lambda\in\Lambda} A_\lambda = X$. An open covering $\{B_\gamma\}_{\gamma\in\Gamma}$ is said to be an open refinement of a covering $\{A_\lambda\}_{\lambda\in\Lambda}$ if for each $\gamma \in \Gamma$ there exists some $\lambda \in \Lambda$ such that $B_\gamma \subseteq A_\lambda$.*

Definition 1.10. *The covering dimension $\dim X$ of a topological space X is the least integer n such that every finite open covering of X has an open refinement of order not exceeding n or ∞, if there is no such integer.*

The following theorem is a useful criterion for characterizing the covering dimension of normal spaces.

Theorem 1.11. *[5, Theorem 3.2.2] If X is a normal space, then $\dim X \leq n$ if and* *only if for each closed subset A of X, each continuous function $f : A \longrightarrow S^n$ has an extension $g : X \longrightarrow S^n$, where S^n is the n-sphere.*

2 Main Results

Theorem 2.1. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that for each $n \in \mathbb{N}$, X_n is locally connected. If A is a uF-algebra on X such that $A^2 = A$ or in other words, if for each $f \in A$, there is $g \in A$ with $f = g^2$, then $A = C(X)$.*

Proof. Let $n \in \mathbb{N}$. By hypothesis for each $f \in A|_{X_n}$ there exists $g \in A|_{X_n}$ such that $f = g^2$, so by [6, Lemma 13.16], for each $f \in A_{X_n}$ there is $g \in A_{X_n}$ such that $f = g^2$. Now [6, Theorem 13.15], shows that $A_{X_n} = C(X_n)$ and so we have

$$A = \varprojlim A_{X_n} = \varprojlim C(X_n) = C(X).$$

□

Now we show that the above theorem is still valid if we assume only that for a dense subset of f's in A, $f = g^2$ for some $g \in A$.

Theorem 2.2. *Let X be a hemicompact k-space with an admissible exhaustion (X_n), such that for each $n \in \mathbb{N}$, X_n is locally connected. Let F and E be subsets of $C(X)$. If for each $f \in F$ there is $\phi \in E$ with $f = \phi^2$, then for each $f \in \overline{F}$ there is $\phi \in \overline{E}$ such that $f = \phi^2$.*

Proof. Let $f \in \overline{F}$. There exists a sequence $\{f_k\}$ in F such that $f_k \longrightarrow f$ as $k \longrightarrow \infty$. By the hypothesis, there is a sequence $\{g_k\}$ in E such that $f_k = g_k^2$ for each $k \in \mathbb{N}$. We will construct the subsequence $\{g_{k_n}\}$ of $\{g_k\}$ and a sequence $\{\phi_n\}$ in $C(X_n)$, by the following argument.

Since $g_k^2 \longrightarrow f$ in $C(X)$, we have $\|g_k^2 - f\|_{X_1} \longrightarrow 0$, whenever $k \longrightarrow \infty$. In a way similar to the proof of [6, Theorem 13.15], we can show that there is a subsequence

$\{g_k^{(1)}\}$ of $\{g_k\}$ which is uniformly Cauchy on X_1 and hence converges to a function $\phi_1 \in C(X_1)$. By the compactness of X_1, $(g_k^{(1)})^2 \longrightarrow \phi_1^2$ on X_1, whenever $k \longrightarrow \infty$. Hence, $\phi_1^2 = f$ on X_1. Let $g_{k_1} \in \{g_k^{(1)} : k \in \mathbb{N}\}$ be such that $\|g_{k_1} - \phi_1\|_{X_1} < 1$. Since $(g_k^{(1)})^2 \longrightarrow f$ on X, we have $\|(g_k^{(1)})^2 - f\|_{X_2} \longrightarrow 0$, whenever $k \longrightarrow \infty$. As before we can find a subsequence $\{g_k^{(2)}\}$ of $\{g_k^{(1)}\}$ and a function $\phi_2 \in C(X_2)$ such that $g_k^{(2)} \longrightarrow \phi_2$ on X_2 and so $(g_k^{(2)})^2 \longrightarrow \phi_2^2$ on X_2. Since $(g_k^{(2)})^2 \longrightarrow f$ on X_2, it follows that $\phi_2^2 = f$ on X_2. On the other hand, $X_1 \subseteq X_2$, so $g_k^{(2)} \longrightarrow \phi_2$ on X_1 which implies $\phi_1 = \phi_2$ on X_1. Let $g_{k_2} \in \{g_k^{(2)}\}$ be such that $\|g_{k_2} - \phi_2\|_{X_2} < \frac{1}{2}$. Continuing in this way, we can find a subsequence $\{g_{k_n}\}$ of $\{g_k\}$ and a function $\phi_n \in C(X_n)$ satisfying the following conditions:

i) $\|g_{k_n} - \phi_n\|_{X_n} < \frac{1}{n}$.

ii) $\phi_n^2 = f$ on X_n .

iii) $\phi_{n+1} = \phi_n$ on X_n .

