*Mathematical Sciences* Vol. 3, No. 3 (2009) 291-300

# Approximation of double integrals using bivariate rational interpolation

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**Abstract**

In this paper a bivariate rational interpolation is developed to create a space surface using both function values and the first order partial derivatives of the function being interpolated as the interpolation data. This method has some characteristics comparing with present interpolation methods. First, the interpolation function has a simple and explicit mathematical representation. Secondly, the interpolation function can be expressed by symmetric bases. We use this method to estimate the value of double integrals. Presented examples show the pertinent features of the method.

**Keywords:** Rational interpolation, Spline interpolation, Numerical integration.



## 1 Introduction

Spline interpolation is a useful and powerful tool in surface design. The common spline interpolation is fix interpolation which mean that shape of the interpolating surface is fixed for the given interpolation data [1]. Then this question arises, "How can we modify the interpolating surface under the condition that the interpolation data are not changed?" This question was answered by Duan et all in [4] and [5].

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In recent years, the univariate rational spline interpolation with parameters has been constructed. By using this method, modification of the interpolating surface is possible and it can be done just by selecting suitable parameters under the condition that interpolation data are not changed [2, 3]. Motivated by the univariate rational spline interpolation, the bivariate rational spline interpolation with parameters based on the function values, has been constructed [4]. In [5] the bivariate interpolation based on both function values and first order derivative values of the function being interpolated was presented.

In this study we use the bivariate rational spline interpolation based on both function values and first order derivative values presented in [5], to approximate double integrals on rectangular domain.

The paper is arranged as follow. In section 2, the bivariate rational spline interpolation with parameters based on the function values and first order derivative values is studied. In section 3, symmetric bases of the bivariate interpolation are derived. Some properties of the interpolation and the concept of integral weights coefficients of the interpolation are given in section 4. In section 5, we study the convergence properties of the method. At last some examples are given in section 6, to show that $\int\int P(x,y)dydx$ approximates $\int\int f(x,y)dydx$ very well.

## 2 Bivariate interpolation

Let $\Omega : [a,b;c,d]$ be the plane region, and

$$\left\{\left(x_i, y_j, f_{ij}, \frac{\partial f_{ij}}{\partial x}, \frac{\partial f_{ij}}{\partial y}\right) i=1,2,...,n;\ j=1,2,...,m\right\}$$

be a given set of data points, where $a = x_1 < x_2 < ... < x_n = b$ and $c = y_1 < y_2 < ... < y_m = d$ are the knot spacings, $f_{ij}, \frac{\partial f_{ij}}{\partial x}, \frac{\partial f_{ij}}{\partial y}$ represent $f(x,y), \frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}$ at the point $(x_i, y_j)$ respectively. Let $h_i = x_{i+1} - x_i,\ l_j = y_{j+1} - y_j$ and for any point $(x,y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ in the $(x,y)$-plane, and let $\theta = \frac{x-x_i}{h_i},\ \eta = \frac{y-y_j}{l_j}$. First, for each

$y = y_j,\ j = 1, 2, ..., m$, construct the $x$-direct interpolating curve $P^*_{i,j}(x)$ in $[x_i, x_{i+1}]$, [3]; this is given by

$$P^*_{i,j}(x) = \frac{p^*_{i,j}(x)}{q^*_{i,j}(x)}, \qquad i = 1, 2, ..., n-1 \tag{1}$$

where

$$\begin{aligned}
p^*_{i,j}(x) &= (1-\theta)^3\alpha^*_{i,j} f_{i,j} + \theta(1-\theta)^2 v^*_{i,j} + \theta^2(1-\theta) w^*_{i,j} + \theta^3\beta^*_{i,j} f_{i+1,j} \\
q^*_{i,j}(x) &= (1-\theta)\alpha^*_{i,j} + \theta\beta^*_{i,j} \\
v^*_{i,j} &= (2\alpha^*_{i,j} + \beta^*_{i,j}) f_{i,j} + h_i\alpha^*_{i,j}\frac{\partial f_{i,j}}{\partial x} \\
w^*_{i,j} &= (\alpha^*_{i,j} + 2\beta^*_{i,j}) f_{i+1,j} - h_i\beta^*_{i,j}\frac{\partial f_{i+1,j}}{\partial x}
\end{aligned}$$

with $\alpha^*_{i,j} > 0,\ \beta^*_{i,j} > 0$. It is easy to prove that this interpolation function exists and is unique for the given data and parameters [2].

