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Solution of Integral Equations by Using Block-Pulse Functions

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Abstract

In this paper, the properties of the hybrid functions which consist of block-pulse functions plus Legendre polynomials are presented. Then, integral equations are converted into an algebraic system by hybrid of general block-pulse functions and the Legendre polynomials. In continue, approximate solutions of integral equations are derived, finally the numerical examples are included to demonstrate the validity and applicability of the algorithm.

Keywords: Integral Equations, Block-Pulse Hybrid Functions, Operational Matrix, Orthogonal Functions, Legendre polynomials.

1 Introduction

Integral equations are a class of important models in applied science, since it is difficult to obtain the analytic solutions of these equations, numerical methods to obtain approximate solutions are of interest. Piecewise constant basis functions [1] were introduced by Alfred Haar in 1910. Orthogonal functions or polynomials, such as block-pulse functions(BPF) [2,3], Walsh functions [4], Fourier series [5], Legendre polynomials [6],

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Chebyshev polynomials [7] and Laguerre polynomials [8], were used to derive solutions of some integral equations. Among all these functions, the BPF set proved to be the most fundamental [9] and it enjoyed immense popularity in different applications in the area of numerical analysis. In recent years the different kinds of hybrid functions [10-14] were applied to solve this problem. In this paper the hybrid functions consisting of general block-pulse functions and Legendre polynomials are used to solve integral equations. The general operational matrices are presented and numerical solutions are derived by hybrid functions.

2 Preliminaries

A set of block-pulse function $b_k(t), k = 1, 2, ..., K$ on the interval $[0, T)$ are defined as

$$b_k(t) = \begin{cases} 1, & t_{k-1} \leq t < t_k, \\ 0, & otherwise, \end{cases}$$

where $t_0 = 0, t_K = T$ and $[t_{k-1}, t_k) \subset [0, T)$, $k = 1, 2, ..., K$.

The Legendre polynomials $L_m(t)$ on the interval $[-1, 1]$ are given by the following recursive formula

$$\begin{cases} L_0(t) = 1, \qquad L_1(t) = t, \\ (m+1)L_{m+1}(t) = (2m+1)tL_m(t) - mL_{m-1}(t), \qquad m = 1, 2, 3, ... \end{cases}$$

The hybrid functions $h_{km}(t), k = 1, 2, ..., K, m = 0, 1, ..., M-1$; on the interval [0,T) are defined as

$$h_{km}(t) = b_k(t) L_m(d_k^{-1}(2t - t_{k-1} - t_k)).$$

So

$$h_{km}(t) = \begin{cases} L_m(d_k^{-1}(2t - t_{k-1} - t_k)), & t_{k-1} \leq t < t_k, \\ 0, & otherwise, \end{cases}$$

where $d_k = t_k - t_{k-1}, k = 1, 2, ..., K$. Let

$$H_k(t) = [h_{k0}(t), \cdots, h_{k,M-1}(t)]^\tau, \qquad H(t) = [H_1^\tau(t), \cdots, H_K^\tau(t)]^\tau.$$

is the transpose.We have the operational properties of hybrid functions

$$\int_0^t H(s)ds = PH(t), \qquad \int_t^\tau H(s)ds = \bar{P}H(t),$$

where

$$P = diag(d_1, \cdots, d_K) \otimes \frac{1}{2}\left[E_{11}^{(M)} + \sum_{k=1}^{M-1}(\frac{1}{2k-1}E_{k,k+1}^{(M)} - \frac{1}{2k+1}E_{k+1,k}^{(M)})\right]$$

$$+\sum_{k=1}^{K-1}\sum_{i=1}^{K-k} d_i E_{i,i+k}^{(K)} \otimes E_{11}^{(M)},$$

$$\bar{P} = d_1 \sum_{i=1}^{K}\sum_{j=1}^{K} E_{ij}^{(K)} \otimes E_{11}^{(M)} - P,$$

$E_{ij}^{(m)}$ is the $m \times m$ matrix with 1 at its entry (i,j) and zeros elsewhere and $\otimes$ denotes Kronecker product.

