Mathematical Sciences Vol. 4, No. 2 (2010) 123-133

More on the regularity property

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Abstract

In this paper, we giving certain properties of the commuting regular semigroups, we get significant results on semigroups. Our investigation involve certain interesting class of commuting regular semigroups.

Keywords: commuting regular semigroups, semigroups, regular semigroups.

1 Introduction

We use S to denote a semigroup. An element a of a semigroup S is called regular if there exists x in S such that $axa = a$. The semigroup S is called regular if all its elements are regular. For elements a and b of a semigroup S, b is called the inverse of a if and only if both of the relations $aba = a$ and $bab = b$ holds. The set of inverses of an element $a \in S$, denote by $V(a)$. An element s of semigroup S is called cancellable if for every r and t, $sr = st$ implies $r = t$. The semigroup S is called cancellative, if all elements of S are cancellable. A semigroup S is called a rectangular band if $aba = a$ for all $a,\ b$ in S, (see [7]).

A semigroup S is called commuting regular (see [4, 5, 11]) if and only if for each $x,\ y \in S$ there exists an element z of S such that $xy = yxzyx$. Also, a two-sided (left,

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right) ideal I of a semigroup is said to be commuting regular two-sided (left, right) ideal if for every $a, b \in I$ there exists an element $c \in I$ such that $ab = bacba$.

Following [8], we note that a semigroup S with 0, is called a quasi -reflexive semigroup with 0 if and only if for all left (right or two sided) ideals A and B of S, $AB = 0$ yields $BA = 0$.
It is known [4] that every commuting regular semigroup with 0 is quasi-reflexive, but not vice-versa and a direct product of a family $\{S_i \mid i \in I\}$ of semigroups with 0 is commuting regular if and only if each S_i is commuting regular.

Proposition 1.1 *([10]) An idempotent $e \neq 0$ of a semigroup S with zero element 0 is primitive if and only if e is the only non-zero idempotent of subsemigroup eSe.*

Proposition 1.2 *([4]) Let S be a commuting semigroup with 0 and $e \neq 0$ is an idempotent element of S. Then the following statements are equivalent:*
(1) Se is a 0-minimal left ideal of S,
(2) eSe is a division subgroup with 0 of S,
(3) eSe is a 0-minimal commuting regular quasi-ideal of S.

We recall the following definitions from [7] and [8]:

Definition 1.3 *If a is an element of a semigroup S, the smallest left ideal of S containing a is $Sa \bigcup \{a\}$ and denoted by S^1a. An equivalence $\pounds$ on S is define by the rule that $a \; \pounds \; b$ if and only if $S^1a = S^1b$. Similarly, we define the equivalence $\Re$ by the rule that $a \; \Re \; b$ if and only if $aS^1 = bS^1$.*

The equivalence $D = \pounds \circ \Re = \Re \circ \pounds$ is a two-sided analogue of $\pounds$ and $\Re$. Also the equivalence $\Im$ by the rule $a \; \Im \; b$ if and only if $S^1aS^1 = S^1bS^1$. Following [1], if S is a commuting regular semigroup, then $D = \Im$.

2 Some properties for commuting regular semigroups

Some new results of the commuting regular semigroups are as follows. We omitted the proofs where they are easy.

Proposition 2.1 *Let S be a rectangular band semigroup. Then S is commutative if and only if S is commuting regular .*

Proposition 2.2 *Let S be a commuting regular semigroup then for every $a \in S$, $aS = Sa$.*

Proposition 2.3 *If S is a commuting regular semigroup, then every left ideal Sa^2 is generated by an idempotent.*

proof Let $a \in S$, there exists $b \in S$ such that $a^2 = a^2ba^2$. So $ba^2 = ba^2ba^2$ and $e = ba^2$ is an idempotent. We show that $Sa^2 = Se$. Let $y \in Sa^2$, there exists $r \in S$ such that $y = ra^2$ and so $y = ra^2 = ra^2ba^2 = ra^2e$. Therefore $Sa^2 \subseteq Se$. Also $e = ba^2 \in Sa^2$ and $Se \subseteq Sa^2$.

Note that if $\alpha : R \to S$ is a homomorphism semigroups and R is a commuting regular semigroup, then $R/Ker(\alpha)$ and $\alpha^{-1}(S)$ are commuting regular.