Now we define the function ϕ on X by $\phi = \phi_n$ on each X_n. By (iii) ϕ is well-defined and since X is a k-space, ϕ is continuous. Also by (ii) $\phi^2 = f$ on X. To prove that $\phi \in \overline{E}$ it is enough to show that $\{g_{k_n}\}$ tends to ϕ. Let $\varepsilon > 0$ and $s \in \mathbb{N}$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$ and $N \geq s$. For each $n \geq N$ we have

$$\|g_{k_n} - \phi\|_{X_s} \leq \|g_{k_n} - \phi\|_{X_n} = \|g_{k_n} - \phi_n\|_{X_n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

i.e., $g_{k_n} \longrightarrow \phi$ on X_s. Since $s \in \mathbb{N}$ was arbitrary, we have $g_{k_n} \longrightarrow \phi$ on X, which completes the proof. □

Corollary 2.3. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that for each $n \in \mathbb{N}$, X_n is locally connected. If each element of a dense subset of $C(X)$ has square root in $C(X)$ then every element of $C(X)$ has this property.*

Corollary 2.4. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that for each $n \in \mathbb{N}$, X_n is locally connected, and let the equalities*

$$G(C(X)) = \exp(C(X)) \ , \ \overline{G(C(X))} = C(X)$$

hold. Then for each $f \in C(X)$ there exists $g \in C(X)$ such that $f = g^2$.

Proof. By the hypothesis, for each $f \in G(C(X))$ there is $h \in C(X)$ such that $f = \exp h$. If we set $g = \exp \frac{h}{2}$ then $f = g^2$. Now by Corollary 2.3, the conclusion holds. □

In [3], authors proved that if X is a locally connected, compact Hausdorff space such that the condition $(*)$ holds for $C(X)$ then $G(C(X)) = \exp C(X)$ [3, Lemma 2.3]. Using this Lemma we get a similar result for hemicompact k-spaces.

Lemma 2.5. *Let X be a hemicompact k-space with an admissible exhaustion (X_n) such that for each $n \in \mathbb{N}$, X_n is locally connected. Suppose for each $f \in C(X)$, there is $g \in C(X)$ such that $f = g^2$. Then $G(C(X)) = \exp C(X)$.*

Proof. We first notice that since X is normal ([2, Remark 3.1.10]), each element of $C(X_n)$ has a continuous extension on X. By the hypothesis, for every $n \in \mathbb{N}$, X_n is locally connected so $G(C(X_n)) = \exp C(X_n)$ by [3, Lemma 2.3]. Now if $f \in G(C(X))$ then $f|_{X_n} \in \exp C(X_n)$, for each $n \in \mathbb{N}$, since X_n has finitely many component, the conclusion holds by [4, Teorem 2.3]. □

In the next theorem we make use of the Countable Sum Theorem which asserts that if X is a normal space and $X = \bigcup_{i\in\mathbb{N}} Y_i$, where each Y_i is closed in X and $\dim Y_i \leq n$, then $\dim X \leq n$ [5, Theorem 3.2.5].

Theorem 2.6. *Let X be a hemicompact k-space. Then $\overline{G(C(X))} = C(X)$ if and only if* $\dim X \leq 1$.

Proof. Assume that $\overline{G(C(X))} = C(X)$. Let $n \in \mathbb{N}$ and $f \in C(X_n)$. Since X is normal, f has a continuous extension $\widetilde{f}$ on X. By the hypothesis for each $\varepsilon > 0$ there exists $\widetilde{g} \in G(C(X))$ such that $\|\widetilde{g} - \widetilde{f}\|_{X_n} < \varepsilon$. If $g = \widetilde{g}|_{X_n}$, then $g \in G(C(X_n))$ and $\|g - f\|_{X_n} < \varepsilon$, i.e., $\overline{G(C(X_n))} = C(X_n)$ for each $n \in \mathbb{N}$. This implies that $\dim X_n \leq 1$. Now using the Countable Sum Theorem we have $\dim X \leq 1$.

Conversely, suppose that $\dim X \leq 1$ and $f \in C(X)$ is non-zero. We will find a sequence $\{f_n\}$ in $G(C(X))$ such that $f_n \longrightarrow f$ as $n \longrightarrow \infty$.