3 Function Approximation

An $l-$dimensional vector function $f(t)$ on the interval $[0,T)$ is expressed as

$$f(t) \simeq \sum_{k=1}^{K}\sum_{m=0}^{M-1} f_{km}h_{km}(t), \tag{1}$$

where

$$f_{km} = \frac{2m+1}{d_k}\int_{t_{k-1}}^{t_k} f(t)L_m(d_k^{-1}(2t - t_{k-1} - t_k))dt.$$

Rewrite $f(t)$ as

$$f(t) \simeq \sum_{k=1}^{K} F_k H_k(t) = FH(t),$$

where

$$F_k = [f_{k0}, \cdots, f_{k,M-1}], \qquad F = [F_1, \cdots, F_K],$$

for corresponding $F_k$and F we denote

$$\hat{F}_k = [\hat{f}_{k0}^{\tau}, \cdots, \hat{f}_{k,M-1}^{\tau}]^{\tau}, \qquad \hat{F} = [\hat{F}_1^{\tau}, \cdots, \hat{F}_K^{\tau}]^{\tau}.$$

Let a matrix function $M(t)$ be appropriate to a vector function$f(t)$. We express $M(t)$ and $f(t)$, respectively, as

$$M(t) \simeq \sum_{k=1}^{K} \sum_{m=0}^{M-1} M_{km} h_{km}(t), \qquad f(t) \simeq \sum_{k=1}^{K} \sum_{m=0}^{M-1} f_{km} h_{km}(t),$$

then

$$M(t)f(t) \simeq \sum_{k=1}^{K} \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} M_{ki} f_{kj} h_{ki}(t) h_{kj}(t).$$

From

$$h_{ki}(t) h_{kj}(t) \simeq \sum_{m=0}^{M-1} d_{km}^{(ij)} h_{km}(t),$$

where

$$d_{km}^{(ij)} = \frac{2}{\pi} \int_{-1}^{1} S_i(t) S_j(t) S_m(t) (1-t^2)^{\frac{1}{2}} dt,$$

we have

$$M(t)f(t) \simeq \sum_{k=1}^{K} \sum_{m=0}^{M-1} \left(\sum_{i=0}^{M-1} \sum_{j=0}^{M-1} d_{km}^{(ij)} M_{ki} f_{kj} \right) h_{km}(t) = \sum_{k=1}^{K} \sum_{m=0}^{M-1} \tilde{M}_{km} h_{km}(t) = \sum_{k=1}^{K} \tilde{M}_k H_k(t), \tag{2}$$

where

$$\tilde{M}_{km} = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} d_{km}^{(ij)} M_{ki} f_{kj} = \hat{M}_{km} \hat{F}_k, \qquad \hat{F}_k = [f_{k0}^{\tau}, \ldots, f_{k,M-1}^{\tau}]^{\tau}, \qquad \hat{\tilde{M}}_k = \hat{M}_k \ \hat{F}_k,$$

$$\hat{\tilde{M}}_k = [\tilde{M}_{k0}^{\tau}, \ldots, \tilde{M}_{k,M-1}^{\tau}]^{\tau}, \qquad \hat{M}_k = [\,\hat{M}_{k0}^{\tau}, \ldots,\, \hat{M}_{k,M-1}^{\tau}]^{\tau},$$

$$\hat{M}_{km} = \left[\sum_{i=0}^{M-1} d_{km}^{(i0)} M_{ki}, \ldots, \sum_{i=0}^{M-1} d_{km}^{(i,M-1)} M_{ki}\right].$$