For each pair $(i, j),\ i = 1, 2, ..., n-1,\ j = 1, 2, .., m-1$, using the interpolation function $P^*_{i,j}(x)$, we define the bivariate rational bicubic interpolation function $P_{i,j}$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follow

$$P_{i,j}(x, y) = \frac{p_{i,j}(x, y)}{q_{i,j}(y)}, \qquad i = 1, 2, ..., n-1;\ j = 1, 2, ..., m-1 \tag{2}$$

where

$$\begin{aligned}
p_{i,j}(x, y) &= (1-\eta)^3\alpha_{i,j} P^*_{i,j}(x) + \eta(1-\eta)^2 v_{i,j} + \eta^2(1-\eta) w_{i,j} + \eta^3\beta_{i,j} P^*_{i,j+1}(x) \\
q_{i,j}(y) &= (1-\eta)\alpha_{i,j} + \eta\beta_{i,j} \\
v_{i,j} &= (2\alpha_{i,j} + \beta_{i,j}) P^*_{i,j}(x) + l_j\alpha_{i,j} f^*_{i,j}(x, y_j) \\
w_{i,j} &= (\alpha_{i,j} + 2\beta_{i,j}) P^*_{i,j+1} - l_j\beta_{i,j} f^*_{i,j+1}(x, y_{j+1})
\end{aligned}$$

with

$$f^*_{i,s}(x, y_s) = (1-\theta)\frac{\partial f_{i,s}}{\partial y} + \theta\frac{\partial f_{i+1,s}}{\partial y}, \quad \theta \in [0, 1], \quad s = j, j+1 \tag{3}$$

and $\alpha_{i,j} > 0,\ \beta_{i,j} > 0$.

The term $P_{i,j}(x, y)$ is called the bivariate rational interpolation based on function values and partial derivative values. It is easy to prove that $P_{i,j}(x, y)$ is unique for given data.

## 3 The bases of the interpolation

In this section, more explicit form of the interpolating function $P_{i,j}(x,y)$ is given. From equation (1) and (3), it can be rewrite as follow

$$P_{i,j}(x,y)=\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}\left[a_{r,s}(\theta,\eta)f_{r,s}+b_{r,s}(\theta,\eta)h_i\frac{\partial f_{r,s}}{\partial x}+c_{r,s}(\theta,\eta)l_j\frac{\partial f_{r,s}}{\partial y}\right] \tag{4}$$

where

$$\begin{aligned}
a_{i,j}(\theta,\eta)&=\frac{(1-\eta)^2(1-\theta)^2((1+\eta)\alpha_{i,j}+\eta\beta_{i,j})((1+\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j})}{((1-\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j})}\\
a_{i+1,j}(\theta,\eta)&=\frac{(1-\eta)^2\theta^2((1+\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j}+(2-\theta)\beta^*_{i,j})}{((1-\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j})}\\
a_{i,j+1}(\theta,\eta)&=\frac{(1-\theta)^2\eta^2((1-\eta)\alpha_{i,j}+(2-\eta)\beta_{i,j})((1+\theta)\alpha^*_{i,j+1}+\theta\beta^*_{i,j+1})}{((1-\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j+1}+\theta\beta^*_{i,j+1})}\\
a_{i+1,j+1}(\theta,\eta)&=\frac{\theta^2\eta^2((1-\eta)\alpha_{i,j}+(2-\eta)\beta_{i,j})((1-\theta)\alpha^*_{i,j+1}+(2-\theta)\beta^*_{i,j+1})}{((1-\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j+1}+\theta\beta^*_{i,j+1})}
\end{aligned} \tag{5}$$

