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Some Inclusion Relations Between Lacunary Methods on Probabilistic Normed Spaces

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Abstract

A lacunary sequence is an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $k_r - k_{r-1} \longrightarrow \infty$ as $r \longrightarrow \infty$. In this paper we introduce lacunary statistical convergence on probabilistic normed space (briefly PN spaces) and establish some inclusion relations between the sets of statistically convergent and lacunary statistically convergent sequences on PN spaces. We also defined lacunary statistical Cauchy sequence in these spaces and prove that it is equivalent to lacunary statistical convergent sequence.

Keywords: Lacunary sequence; statistical convergence; continuous t-norm and probabilistic normed spaces.

1 Introduction

The notion of statistical convergence was introduced by Fast [3] and Schoenberg [10] independently. Over the years statistical convergence has been discussed by several authors. Fridy [5]introduced a related concept of convergence with the help of lacunary

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sequence $\theta = (k_r)$ and called it lacunary statistically convergence. In present paper we define lacunary statistical convergence and lacunary statistical Cauchy sequences on PN spaces and establish some inclusion relations between lacunary methods on PN-Spaces.

2 Preliminaries

In this section we recall some definitions and terminology regarding PN-spaces, convergence structure in PN Spaces and lacunary statistical convergence.

Throughout of this paper, the system of real numbers will be denoted by R and $R_0^+ = [0, \infty)$.

Definition 2.1 [11] A function $f : R \longrightarrow R_0^+$ is called a distribution function if it is non-decreasing and left-continuous with $inf_{t \in R} f(t) = 0$, and $inf_{t \in R} f(t) = 1$.

We will denote the set of all distribution functions by D.

Definition 2.2 [11] A triangular norm, briefly t-norm, is a binary operation T on $[0, 1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, i.e., it is the continuous mapping $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ such that for all $a, b, c \in [0, 1]$:

(i) $T(a, 1) = a$,

(ii) $T(a, b) = T(b, a)$,

(iii) $T(c, d) \geq T(a, b)$ if $c \geq a$ and $d \geq b$,

(iv) $T(T(a, b), c) = T(a, T(b, c))$.

The T operations $T(a, b) = max\{a + b - 1, 0\}, T(a, b) = ab$, and $T(a, b) = min\{a, b\}$ on $[0, 1]$ are t-norms. We will denote the t-norm min by $\wedge$.

With the help of Definition 2.1 and Definition 2.2, a probabilistic normed space is defined as follows.

Definition 2.3[1] A triplet (X, F, T) is called a probabilistic normed space (briefly, a PN-Space) if X is a real vector space, F is a mapping from X into D (for $x \in X$, the distribution function $F(x)$ is denoted by F_x, and $F_x(t)$ is the value F_x at $t \in R$) and T

is a t-norm satisfying the following conditions:

(PF-1) $F_x(0)$ =0,

(PF-2) $F_x(t) = 1$ for all $t > 0$ if and only if $x = 0$,

(PF-3) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha \in R - \{0\}$,

(PF-4) $F_{x+y}(s+t) \geq T(F_x(s), F_y(t))$ for all $x, y \in X$, and $s, t \in R_0^+$.

The mapping $x \longrightarrow F_x$ is called a T-norm or a T-probabilistic norm on X.

Suppose that $(X, \|.\|)$ be a normed space. Let $\mu \in D$ such that $\mu(0)$=0, and $\mu \neq \epsilon_0$, where ϵ_0 is defined by

$$\epsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1 & t > 0. \end{cases}$$