Therefore

$$N(s,t)f(t) \simeq \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{N}_{km}(s) h_{km}(t), \tag{3}$$

where

$$\tilde{N}_{km}(s) = \hat{N}_{km}(s)\hat{F}_k, \qquad \hat{F}_k = [f_{k0}^{\tau}, \ldots, f_{k,M-1}^{\tau}]^{\tau},$$

$$\hat{N}_{km}(s) = \left[\sum_{i=0}^{M-1} d_{km}^{(i0)} N_{ki}(s), \ldots, \sum_{i=0}^{M-1} d_{km}^{(i,M-1)} N_{ki}(s)\right],$$

$$N_{ki}(s) = \frac{2}{\pi}\int_{-1}^{1} N(s, 2^{-1}(d_k t + t_{k-1} + t_k))\, S_i(t)\, (1-t^2)^{\frac{1}{2}}\, dt.$$

Let

$$\begin{aligned} w(t) &= \textstyle\int_{t_0}^{t_f} N(t,s) f(s) ds \simeq \sum_{k=1}^{K} \sum_{m=0}^{M-1} w_{km} h_{km}(t),\\ \hat{N}_{km}(t) &\simeq \textstyle\sum_{j=1}^{K}\sum_{l=0}^{M-1} \hat{N}_{km}^{(jl)} h_{jl}(t), \end{aligned} \tag{4}$$

where

$$\hat{N}_{km}^{(jl)} = \left[\sum_{i=0}^{M-1} d_{km}^{(i0)} M_{ki}^{(jl)}, \ldots, \sum_{i=0}^{M-1} d_{km}^{(i,M-1)} M_{ki}^{(jl)}\right],$$

$$N_{ki}^{(jl)}(s) = \frac{2}{\pi}\int_{-1}^{1} N_{ki}(2^{-1}(d_j t + t_{j-1} + t_j))\, S_l(t)\, (1-t^2)^{\frac{1}{2}}\, dt.$$

By Eq. (3) we have

$$w(t) \simeq \sum_{j=1}^{K}\sum_{l=0}^{M-1}\sum_{k=1}^{K}\sum_{m=0}^{M-1} \hat{N}_{km}^{(jl)} \left(\int_{t_{k-1}}^{t_k} h_{km}(t) dt\right) \hat{F}_k h_{jl}(t).$$

So

$$\hat{W} = \sum_{k=1}^{[\frac{M+1}{2}]} \sum_{i=1}^{K}\sum_{j=1}^{K} E_{ij}^{(K)} \otimes \frac{d_j}{2k-1} N_{j,2k-1}^{(i)} \hat{F}, \tag{5}$$

where

$$N_{j,2k-1}^{(i)} = \left[N_{j,2k-1}^{(i0)^\tau}, \ldots, N_{j,2k-1}^{(i,M-1)^\tau}\right]^\tau.$$

4 Analysis of Systems of Integral Equations

Consider the following integral equations system described by:

$$x(t) + \int_{t_0}^{t_k} K(t,s)x(s)ds + u(t) = 0\,, \qquad t \in [t_0, t_k], \tag{6}$$

where $x(t)$and $u(t)$ are known and unknown n-dimensional vector function.
By using Eqs. (1) –(4) we approximate the quantities $x(t)$, $\int_{t_0}^{t_k} K(t,s)x(s)ds$ and $u(t)$with:

$$x(t) \simeq \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{A}_{km} h_{km}(t), \qquad u(t) \simeq \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{B}_{km} h_{km}(t),$$
$$int_{t_0}^{t_k} K(t,s)x(s)ds \simeq \sum_{k=1}^{K}\sum_{m=0}^{M-1} w_{km} h_{km}(t).$$

Integrating Eq. (6) from $t_0$to t and combining Eqs. (2),(3) and (5) we obtain

$$[X_1,\ldots,X_K]H(t) - [X_{01},\ldots,X_{0K}]H(t) = \int_{t_0}^{t} \{[\tilde{A}_1,\ldots,\tilde{A}_K] + [W_1,\ldots,W_K] + [\tilde{B}_1,\ldots,\tilde{B}_K]\}H(s)ds,$$

where

$$X_k = [x_{k0},\ldots,x_{k,M-1}], \qquad X_{0k} = [x_0, 0, \ldots, 0], \qquad k = 1,2,\ldots K.$$