$$\begin{aligned}
b_{i,j}(\theta,\eta)&=\frac{\theta(1-\theta)^2(1-\eta)^2\alpha^*_{i,j}((1+\eta)\alpha_{i,j}+\eta\beta_{i,j})}{((1-\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j})}\\
b_{i+1,j}(\theta,\eta)&=\frac{-\theta^2(1-\theta)(1-\eta)^2\beta^*_{i,j}((1+\eta)\alpha_{i,j}+\eta\beta_{i,j})}{((1-\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j})}\\
b_{i,j+1}(\theta,\eta)&=\frac{\theta(1-\theta)^2\eta^2\alpha^*_{i,j+1}((1-\eta)\alpha_{i,j}+(2-\eta)\beta_{i,j})}{((1-\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j+1}+\theta\beta^*_{i,j+1})}\\
b_{i+1,j+1}(\theta,\eta)&=\frac{-\theta^2(1-\theta)\eta^2\beta^*_{i,j+1}((1-\eta)\alpha_{i,j}+(2-\eta)\beta_{i,j})}{((1-\eta)\alpha_{i,j}+\eta\beta_{i,j})((1-\theta)\alpha^*_{i,j+1}+\theta\beta^*_{i,j+1})}
\end{aligned} \tag{6}$$

$$\begin{aligned}
c_{i,j}(\theta,\eta)&=\frac{(1-\eta)^2(1-\theta)\eta\alpha_{i,j}}{(1-\eta)\alpha_{i,j}+\eta\beta_{i,j}}\\
c_{i+1,j}(\theta,\eta)&=\frac{(1-\eta)^2\theta\eta\alpha_{i,j}}{(1-\eta)\alpha_{i,j}+\eta\beta_{i,j}}\\
c_{i,j+1}(\theta,\eta)&=\frac{-(1-\eta)(1-\theta)\eta^2\beta_{i,j}}{(1-\eta)\alpha_{i,j}+\eta\beta_{i,j}}\\
c_{i+1,j+1}(\theta,\eta)&=\frac{-(1-\eta)\theta\eta^2\beta_{i,j}}{(1-\eta)\alpha_{i,j}+\eta\beta_{i,j}}
\end{aligned} \tag{7}$$

The terms $a_{r,s}(\theta,\eta), b_{r,s}(\theta,\eta), c_{r,s}(\theta,\eta),\ r=i,i+1,\ s=j,j+1$ are the bases of the interpolation defined by 2. If $\alpha^*_{i,j}=\beta^*_{i,j},\ \alpha^*_{i,j+1}=\beta^*_{i,j+1}$ and $\alpha_{i,j}=\beta_{i,j}$, (5-7) becomes

$$
\begin{aligned}
a_{i,j}(\theta,\eta) &= (1-\eta)^2(1+2\eta)(1-\theta)^2(1+2\theta)\\
a_{i+1,j}(\theta,\eta) &= (1-\eta)^2(1+2\eta)\theta^2(3-2\theta)\\
a_{i,j+1}(\theta,\eta) &= \eta^2(3-2\eta)(1-\theta)^2(1+2\theta)\\
a_{i+1,j+1}(\theta,\eta) &= \eta^2(3-2\eta)\theta^2(3-2\theta)\\
b_{i,j}(\theta,\eta) &= (1-\eta)^2(1+2\eta)\theta(1-\theta)^2\\
b_{i+1,j}(\theta,\eta) &= -(1-\eta)^2(1+2\eta)\theta^2(1-\theta)\\
b_{i,j+1}(\theta,\eta) &= \eta^2(3-2\eta)(1-\theta)^2\theta\\
b_{i+1,j+1}(\theta,\eta) &= -\eta^2(3-2\eta)(1-\theta)\theta^2\\
c_{i,j}(\theta,\eta) &= (1-\theta)(1-\eta)^2\eta\\
c_{i+1,j}(\theta,\eta) &= \theta(1-\eta)^2\eta\\
c_{i,j+1}(\theta,\eta) &= -(1-\eta)\eta^2(1-\theta)\\
c_{i+1,j+1}(\theta,\eta) &= -(1-\eta)\eta^2\theta
\end{aligned}
\tag{8}
$$