For $x \in X$ and $t \in R$, if we define

$$F_x(t) = \begin{cases} \epsilon_0(t), & \text{if } x = 0, \\ \mu(\frac{t}{\|x\|}) & x \neq 0; \end{cases}$$

then it is shown in [1] that (X, F, T) be a PN-space in the sense of Definition 2.3. For example if we define the functions μ and μ' on R by

$$\mu(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{x}{x+1} & x > 0; \end{cases} \quad and \ \mu' = \begin{cases} 0, & \text{if } x \leq 0, \\ \exp(-\frac{1}{x}) & x > 0; \end{cases}$$

then we obtain the following well-known T-norms

$$F_x(t) = \begin{cases} \epsilon_0(t), & \text{if } x = 0, \\ \frac{t}{t+\|x\|} & x \neq 0; \end{cases} \quad and \ F'_x(t) = \begin{cases} \epsilon_0(t), & \text{if } x = 0, \\ \exp(\frac{-\|x\|}{t}) & x \neq 0. \end{cases}$$

Let (X, F, T) be a PN-Space. For $x \in X$, $t > 0$ and $0 < r < 1$, the ball centered at x with radius r is defined by $\{y \in X : F_{x-y}(t) > 1 - r\}$ and is denoted by $B(x, r, t)$.

A sequence (x_n) in X is said to be convergent to some point $\xi \in X$ with respect to the probabilistic norm F if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer m such that $F_{x_n - \xi}(\epsilon) > 1 - \lambda$ whenever $n \geq m$.

A sequence (x_n) in X is said to be Cauchy with respect to the probabilistic norm F if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer k such that $F_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq k$.

A sequence (x_k) in X is said to be statistical convergent to some point $\xi \in X$ with respect to the probabilistic norm F, denoted by $S_F - \lim x_k = \xi$, if for every $\epsilon > 0$ and $\lambda \in (0,1)$,

$lim_{n \longrightarrow \infty} \frac{1}{n} |\{k \in N : F_{x_k - \xi}(\epsilon) \leq 1 - \lambda\}| = 0.$

Let $S_F(X)$ denotes the set of all statistical convergent sequences in X.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \longrightarrow \infty$ as $r \longrightarrow \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1} k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

A lacunary sequence $\theta' = (k'_r)$ is called a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subset (k'_r)$.

For a lacunary sequence $\theta = (k_r)$, the number sequence (x_k) is said to be lacunary statistical convergence to a number ξ, denoted by $S_\theta - lim_{k \longrightarrow \infty} x_k = \xi$, provided that for each $\epsilon > 0$,

$lim_{r \longrightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - \xi| \geq \epsilon\}| = 0.$

Some work on lacunary statistical convergence can be found in [2], [4], and [6-9].

3 Lacunary Statistical Convergence

In this section, we define and study lacunary statistical convergence on PN-Spaces.

Definition 3.1 Let (X, F, T) be a PN-Space and θ be a lacunary sequence. A sequences $x = (x_k)$ in X is said to be lacunary statistically convergent to $\xi \in X$ with respect to the probabilistic norm F provided that for each $\epsilon > 0$ and $\lambda \in (0,1)$, we have

$\lim_{r \longrightarrow \infty} \frac{1}{h_r} |\{k \in I_r : F_{x_k - \xi}(\epsilon) \leq 1 - \lambda\}| = 0.$

In this case ξ is called the lacunary statistical limit of the sequence $x = (x_k)$ and we write $S_F^\theta - \lim x_k = \xi$ or $x_k \longrightarrow \xi(S_F^\theta)$.

Let, $S_F^\theta(X)$ denotes the set of all lacunary statistically convergent sequences in a PN-Space (X, F, T).

In next Theorem, we show that for any fixed θ, the S_F^θ-limit is unique provided it exists.

Theorem 3.1 Let (X, F, T) be a PN-Space and θ be a fixed lacunary sequence. If $x = (x_n)$ be a sequence in X such that $S_F^\theta - \lim x_k$ exists, then it must be unique.