Let $K_n = \{x \in X : |f(x)| \leq \frac{1}{n}\}$ and $E_n = \overline{K_n^c} \cap X_n$. Note that since $f \neq 0$, there exists $s \in \mathbb{N}$ such that $E_n \neq \emptyset$ for all $n \geq s$. Let $n \geq 2$. We will construct a function $v_n \in C(X)$ such that $v_n|_{E_n} = |f|$, $v_n(X) \subseteq [\frac{1}{n}\ ,\ \|f\|_{X_n} + \frac{1}{n}]$ and $\frac{1}{n} \leq v_n \leq \frac{2}{n-1}$ on K_n. The procedure is as follows:

Since X is normal by Tietze's extension theorem there is $v \in C(X)$ such that $v|_{E_n} = |f|$ and $v(X) \subseteq [\frac{1}{n}\ ,\ \|f\|_{X_n}]$. Set $A = \{x \in K_n : v(x) \geq \frac{1}{n-1}\}$, and $B = \{x \in K_n : v(x) \leq \frac{1}{2}(\frac{1}{n-1} + \frac{1}{n})\}$. Clearly $A \cap (B \cup E_n) = \emptyset$. Also A and $B \cup E_n$ are closed subsets of X, hence Urysohn's lemma provides a function $g \in C(X)$ such that $g = 0$ on A , $g = 1$ on $B \cup E_n$ and $0 \leq g \leq 1$ elsewhere. So the function $gv \in C(X)$ has the following properties:

i) $gv = 0$ on A.

ii) $gv = |f|$ on E_n.

iii) $\frac{1}{n} \leq gv \leq \frac{1}{2}(\frac{1}{n} + \frac{1}{n-1})$ on B.

iv) $0 \leq gv \leq \frac{1}{n-1}$ on $C = \{x \in K_n : \frac{1}{2}(\frac{1}{n} + \frac{1}{n-1}) \leq v(x) \leq \frac{1}{n-1}\}$.

On the other hand, $A \cup C$ is closed and $(D \cup C) \cap E_n = \emptyset$, where $D = \{x \in X : g(x) = 0\}$. By applying Urysohn's lemma once more, we obtain a function $h \in C(X)$ such that $h = 0$ on E_n, $h = \frac{1}{n}$ on $D \cup C$ and $0 \leq h \leq \frac{1}{n}$ elsewhere. Let $v_n = gv + h$. It is easy to see that v_n has the desired properties.

Now since $\dim X \leq 1$ by Theorem 1.11, there is a function $w_n \in C(X)$ such that $w_n|_{E_n} = \frac{f|_{E_n}}{|f|_{E_n}|}$ and $w_n(X) \subseteq S^1$, where S^1 is the unit circle in the plane. Set $f_n = v_n w_n$. Obviously $f_n \in G(C(X))$. It remains to show that $f_n \longrightarrow f$ as $n \longrightarrow \infty$. Let $r \in \mathbb{N}$

and $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $\frac{3}{N-1} < \varepsilon$ and $N \geq r$. Let $n \geq N$. For each $x \in X_r$, if $|f(x)| \leq \frac{1}{n}$ then

$$|f(x) - f_n(x)| \leq |f(x)| + |v_n(x)w_n(x)| \leq \frac{1}{n} + \frac{2}{n-1} \leq \frac{3}{N-1} < \varepsilon.$$

And if $|f(x)| \geq \frac{1}{n}$ then $|f(x) - f_n(x)| = 0$, i.e., $\|f - f_n\|_{X_r} < \varepsilon$ for all $n \geq N$. This completes the proof of the theorem. □

Theorem 2.7. *Let X be a hemicompact k-space and (X_n) be an admissible exhaustion of X such that for each $n \in \mathbb{N}$, X_n is locally connected. Then the following conditions are equivalent:*

i) For each $f \in C(X)$ there is $g \in C(X)$ such that $f = g^2$.

ii) $\pi^1(X)$ is trivial and $\dim X \leq 1$.

Proof. Let (i) hold. By Lemma 2.5, we have $G(C(X)) = \exp C(X)$, so by [4, Teorem 3.1], $\pi^1(X)$ is trivial. On the other hand, since X is normal, (i) shows that for each $n \in \mathbb{N}$ and $f \in C(X_n)$, there is $g \in C(X_n)$ such that $f = g^2$. So [3, Teorem 2.2], shows that $\dim X_n \leq 1$. Hence by the Countable Sum Theorem we have $\dim X \leq 1$.

Now let (ii) hold. Since $\pi^1(X)$ is trivial we have $G(C(X)) = \exp(C(X))$ and since $\dim X \leq 1$ by Theorem 2.6, we have $\overline{G(C(X))} = C(X)$. Hence (i) follows from Corollary 2.4. □

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