Proposition 2.4 *If S be a commuting regular semigroup, then D_{a^2} is regular for all $a \in S$.*

Proof Since $a \in S$ then $\exists x \in S$ such that $a^2 = a^2xa^2$ so a^2 is regular element of S and by proposition 3.1 of [9], D_{a^2} is regular.

Proposition 2.5 *If S be a commuting regular semigroup with zero. If S has no zero divisor, then a^2 has inverse, for all $a \in S$.*

Proof $a^2 = a^2xa^2$, if $a^2 = b$ then $b = bxb$. Also $xb = xbxb$, so $(x - xbx)b = 0$ and therefore $x = xbx$. Thus x is an inverse of $b = a^2$.

Proposition 2.6 *If S is a commuting regular semigroup, then so is each homomorphic image of S.*

Proof Let α be an epimorphism from semigroup S into a semigroup S' and $a,\ b \in S'$, then there exist $r,\ s \in S$ such that $a = \alpha(r)$ and $b = \alpha(s)$. Since S is a commuting regular, there exists t such that $rs = srtsr$ and

$$ab = \alpha(r)\alpha(s) = \alpha(rs) = \alpha(srtsr) = \alpha(s)\alpha(r)\alpha(t)\alpha(s)\alpha(r) = bacba,$$

where $c = \alpha(t)$.

Lemma 2.7 *The center of a commuting regular semigroup S is commuting regular.*

Proof Let $a, b \in Z(S)$, there exists $x \in S$ such that $ab = baxba = (ba)^2x = x(ba)^2$. So $abx = (ba)^2x^2 = x^2(ba)^2$. Therefore $ab = baxba = (ba)^2x^2ba = ba(bax^2)ba$. We show that $bax^2 \in Z(S)$. Note that $bax \in Z(S)$ because if $y \in S$ then,

$$(bax)y = ba(xy) = (xy)ba = (xy)(baxba) = (xba)y(xba) = x(ba)^2yx = bayx = y(bax)$$

and so

$$(bax^2)y = (bax)(xy) = xy(bax) = x(yba)x = x(bay)x = (xba)yx = y(xba)x = y(bax^2).$$

If we consider $bax^2 = t$, then $ab = ba(bax^2)ba = batba$ and $Z(S)$ is a commuting regular semigroup.

Example 2.8 *If the semigroup $S = \{0, a, b, c\}$ is defined by the multiplication table,*

	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	c	b	c
c	0	c	0	0

then S is not commuting regular semigroup but $I = aS = \{0, a\}$ is a commuting regular ideal.

Example 2.9 *Consider the bicyclic semigroup $B = \langle a,\ b|ab = 1\rangle$, (see [2, 7]). It is clear that for every positive integer m, n, p and q we have,*

$$(b^m a^n)(b^p a^q) = b^{m-n+t} a^{q-p+t} \quad (t = max(n, p)).$$

Suppose that $x = b^m a^n$ and $y = b^p a^q$ are arbitrary elements of B. If $q \geq p$ and $m \geq n$, then we get $t = b^{t+q-p} a^{t+m-n}$ where, $t = max(n, p)$. So,

$$\begin{aligned} yxtyx &= (b^p a^q)(b^m a^n)(b^{t+q-p} a^{t+m-n})(b^p a^q)(b^m a^n) \\ &= (b^p a^q) b^m (a^n b^{t+q-p})(a^{t+m-n} b^p) a^q (b^m a^n) \\ &= (b^p a^q)(b^m b^{t+q-p-n})(a^{t+m-n-p} a^q)(b^m a^n) \quad (\textit{for, } t+q-p \geq n \textit{ and } t+m-n \geq p) \\ &= (b^p a^q)(b^{m+t+q-p-n} a^{t+m-n-p+q})(b^m a^n) \\ &= b^p (a^q b^{m+t+q-p-n})(a^{t+m-n-p+q} b^m) a^n \\ &= b^p b^{m+t-p-n} a^{t-n-p+q} a^n \quad (\textit{for, } q+t-p+m-n \geq q \textit{ and } m+t-n+q-p \geq m) \\ &= b^{m-n+t} a^{q-p+t} \\ &= xy. \end{aligned}$$

If $p \geq q$ or $n \geq m$ then with a similar method, for every elements x and y of B, there exists $t \in B$, such that $xy = yxtyx$. Thus B is a commuting regular semigroup.