Proof Suppose that there exists ξ and η in X with $\xi \neq \eta$, $S_F^\theta - \lim x_k = \xi$ and $S_F^\theta - \lim x_k = \eta$. Since for $\xi \neq \eta$,

$$\{k \in I_r : F_{x_k - \eta}(\epsilon) > 1 - \lambda\} \cap \{k \in I_r : F_{x_k - \xi}(\epsilon) > 1 - \lambda\} = \emptyset,$$

and therefore

$$\lim_{r \longrightarrow \infty} \tfrac{1}{h_r} |\{k \in I_r : F_{x_k - \eta}(\epsilon) > 1 - \lambda\}| \leq \lim_{r \longrightarrow \infty} \tfrac{1}{h_r} |\{k \in I_r : F_{x_k - \xi}(\epsilon) \leq 1 - \lambda\}|.$$

Since $S_F^\theta - \lim x_k = \xi$, it follows that

$$\lim_{r \longrightarrow \infty} \tfrac{1}{h_r} |\{k \in I_r : F_{x_k - \eta}(\epsilon) > 1 - \lambda\}| \leq 0.$$

and consequently

$$\lim_{r \longrightarrow \infty} \tfrac{1}{h_r} |\{k \in I_r : F_{x_k - \eta}(\epsilon) > 1 - \lambda\}| = 0\ ;$$

as it can not be negative.

But then we have a contradiction to the fact that $S_\theta^F - \lim x_k = \eta$. Hence $\xi = \eta$.

Next Theorem gives the algebraic characterization of lacunary statistical convergence on PN-Spaces.

Theorem 3.2 Let (X, F, T) be a PN-Space, θ be lacunary sequence and $x = (x_k)$, $y = (y_k)$ be two sequences in X.

(i) If $S_F^\theta - \lim x_k = \xi$ and $0 \neq$ c $\in R$, then $S_F^\theta - \lim cx_k = c\xi$.

(ii) If $S_F^\theta - \lim x_k = \xi$ and $S_F^\theta - \lim y_k = \eta$, then $S_F^\theta - \lim(x_k + y_k) = \xi + \eta$.

Proof (i) Let $\epsilon > 0$ and $\lambda \in (0, 1)$. Since, for $0 \neq$ c $\in R$

$$F_{cx_k - c\xi}(\epsilon) = F_{x_k - \xi}(\tfrac{\epsilon}{|c|}),$$

it follows that

$$\tfrac{1}{h_r} |\{k \in I_r : F_{cx_k - c\xi}(\epsilon) \leq 1 - \lambda\}| \leq \tfrac{1}{h_r} |\{k \in I_r : F_{x_k - \xi}(\tfrac{\epsilon}{|c|}) \leq 1 - \lambda\}|.$$

Since $S_F^\theta - \lim x_k = \xi$, therefore we have $S_F^\theta \longrightarrow \lim cx_k = c\xi$.

(ii) Assume that $S_F^\theta \longrightarrow \lim x_k = \xi$ and $S_F^\theta \longrightarrow \lim y_k = \eta$.

Let $\epsilon > 0$ and $\lambda \in (0, 1)$. Choose $\gamma \in (0,1)$ such that

$$T((1-\gamma),(1-\gamma)) > (1-\lambda). \tag{1}$$

Since

$$\lim_r \frac{1}{h_r}|\{k \in I_r : F_{(x_k+y_k)-(\xi+\eta)}(\epsilon) \le 1-\lambda\}| \le \lim_r \frac{1}{h_r}|\{k \in I_r : F_{x_k-\xi}(\frac{\epsilon}{2}) \le 1-\gamma\}| + \lim_r \frac{1}{h_r}|\{k \in I_r : F_{y_k-\eta}(\frac{\epsilon}{2}) \le 1-\gamma\}|,$$

it follows by assumption that $S_F^\theta - \lim(x_k + y_k) = \xi + \eta$.

Theorem 3.3 Let (X, F, T) be a PN-Space. For any lacunary sequence θ, $S_F(X) \subset S_F^\theta(X)$, if and only if, $\liminf_r q_r > 1$.