We define

$$
\begin{aligned}
w_{0,0}(\theta,\alpha^*_{i,j},\beta^*_{i,j}) &= \frac{(1-\theta)^2((1+\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j})}{(1-\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j}}\\
w_{1,0}(\theta,\alpha^*_{i,j},\beta^*_{i,j}) &= \frac{\theta^2((1-\theta)\alpha^*_{i,j}+(2-\theta)\beta^*_{i,j})}{(1-\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j}}\\
w_{0,1}(\theta,\alpha^*_{i,j},\beta^*_{i,j}) &= \frac{\theta(1-\theta)^2\alpha^*_{i,j}}{(1-\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j}}\\
w_{1,1}(\theta,\alpha^*_{i,j},\beta^*_{i,j}) &= \frac{-\theta^2(1-\theta)\beta^*_{i,j}}{(1-\theta)\alpha^*_{i,j}+\theta\beta^*_{i,j}}
\end{aligned}
$$

Then we can rewrite the bases of the interpolation defined by (2) as

$$
\begin{aligned}
a_{i,j}(\theta,\eta) &= w_{0,0}(\eta,\alpha_{i,j},\beta_{i,j})w_{0,0}(\theta,\alpha^*_{i,j},\beta^*_{i,j})\\
a_{i+1,j}(\theta,\eta) &= w_{0,0}(\eta,\alpha_{i,j},\beta_{i,j})w_{1,0}(\theta,\alpha^*_{i,j},\beta^*_{i,j})\\
a_{i,j+1}(\theta,\eta) &= w_{1,0}(\eta,\alpha_{i,j},\beta_{i,j})w_{0,0}(\theta,\alpha^*_{i,j+1},\beta^*_{i,j+1})\\
a_{i+1,j+1}(\theta,\eta) &= w_{1,0}(\eta,\alpha_{i,j},\beta_{i,j})w_{1,0}(\theta,\alpha^*_{i,j+1},\beta^*_{i,j+1})\\
b_{i,j}(\theta,\eta) &= w_{0,0}(\eta,\alpha_{i,j},\beta_{i,j})w_{0,1}(\theta,\alpha^*_{i,j},\beta^*_{i,j})\\
b_{i+1,j}(\theta,\eta) &= w_{0,0}(\eta,\alpha_{i,j},\beta_{i,j})w_{1,1}(\theta,\alpha^*_{i,j},\beta^*_{i,j})\\
b_{i,j+1}(\theta,\eta) &= w_{1,0}(\eta,\alpha_{i,j},\beta_{i,j})w_{0,1}(\theta,\alpha^*_{i,j+1},\beta^*_{i,j+1})\\
b_{i+1,j+1}(\theta,\eta) &= w_{1,0}(\eta,\alpha_{i,j},\beta_{i,j})w_{1,1}(\theta,\alpha^*_{i,j+1},\beta^*_{i,j+1})
\end{aligned}
$$

$$c_{i,j}(\theta,\eta) = (1-\theta)w_{0,1}(\eta,\alpha_{i,j},\beta_{i,j})$$
$$c_{i+1,j}(\theta,\eta) = \theta w_{0,1}(\eta,\alpha_{i,j},\beta_{i,j})$$
$$c_{i,j+1}(\theta,\eta) = (1-\theta)w_{1,1}(\eta,\alpha_{i,j},\beta_{i,j})$$
$$c_{i+1,j+1}(\theta,\eta) = \theta w_{1,1}(\eta,\alpha_{i,j},\beta_{i,j})$$