Lemma 2.10 *Suppose that S is a commuting regular semigroup then:*
(i) Every idempotent element is central, i.e., $Id(S) \subseteq Z(S)$.
(ii) For each $x,\ y \in S$, there exist $s,\ t \in S$, such that $xy = sx = yt$.

Proof The proof is similar methods used in the Theorem *I*, of [11]. For idempotents e and f of a semigroups S the intersection of the right ideals Se and Sf is not exactly equal to the ideal Sef, but for commuting regular semigroups we have:

Proposition 2.11 *If e and f are idempotents in a commuting regular semigroup S, then $Se \bigcap Sf = Sef$.*

Proof If $z = sef \in Sef$, then $z \in Sf$ and $z = sfe \in Se$, by the Lemma 2.10. Hence $z \in Se \bigcap Sf$.

Conversely, if $z = xe = yf \in Se \bigcap Sf$ then $z = yf = (yf)f = zf = xef$ and so $z \in Sef$. Following [6, 7] and the definition of an ideal of a semigroup. Then,

Theorem 2.12 *Let S be a cancellative semigroup and I be an ideal of S such that non zero idempotent e belong to I, S is a commuting regular semigroup if and only if I is a commuting regular.*

Proof Let S be a commuting regular semigroup and $a, b \in I$, there exists $c \in S$ such that $ab = bacba$. By the Lemma 2.10, $abe = (bacba)e = ba(ce)ba$. Therefore I is a commuting regular semigroup.

Conversely, let $a, b \in S$, then $ae, be \in I$ and there exists $c \in R$ such that $aebe = (be)(ae)c(be)(ae)$. So $abe = (bacba)e$, by the Lemma 2.10. Then $ab = bacba$. Therefore S is a commuting regular semigroup.

Proposition 2.13 *Let e be an idempotent element of a semigroup S. If S is a commuting regular semigroup, then $S' = eSe$ is a commuting regular semigroup with identity.*

Proof Clearly S' is a semigroup with identity. Let exe and eye are arbitrary elements in S'. Then, there are t_1, t_2, t_3, t_4 in S such that

$$\begin{aligned}(exe)(eye) &= e(xey)e = e(yxet_1yxe)e \\ &= e(y(ext_2ex)t_1y(ext_2ex))e \\ &= eyex(t_2ext_1yext_2)exe \\ &= eyex((et_2t_3et_2)xt_1(ext_2yt_3ext_2y))exe \\ &= eyexe(t_2t_3et_2xt_1ext_2yt_3)(xt_2et_4xt_2e)yexe \\ &= (eye)(exe)(et_5e)(eye)(exe),\end{aligned}$$

where, $t_5 = t_2t_3et_2xt_1ext_2yt_3xt_2et_4xt_2$.

Proposition 2.14 *Let S be a commuting regular semigroup with the set E of the idempotents. Let $e,\ f \in E$ and $a,\ b \in S$. We define the sandwich set $S(e, f)$, by*

$$S(e, f) = \{g \in V(ef) \cap E \ : \ ge = fg = g\}.$$

Then, $S(e,f)$ is a subsemigroup of S with exactly one element.

Proof By Theorem 3.9 of [1], $S(e,f)$ is a regular subsemigroup of S.
Now, Let $x \in S(e,f)$, then

$$\begin{aligned} x &= x(ef)x = x(xef) = (xefx)(xef) \\ &= (efxx)(xef) = ef(xxx)ef \quad \text{(for, S(e,f) is a semigroup)} \\ &= ef \end{aligned}$$

This shows that, $S(e,f)$ is a subsemigroup of S with exactly one element.

Proposition 2.15 *Let e be a non-zero idempotent of a commuting regular semigroup S with zero element 0. If eS is a 0-minimal right ideal of S, then e is primitive in S.*

Proof Assume eS is a 0-minimal right ideal of S. By proposition 1.2, eSe is a subgroup with 0 of S. Obviously e is the only non-zero idempotent of eSe. By proposition 1.1, e is primitive.

Proposition 2.16 *Let S be a commutative semigroup. If S is 0-simple then S is commuting regular semigroup.*

Proof Suppose that S is 0-simple. Then S^2 is an ideal of S and hence, since $S^2 \neq 0$, we must have $S^2 = S$. Hence, $S^5 = S$. Now for any $a,\ b$ in $S - \{0\}$, the subset $abSab$ is an ideal of S, hence $abSab = 0$ or $abSab = S$. If $abSab = 0$ then $I = \{x|\ axSax = 0\}$ contains elements other 0. So $I = S$ and therefore, $aSaS = aS^2aS = 0$. Now, Let $J = \{x \in S|\ aSxS = 0\}$. So $J = 0$, and therefore, $S = S^5 = 0$, contradiction. Thus, $abSab = S$, but $x = ba \in S$, so, there exist $t \in S$ such that $abtab = ba$.