Proof Sufficiency- Suppose that $\liminf_r q_r > 1$. Then there exists a $\delta > 0$ such that $q_r \ge 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \ge \frac{\delta}{1+\delta}.$$

If (x_k) is statistically convergent to ξ with respect to the probabilistic norm F, then for each $\epsilon > 0$, $\lambda \in (0, 1)$ and sufficiently large r, we have

$$\frac{1}{k_r}|\ \{k \le k_r : F_{x_k-\xi}(\epsilon) \le 1-\lambda\}\ | \ge \frac{1}{k_r}|\ \{k \in I_r : F_{x_k-\xi}(\epsilon) \le 1-\lambda\}\ | \frac{\delta}{1+\delta}.\frac{1}{h_r}|\ \{k \in I_r : F_{x_k-\xi}(\epsilon) \le 1-\lambda\}\ |.$$

Since, $x_k \in S_F(X)$, it follows, $x_k \longrightarrow \xi(S_F^\theta)$ and hence $S_F(X) \subset S_F^\theta(X)$.

Necessity- Suppose that $\liminf_r q_r = 1$. Then we can select a subsequence $(k_{r(j)})$ of the lacunary sequence θ such that

$$\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j} \quad and \quad \frac{k_{r(j)-1}}{k_{r(j-1)}} > j, \ \ where\ r(j) \ge r(j-1) + 2.$$

Let $\mathbf{0} \ne \xi \in$ X. We define a sequence $x = (x_k)$ as follow:

$$x_k = \begin{cases} \xi, & \text{if } k \in I_{r(j)} \ \ for\ some\ j = 1, 2, \cdots, \\ \mathbf{0} & otherwise. \end{cases}$$

We shall show that (x_k) is statistically convergent to $\mathbf{0}$ with respect to the probabilistic norm F. Let $\epsilon > 0$ and $\lambda \in (0,1)$. Choose $\epsilon_1 > 0$ and $\lambda_1 \in (0,1)$ such that $B(\mathbf{0},\epsilon_1,\lambda_1) \subset B(\mathbf{0},\epsilon,\lambda)$ and $\xi \notin B(\mathbf{0},\epsilon_1,\lambda_1)$.

Also for each n we can find a positive number j_n such that $k_{r(j_n)} < n \le k_{r(j_n)+1}$. Then, for each $n \in N$, we have

$$\frac{1}{n}|\ \{k \le n : F_{x_k}(\epsilon) \le 1-\lambda\}\ | \ \le\ \frac{1}{k_{r(j_n)}}|\ \{k \le n : F_{x_k}(\epsilon_1) \le 1-\lambda_1\}\ |$$
$$\le \frac{1}{k_{r(j_n)}}\{\ |\ \{k \le k_{r(j_n)} : F_{x_k}(\epsilon_1) \le 1-\lambda_1\}\ | + |\ \{k_{r(j_n)} < k \le n : F_{x_k}(\epsilon_1) \le 1-\lambda_1\}\ |\ \}$$
$$\le \frac{1}{k_{r(j_n)}}\{\ |\ \{k \le k_{r(j_n)} : F_{x_k}(\epsilon_1) \le 1-\lambda_1\}\ | + \frac{1}{k_{r(j_n)}}(k_{r(j_n)+1} - k_{r(j_n)})$$
$$< \frac{1}{j_n+1} + 1 + \frac{1}{j_n} - 1 = \frac{1}{j_n+1} + \frac{1}{j_n}.$$

It follows that $S_F^\theta - \lim x_k = \mathbf{0}$. Next we shall show that (x_k) is not lacunary statistical convergent with respect to the probabilistic norm F. Since $\mathbf{0} \ne \xi$ so we can choose $\epsilon > 0$ and $\lambda \in (0,1)$ such that $\xi \notin B(\mathbf{0},\epsilon,\lambda)$. Thus

$$\lim_{j\longrightarrow\infty} \frac{1}{h_{r(j)}}|\ \{k_{r(j)-1} < k \le k_{r(j)} : F_{x_k}(\epsilon) \le 1-\lambda\}\ | = \lim_{j\longrightarrow\infty} \frac{1}{h_{r(j)}}(k_{r(j)} - k_{r(j)-1})$$
$$= \lim_{j\longrightarrow\infty} \frac{1}{h_{r(j)}}(h_{r(j)}) = 1,$$

and

$$\lim_{j\longrightarrow\infty,\ r\ne r(j), j=1,2,\cdots} \frac{1}{h_r}|\ \{k_{r-1} < k \le k_r : F_{x_k-\xi}(\epsilon) \le 1-\lambda\}\ | = 1 \ne 0.$$