## 4 Application to double integrals

In this section we use the above mentioned interpolation to evaluate integrals

$$\int\int f(x,y)\ dy\ dx \qquad\qquad (x,y)\in D=[a,b;c,d] \tag{9}$$

Consider $f(x,y)$ be any continuous function with continuous first order partial derivatives on the domain $D$. Let $P_{i,j}(x,y)$ be the bivariate rational interpolation of $f(x,y)$ on the subregion $[x_i,x_{i+1};y_j,y_{j+1}]$ as defined in (2). Thus

$$\int\int f(x,y)\ dy\ dx \simeq \int\int P_{i.j}(x,y)\ dy\ dx \qquad (x,y)\in D_{i,j} \tag{10}$$

where $D_{i,j}$ denotes the subregion $[x_i,x_{i+1};y_j,y_{j+1}]$. Then from (2) we have

$$\int\int P_{i.j}(x,y)\ dy\ dx = h_i l_j \sum_{r=i}^{i+1}\sum_{s=j}^{j+1}\left[a^*_{r,s}f_{r,s} + b^*_{r,s}h_i\frac{\partial f_{r,s}}{\partial x} + c^*_{r,s}l_j\frac{\partial f_{r,s}}{\partial y}\right] \tag{11}$$

where

$$a^*_{r,s} = \int_0^1\int_0^1 a_{r,s}(\theta,\eta)\ d\theta\ d\eta, \qquad r=i,i+1,\ s=j,j+1$$
$$b^*_{r,s} = \int_0^1\int_0^1 b_{r,s}(\theta,\eta)\ d\theta\ d\eta, \qquad r=i,i+1,\ s=j,j+1$$
$$c^*_{r,s} = \int_0^1\int_0^1 c_{r,s}(\theta,\eta)\ d\theta\ d\eta, \qquad r=i,i+1,\ s=j,j+1$$

Calling $a^*_{r,s}, b^*_{r,s}$ and $c^*_{r,s}$ the integral weights of $P_{i,j}$ defined by (2). With (11), (10) and (9) we have

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\ dy\ dx &\simeq \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} P_{i.j}(x,y)\ dy\ dx\\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}\left[a^*_{r,s}f_{r,s} + b^*_{r,s}h_i\frac{\partial f_{r,s}}{\partial x} + c^*_{r,s}l_j\frac{\partial f_{r,s}}{\partial y}\right]\end{aligned} \tag{12}$$

## 5 Convergence properties

Consider the interpolating function $P_{i,j}(x,y)$ defined by (2). Let $(x,y)$ be any point in the subregion $[x_i, x_{i+1}; y_j, y_{j+1}]$ and $h_i = (x_{i+1} - x_i)/N$, $l_j = (y_{j+1} - y_j)/M$, for $i = 1, 2, ..., N-1, \quad j = 1, 2, ..., M-1$. By the Taylor's expansion, there exists point $(\delta_r, \xi_s) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ such that

$$f(x,y) = f(x_{i+r}, y_{j+s}) + ((x - x_{i+r})\frac{\partial}{\partial x} + (y - y_{j+s})\frac{\partial}{\partial y})f(\delta_r, \xi_s), \quad r, s = 0, 1$$

Denote

$$\left\|\frac{\partial f}{\partial x}\right\| = \max_{x\in[a,b]}\left|\frac{\partial f(x,y)}{\partial x}\right|,$$

$$\left\|\frac{\partial f}{\partial y}\right\| = \max_{y\in[c,d]}\left|\frac{\partial f(x,y)}{\partial y}\right|,$$

Then

$$|f(x,y) - f(x_{i+r}, y_{j+s})| \leq \left(2h_i\left\|\frac{\partial f}{\partial x}\right\| + 2l_j\left\|\frac{\partial f}{\partial y}\right\|\right), \quad r, s = 0, 1$$

From [4], since

$$\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|a_{r,s}| = 1, \qquad \sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|b_{r,s}| \leq \frac{1}{4}, \quad \sum_{r=i}^{i+1}\sum_{s=j}^{j+1}|c_{r,s}| \leq \frac{1}{4}$$

Then

$$|f(x,y) - P_{i,j}(x,y)| \leq \left(2h_i\left\|\frac{\partial f}{\partial x}\right\| + 2l_j\left\|\frac{\partial f}{\partial y}\right\|\right) + \frac{h_i}{4}\left\|\frac{\partial f}{\partial x}\right\| + \frac{l_j}{4}\left\|\frac{\partial f}{\partial y}\right\|$$

Thus the error of the interpolation satisfies

$$|f(x,y) - P_{i,j}(x,y)| \leq \frac{9}{4}\left(h_i\left\|\frac{\partial f}{\partial x}\right\| + l_j\left\|\frac{\partial f}{\partial y}\right\|\right) \tag{13}$$