Thus

$$[X_1,\ldots,X_K] - [X_{01},\ldots,X_{0K}] = \{[\tilde{A}_1,\ldots,\tilde{A}_K] + [W_1,\ldots,W_K] + [\tilde{B}_1,\ldots,\tilde{B}_K]\}P.$$

Using Kronecker product we rewrite the above equation as

$$\hat{X} - \hat{X}_0 = (P^\tau \otimes I_n)\,\{[\hat{\tilde{A}}_1^\tau, \ldots, \hat{\tilde{A}}_K^\tau]^\tau + [\hat{W}_1^\tau, \ldots, \hat{W}_K^\tau]^\tau + [\hat{\tilde{B}}_1^\tau, \ldots, \hat{\tilde{B}}_K^\tau]^\tau\},$$

where

$$\hat{X} = [\hat{X}_1^\tau, \ldots, \hat{X}_K^\tau]^\tau, \qquad \hat{X}_k = [x_{k0}^\tau, \ldots, x_{k,M-1}^\tau]^\tau, \qquad \hat{\tilde{A}}_k = [\tilde{A}_{k0}^\tau, \ldots, \tilde{A}_{k,M-1}^\tau]^\tau,$$
$$\hat{X}_0 = [\hat{X}_{01}^\tau, \ldots, \hat{X}_{0K}^\tau]^\tau, \qquad \hat{X}_{0k} = [x_0^\tau, 0^\tau, \ldots, 0^\tau]^\tau, \qquad k = 1, 2, \ldots, K.$$

$\hat{W}_K$ and $\hat{\tilde{B}}_K$ have the similar meaning as $\hat{\tilde{A}}_K$. So

$$l\hat{X} - \hat{X}_0 = (P^\tau \otimes I_n)\,\{[(\hat{A}_1\hat{X}_1)^\tau, \ldots, (\hat{A}_K\hat{X}_K)^\tau]^\tau + \hat{W} + [(\hat{B}_1\hat{U}_1)^\tau, \ldots, (\hat{B}_K\hat{U}_K)^\tau]^\tau\}$$

$$= (P^\tau \otimes I_n)\left[\sum_{k=1}^{K}(E_{kk}^{(K)} \otimes \hat{A}_k)\hat{X} + \sum_{k=1}^{[\frac{M+1}{2}]}\sum_{i=1}^{K}\sum_{j=1}^{K} E_{ij}^{(K)} \otimes \frac{d_j}{2k-1} N_{j,2k-1}^{(i)}\hat{X} + \sum_{k=1}^{K}(E_{kk}^{(K)} \otimes \hat{B}_k)\hat{U}\right].$$

Therefore

$$\hat{X} = \Gamma\hat{U} + \Omega, \tag{7}$$

where

$$\Gamma = [I_{MKn} - (P^\tau \otimes I_n)\phi]^{-1}\,(P^\tau \otimes I_n)\sum_{k=1}^{K}(E_{kk}^{(K)} \otimes \hat{B}_k),$$

$$\phi = \sum_{k=1}^{K}(E_{kk}^{(K)} \otimes \hat{A}_k) + \sum_{k=1}^{[\frac{M+1}{2}]}\sum_{i=1}^{K}\sum_{j=1}^{K} E_{ij}^{(K)} \otimes \frac{d_j}{2k-1} N_{j,2k-1}^{(i)},$$

$$\Omega = [I_{MKn} - (P^\tau \otimes I_n)\phi]^{-1}\,\hat{X}_0.$$

5 Numerical Examples

In this section we want to use this method to solve integral equations.