Hence neither ξ nor $\mathbf{0}$ can be lacunary statistical limit of the sequence (x_k) with respect to the probabilistic norm F. No other point of X can be lacunary statistical limit of the sequence as well. Hence, $(x_k) \notin S_\theta^F(X)$ and the proof of the Theorem is complete.

Theorem 3.4 Let (X,F,T) be a PN-Space. For any lacunary sequence θ, $S_F^\theta(X) \subset S_F(X)$, if and only if, $\limsup_r q_r < \infty$.

Proof Sufficiency- If $\limsup_r q_r < \infty$, then there is an $H > 0$ such that $q_r < H$ for all r. Suppose that $x_k \longrightarrow \xi(S_F^\theta)$, and let $N_r = |\ \{k \in I_r : F_{x_k-\xi}(\epsilon) \le 1-\lambda\}\ |$. By definition of a lacunary statistical convergent sequence there is a positive integer r_0 such that

$$\frac{N_r}{h_r} < \epsilon \ for\ all\ r > r_0. \tag{2}$$

Now let $K = max\{N_r : 1 \leq r \leq r_0\}$ and n be any integer satisfying $k_{r-1} < n \leq k_r$; then we can write

$$\begin{aligned}
\frac{1}{n}|\ \{k \leq n : F_{x_k-\xi}(\epsilon) \leq 1-\lambda\}\ | &\leq \frac{1}{k_{r-1}}|\ \{k \leq k_r : F_{x_k-\xi}(\epsilon) \leq 1-\lambda\}\ | \\
&= \frac{1}{k_{r-1}}\{N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r\} \\
&\leq \frac{K}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\{h_{r_0+1}\frac{N_{r_0+1}}{h_{r_0+1}} + \cdots + h_r\frac{N_r}{h_r}\} \\
&\leq \frac{r_0 K}{k_{r-1}} + \frac{1}{k_{r-1}}(\sup_{r>r_0}\frac{N_r}{h_r})\{h_{r_0+1} + \cdots + h_r\} \\
&\leq \frac{r_0 K}{k_{r-1}} + \epsilon\frac{k_r - k_{r_0}}{k_{r-1}}, by (2), \\
&\leq \frac{r_0 K}{k_{r-1}} + \epsilon q_r \leq \frac{r_0 K}{k_{r-1}} + \epsilon H;
\end{aligned}$$

and the sufficiency follows immediately.

Necessity- Suppose that $\limsup_r q_r = \infty$. Let $\mathbf{0} \neq \xi \in X$. Select a subsequence $(k_{r(j)})$ of the lacunary sequence $\theta = (k_r)$ such that $q_{r(j)} > j$, $k_{r(j)} > j + 3$. Define a sequence $x = (x_k)$ as follows.

$$x_k = \begin{cases} \xi, & \text{if } k_{r_{(j)}-1} < k \leq 2k_{r_{(j)}-1}\ \ for\ some\ j = 1, 2, \cdots, \\ \mathbf{0} & otherwise. \end{cases}$$

Since $\mathbf{0} \neq \xi$ so we can choose $\epsilon > 0$ and $\lambda \in (0,\, 1)$ such that $\xi \notin B(\mathbf{0}, \epsilon, \lambda)$. Now for $j > 1$,

$$\frac{1}{h_{r(j)}}|\ \{k \leq k_{r(j)} : F_{x_k}(\epsilon) \leq 1-\lambda\}\ | < \frac{1}{h_{r(j)}}(k_{r(j)-1}) = \frac{1}{(k_{r(j)} - k_{r(j)-1})}(k_{r(j)-1}) < \frac{1}{j-1}.$$