Thus

$$\begin{aligned}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}|f(x,y) - P_{i,j}(x,y)|\ dy\ dx &\leq \frac{9}{4}\left(h_i\left\|\frac{\partial f}{\partial x}\right\| + l_j\left\|\frac{\partial f}{\partial y}\right\|\right)h_i l_j \\ &\leq \frac{9}{4}\triangle^3\left(\left\|\frac{\partial f}{\partial x}\right\| + \left\|\frac{\partial f}{\partial y}\right\|\right)\end{aligned} \tag{14}$$

Where $\triangle = \max_{i,j}(h_i, l_j)$.

## 6 Numerical illustrations

In this section some examples shows the method's pertinent features. In this section we choose $\alpha^*_{i,j} = \beta^*_{i,j} = \alpha_{i,j} = \beta_{i,j} = 1$

**Example 6.1** *Assume* $f(x, y) = \sqrt{1-(1-x)^2-(1-y)^2}$ *with* $D = [0.5, 1.5; 0.5, 1.5]$. *Let* $h = 1/N$, $l = 1/M$, $x_i = 0.5+ih$ *and* $y_j = 0.5+jl$ *for* $i = 0, 1, ..., N,\ j = 0, 1, ..., M$. *Table 1 shows numerical result of the method for different values of* $N$ *and* $M$. *The exact value of the integral is 0.9109658470190631.*

**Example 6.2** *Consider* $f(x, y) = y(x-x^2+3y)/(1+y)x^2$ *where* $D = [0.5, 1.5; 0.5, 1.5]$. *Table 2 shows numerical result of the method for different values of* $N$ *and* $M$. *The exact value of the integral is 2.0915410998621934.*

**Example 6.3** *Consider* $f(x, y) = 1/(1 + x^2 + y^2)$ *where* $D = [0, 1; 0, 1]$. *Table 3 shows numerical result of the method for different values of* $N$ *and* $M$. *The exact value of the integral is 0.6395103518703110.*

Table 1 : Numerical results of Example 1

| $N$ | $M$ | Our Method | error |
|---|---|---|---|
| 10 | 10 | 0.9109699713 | 0.41e-5 |
| 20 | 20 | 0.9109661055 | 0.26e-6 |
| 50 | 50 | 0.9109658530 | 0.60e-8 |
| 100 | 100 | 0.9109658474 | 0.41e-9 |
| 200 | 200 | 0.9109658470 | 0.26e-10 |

Table 2 : Numerical results of Example 2

| $N$ | $M$ | Our Method | error |
|---|---|---|---|
| 10 | 10 | 2.091367427932 | 0.17e-3 |
| 20 | 20 | 2.091530034977 | 0.11e-4 |
| 50 | 50 | 2.091540815015 | 0.28e-6 |
| 100 | 100 | 2.091541082044 | 0.18e-7 |
| 200 | 200 | 2.091541098748 | 0.11e-8 |
| 500 | 500 | 2.091541099833 | 0.29e-10 |

Table 3 : Numerical results of Example 3

| $N$ | $M$ | Our Method | error |
|---|---|---|---|
| 10 | 10 | 0.639510092354 | 0.26e-6 |
| 20 | 20 | 0.639510335623 | 0.16e-7 |
| 50 | 50 | 0.639510351454 | 0.42e-9 |
| 100 | 100 | 0.639510351844 | 0.26e-10 |
| 200 | 200 | 0.639510351869 | 0.16e-11 |

## 7 Conclusion

This paper gives a kind of bivariate spline interpolation, depends on the function values and partial derivatives of the function being interpolated. The interpolating function has simple and explicit expression with parameters. It is better to use bivariate spline described by (2), in the practical applications. Equation (4) is more explicit form of the interpolation function and it is convenient for the theoretical analysis. We used this interpolating function for evaluating two dimensional integrals. There is some suitable methods to adjusting the interpolating surface to achieve better estimations [5]. This methods can be use for integration to get better approximation.

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