Example 1. In this example we solve equation

$$x(t) = \int_0^{\frac{\pi}{2}} k(t,s)f(s)ds + u(t)$$

where

$$
\begin{aligned}
k(t,s) &= t\sin(s+t) - s\cos(s-t)\\
g(t) &= \sin(t) + \frac{1}{16}(-2\pi(2t-1)\cos(t) + (4+\pi^2-8t)\sin(t))
\end{aligned}
$$

and the exact solution is $x(s) = \sin(s)$.
By using Eq. (7) and choosing $M = 3, K = 2$ and $M = 4, K = 3$ we have the numerical results that shown in Table 1.

t Values	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$E_{(M=3,K=2)}$	3.11×10^{-11}	6.41×10^{-9}	2.41×10^{-9}	5.81×10^{-10}	1.53×10^{-8}
$E_{(M=4,K=3)}$	0	4.27×10^{-11}	8.82×10^{-10}	1.36×10^{-10}	9.11×10^{-10}

Table 1: Numerical results for Example 1

Example 2. Consider the following system of Fredholm integral equation

$$
\begin{cases}
u_1(t) = \int_{-1}^{1}\sin(t^2+s)x_1(s)ds - \int_{-1}^{1}3se^{t^2s^2}x_2(s)ds,\\
u_2(t) = -\int_{-1}^{1}3\cos(ts)x_1(s)ds + \int_{-1}^{1}ste^{3ts^2}x_2(s)ds,
\end{cases}
$$

where

$$
\begin{cases}
u_1(t) = -\frac{3e^{t^2-1}(-e+e^{t^4})}{2(1+t^2)} - \frac{\sin(t^2+3t)+\sin(t^2-3)}{6} + \frac{(\sin(t-t^2)+\sin(t^2+1))}{2},\\
\\
u_2(t) = \frac{e^{3t^3+t^2}-e^{3t+1}}{2(e+3e^t)} + \frac{-6\cos 2t\cos t^2+3t\sin 2\sin t+6\cos t(\cos 2-t\sin t\sin t^2)}{t^2-4},
\end{cases}
$$

and the exact solutions are $x_1(s) = \sin 2s$ and $x_2(s) = e^{s^2-1}$. Numerical results for hybrid solutions of this equation with $M = 4, K = 3$ and $M = 5, K = 3$ are shown in Table 2.

Conclusion

Using the excellent properties of operational matrices of the hybrid function of general block-pulse functions and the Legendre polynomials, the general algorithms for system of integral equations are derived. This paper's method has some advantages, method is easy to apply for first and second kind system of integral equations, also we need less computations than other methods. In some methods kernels of system must be satisfied in some conditions such as being

	E_{x_1}		E_{x_2}	
x	$M=4, K=3$	$M=5, K=3$	$M=4, K=3$	$M=5, K=3$
-1	3.2154×10^{-6}	2.0180×10^{-8}	8.1426×10^{-4}	1.0021×10^{-6}
-0.75	1.5841×10^{-8}	5.6211×10^{-10}	3.1102×10^{-6}	5.2984×10^{-7}
-0.5	1.9427×10^{-8}	8.1247×10^{-9}	6.3523×10^{-8}	7.9416×10^{-8}
-0.25	5.3974×10^{-7}	2.1144×10^{-9}	1.2488×10^{-8}	6.3529×10^{-10}
0	2.3819×10^{-8}	3.6284×10^{-11}	7.1515×10^{-6}	1.3025×10^{-9}
0.25	5.1982×10^{-7}	1.1544×10^{-11}	1.2343×10^{-7}	8.2474×10^{-10}
0.5	1.2014×10^{-6}	4.2931×10^{-9}	5.2110×10^{-8}	4.1444×10^{-9}
0.75	7.3618×10^{-6}	3.3213×10^{-9}	6.1024×10^{-7}	4.1625×10^{-9}
1	5.7211×10^{-5}	2.8417×10^{-7}	7.1111×10^{-5}	3.8171×10^{-7}

Table 2: Numerical results for Example 2

separable but the method of this paper do not have such conditions. The illustrative examples demonstrate that this technique is convenient for application.

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