From which we have $(x_k) \in S_F^\theta(X)$. But $(x_k) \notin S_F(X)$. For

$$\begin{aligned}
&\frac{1}{2k_{r_{(j)}-1}}|\ \{k \leq 2k_{r_{(j)}-1} : F_{x_k}(\epsilon) \leq 1-\lambda\}\ | \\
&= \frac{1}{2k_{r_{(i)}-1}}\{k_{r_{(1)}-1} + k_{r_{(2)}-1} + \cdots + k_{r_{(j)}-1}\} > \frac{1}{2}
\end{aligned}$$

This shows that (x_k) cannot be statistically convergent with respect to the probabilistic norm F. This completes the proof.

Theorem 3.3 and Theorem 3.4 immediately gives the following corollary.

Corollary Let (X, F, T) be a PN-Space. For any lacunary sequence θ, $S_F^\theta(X) = S_F(X)$, if and only if, $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$.

Theorem 3.5 Let (X, F, T) be a PN-Space. If $x = (x_k) \in S_F(X) \bigcap S_F^\theta(X)$, then $S_F^\theta - lim_{k\longrightarrow\infty} x_k = S_F - lim_{k\longrightarrow\infty} x_k$.

Proof Suppose that $S_F - lim_{k\longrightarrow\infty} x_k = \xi$ and $S_F^\theta - lim_{k\longrightarrow\infty} x_k = \eta$ where $\xi \neq \eta$. Let $\epsilon > 0$ and $\lambda \in (0,1)$. Since for $\xi \neq \eta$

$\{k \in N : F_{x_k-\eta}(\epsilon) > 1-\lambda\} \bigcap \{k \in N : F_{x_k-\xi}(\epsilon) > 1-\lambda\} = \emptyset$;

it follows that

$\{k \in N : F_{x_k-\eta}(\epsilon) > 1-\lambda\} \subseteq \{k \in N : F_{x_k-\xi}(\epsilon) \leq 1-\lambda\}$.

and therefore we have

$lim_{n\longrightarrow\infty} \frac{1}{n} |\{k \in N : F_{x_k-\eta}(\epsilon) > 1-\lambda\}| \leq lim_{n\longrightarrow\infty} \frac{1}{n} |\{k \in N : F_{x_k-\xi}(\epsilon) \leq 1-\lambda\}|$

Since $S_F - lim_{k\longrightarrow\infty} x_k = \xi$, therefore

$$lim_{n\longrightarrow\infty} \frac{1}{n} |\{k \in N : F_{x_k-\eta}(\epsilon) \leq 1-\lambda\}| = 1. \tag{3}$$

Consider the $k_m th$ term of the statistical expression $\frac{1}{n}|\{k \in N : F_{x_k-\eta}(\epsilon) \leq 1-\lambda\}|$:

$$\frac{1}{k_m} |\{k \in \bigcup_{r=1}^{m} I_r : F_{x_k-\eta}(\epsilon) \leq 1-\lambda\}| = \frac{1}{k_m} \sum_{r=1}^{m} |\{k \in I_r : F_{x_k-\eta}(\epsilon) \leq 1-\lambda\}|$$

$$= \frac{1}{\sum_{r=1}^{m} h_r} \sum_{r=1}^{m} h_r t_r, \tag{4}$$

where $t_r = \frac{1}{h_r}|\{k \in I_r : F_{x_k-\eta}(\epsilon) \leq 1-\lambda\}| \longrightarrow 0$ as $x_k \longrightarrow \eta(S_F^\theta)$. Since θ is a lacunary sequence and (4) is a regular weighted mean transform of t_r, therefore it too tends to zero as $m \longrightarrow \infty$. Since this is a subsequence of $(\frac{1}{n}|\{k \in N : F_{x_k-\eta}(\epsilon) \leq 1-\lambda\}|)_{n=1}^{\infty}$, it follows that $(\frac{1}{n}|\{k \in N : F_{x_k-\eta}(\epsilon) \leq 1-\lambda\}|)$ cannot converges to 1 and therefore we obtained a contradiction to (3). Hence $\xi = \eta$.

Theorem 3.6 Let (X, F, T) be a PN-Space. If θ' be a lacunary refinement of θ and

$x_k \longrightarrow \xi(S_F^{\theta'})$, then $x_k \longrightarrow \xi(S_F^{\theta})$.

Proof Suppose each I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{v(r)}$ of θ', so that $k_{r-1} < k'_{r,1} < k'_{r,2} \ldots < k'_{r,v(r)} = k_r$, where $I'_{r,i} = (k'_{r,i-1}, k'_{r,i}]$. Note that, for all r, $v(r) \geq 1$ because $\{k_r\} \subset \{k'_r\}$. Let $(I_j^{\star})_{j=1}^{\infty}$ be the sequence of abutting interval $\{I'_{r,i}\}$ ordered by increasing right end points. Since $x_k \longrightarrow \xi(S_F^{\theta'})$, so for each $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_j \sum_{I_j^{\star} \subseteq I_r} \frac{1}{h_r^{\star}} |\{k \in I_j^{\star} : F_{x_k - \xi}(\epsilon) \leq 1 - \lambda\}| = 0. \tag{5}$$

As before, we write $h_r = k_r - k_{r-1}$, $h'_{r,i} = k'_{r,i} - k'_{r,i-1}$ and $h'_{r,1} = k'_{r,1} - k_{r-1}$.

For each $\epsilon > 0$ and $\lambda \in (0,1)$, we have

$$\frac{1}{h_r} |\{k \in I_r : F_{x_k - \xi}(\epsilon) \leq 1 - \lambda\}| = \frac{1}{h_r} \sum_{I_j^{\star} \subseteq I_r} h_j^{\star} (\frac{1}{h_j^{\star}} |\{k \in I_j^{\star} : F_{x_k - \xi}(\epsilon) \leq 1 - \lambda.\}|) \tag{6}$$

By (5) $(\frac{1}{h_j^{\star}} |\{k \in I_j^{\star} : F_{x_k - \xi}(\epsilon) \leq 1 - \lambda.\}|)$ is a null sequence and (6) is its regular weighted mean transform. Hence the transform (6) also tends to zero as $r \longrightarrow \infty$ and consequently $x_k \longrightarrow \xi(S_F^{\theta})$.

Finally, we define lacunary statistical Cauchy sequences on PN-Spaces and present the Cauchy convergence criterion for lacunary statistical convergence in these spaces. We first recall δ_θ-density of subsets of N. For any $K \subseteq N$, the δ_θ-density of K is denoted by $\delta_\theta(K)$ and is defined by

$\delta_\theta(K) = \lim_r \frac{1}{h_r} |\{k \in I_r : k \in K\}| = 0.$

Note that, $\delta_\theta(K) = 0$ if K is a finite set; $\delta_\theta(K^C) = 1 - \delta_\theta(K)$; if $K \subseteq S$, then $\delta_\theta(K) \leq \delta_\theta(S)$.

Definition 3.2 Let (X, F, T) be a PN-Space and θ be a lacunary sequence. A sequence $x = (x_k)$ in X is said to be lacunary statistically Cauchy with respect to the probabilistic norm F or S_F^{θ}-Cauchy provided that or each $\epsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer m such that $\delta_\theta(\{k \in I_r : F_{x_k - x_m}(\epsilon) \leq 1 - \lambda\}) = 0$, i.e.,

$\lim_{r \longrightarrow \infty} \frac{1}{h_r} |\{k \in I_r : F_{x_k - x_m}(\epsilon) \leq 1 - \lambda\}| = 0.$

Theorem 3.7 Let (X, F, T) be a PN-Space and θ be a lacunary sequence. A sequence

$x = (x_k)$ in X is S_F^θ-convergent, if and only if, it is S_F^θ-Cauchy.

Proof First assume that $S_F^\theta - \lim_{k \longrightarrow \infty} x_k = \xi$. Let $\epsilon > 0$ be given and $\lambda \in (0,1)$. Choose $\gamma \in (0,1)$ such that

$$T((1-\gamma),(1-\gamma)) > (1-\lambda). \tag{7}$$

Since, $S_F^\theta - \lim_{k \longrightarrow \infty} x_k = \xi$, therefore we have

$\delta_\theta(A) = \delta_\theta(\{k \in I_r : F_{x_k-\xi}(\frac{\epsilon}{2}) \leq 1-\gamma\}) = 0;$

which implies that

$\delta_\theta(A^C) = \delta_\theta(\{k \in I_r : F_{x_k-\xi}(\frac{\epsilon}{2}) > 1-\gamma\}) = 1.$

Let, $m \in A^C$, then $F_{x_m-\xi}(\frac{\epsilon}{2}) > 1-\gamma$. If we take $B = \{k \in I_r : F_{x_k-x_m}(\epsilon) \leq 1-\lambda\}$, then we need to show that $B \subseteq A$. Let $k \in B - A$. Then we have $F_{x_k-x_m}(\epsilon) \leq 1-\lambda$ and $F_{x_k-\xi}(\frac{\epsilon}{2}) > 1-\gamma$. Now using (PF-4) of Definition 2.3 and (7), we have

$$1-\lambda \geq F_{x_k-x_m}(\epsilon) \geq T(F_{x_k-\xi}(\frac{\epsilon}{2}), F_{x_m-\xi}(\frac{\epsilon}{2})) > T(1-\gamma, 1-\gamma) > 1-\lambda$$

which is not possible. Hence, $B \subseteq A$. But then we have

$$\lim_{r \longrightarrow \infty} \frac{1}{h_r}|\{k \in I_r : F_{x_k-x_m}(\epsilon) \leq 1-\lambda\}| \leq \lim_{r \longrightarrow \infty} \frac{1}{h_r}|\{k \in I_r : F_{x_k-\xi}(\frac{\epsilon}{2}) \leq 1-\gamma\}| = 0;$$

which implies that $\lim_{r \longrightarrow \infty} \frac{1}{h_r}|\{k \in I_r : F_{x_k-x_m}(\epsilon) \leq 1-\lambda\}| = 0$ as it can not be negative. This shows that $x = (x_k)$ is S_F^θ-Cauchy.

Conversely, Suppose that $x = (x_k)$ is S_F^θ-Cauchy but not S_F^θ-convergent. Let $\epsilon > 0$ be given and $\lambda \in (0,1)$. Choose $\gamma \in (0,1)$ such that (7) is satisfied. By assumption, there exist positive integer m such that $\delta_\theta(A) = 0$ where

$A = \{k \in I_r : F_{x_k-x_m}(\epsilon) \leq 1-\lambda\}$

and also $\delta_\theta(B) = 0$, where $B = \{k \in I_r : F_{x_k-\xi}(\frac{\epsilon}{2}) > 1-\gamma\}$.

Since

$F_{x_k-x_m}(\epsilon) \geq T(F_{x_k-\xi}(\frac{\epsilon}{2}), F_{x_m-\xi}(\frac{\epsilon}{2})) \geq T((1-\gamma),(1-\gamma)) > 1-\lambda,$

if $F_{x_k-\xi}(\frac{\epsilon}{2}) > 1-\gamma$ and $F_{x_m-\xi}(\frac{\epsilon}{2}) > 1-\gamma$. Therefore

$\delta_\theta(\{k \in I_r : F_{x_k-x_m}(\epsilon) > 1-\lambda\}) = 0$ i.e, $\delta_\theta(A^C) = 0$ and hence $\delta_\theta(A) = 1$ which leads to a contradiction. Hence $x = (x_k)$ in X is S_F^θ-convergent.

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