Corollary 2.17 *Let S be a commutative semigroup. If S is simple then S is commuting regular.*

Proposition 2.18 *A semigroup S with zero is a 0-group if and only if $aS = Sa = S,\ \ \forall a \in S - \{0\}$.*

Example 2.19 *Every* 0*-group is a commuting regular semigroup.*

Proposition 2.20 *Let S be a commuting regular semigroup. If S is a* 0*-simple. then S is a* 0*-group.*

Proof We have $aS = Sa$, aS is an ideal in S so $aS = 0$ or $aS = S$, but $S^2 \neq 0$, therefore, $aS = S$ and by proposition 1.6 of [7], S is a 0-group.

3 Commuting regular and lattice of Congruences

Firstly, we recall the following definitions from [7]:

Definition 3.1 *Let S be a semigroup. A relation R on the set S is called compatible if*

$$(\forall s, t, a \in S) \quad [(s,t) \in R \text{ and } (s',t') \in R] \Rightarrow (ss', tt') \in R.$$

A compatible equivalence relation is called congruence.

If ρ is a congruence on a semigroup S then we can define a binary operation on the quotient set $\frac{S}{\rho}$ in a natural way as follows:

$$(a\rho)(b\rho) = (ab)\rho.$$

Proposition 3.2 *Let ρ be a congruence on commuting regular semigroup S. Then $\frac{S}{\rho}$ is a commuting regular semigroup.*

Remark: If Y is a non-empty subset of a partially ordered set X, then element c in X is a lower bound for Y if $c \leq y$ for every $y \in Y$. If the set of lower bounds of Y is non-empty and has a maximum element d, then d is called the greatest lower bound and denoted by $d = \wedge\{y \ : \ y \in Y\}$. If $Y = \{a, \ b\}$, $d = a \wedge b$. If $(X, \ \leq)$ is such that $a \wedge b$ exists for every $a, \ b \in X$, $(X, \ \leq)$ is called a lower semilattice. Analogous definitions exists for the least upper bound $\vee\{y : \ y \in Y\}$ of a non-empty subset Y of

X and for an upper semilattice. If $(X, \ \leq)$ is both a lower semilattice and an upper semilattice, then it is a lattice. If non-empty M of lattice $L = (L, \leq, \wedge, \vee)$ such that

$$a,\ b \in M \Longrightarrow a \wedge\ b, \ \ a \vee b \in M$$

then M is a sublattice [9].

Proposition 3.3 *Let $(E, \leq)$ be a lower semilattic. Then $(E, \wedge)$ is commuting regular semigroup.*

Proof By proposition 3.3 of [7], $(E, \wedge)$ is a commutative semigroup of idempotents and so for all $a,\ b$ in E, if $a \leq b$ then $ab = a = ba$. So, $ab = (ba)a(ba)$.

Corollary 3.4 *If ρ and δ are equivalences on semigroup S (congruence on semigroup S) such that $\rho \circ \delta = \delta \circ \rho$, then ρ and delta are commuting regular elements.*

Proof

$$(\rho \circ \delta)^2 = \rho \circ (\delta \circ \rho) \circ \delta = (\rho \circ \rho) \circ (\delta \circ \delta) = (\rho \circ \delta),$$

so $(\rho \circ)^n = \rho \circ \delta$ for every n in N. Therefore,

$$\rho \circ \delta = (\delta \circ \rho)(\delta \circ \rho)(\delta \circ \rho).$$

Proposition 3.5 *Let G be a group, then $\zeta(G)$ (the set of congruences on G), is a commuting regular semigroup.*

Proof Let $(a, b) \in \rho \circ \delta$. Then, There exists c in G such that $(a, c) \in \rho,\ (c, b) \in \delta$. It follows that

$$a = cc^{-1}a \equiv bc^{-1}a \quad (mod\ \delta),$$

$$bc^{-1}a \equiv bc^{-1}c = c \ \ (mod\ \rho).$$

Thus $(a, b) \in \delta \circ \rho$ and so $\rho \circ \delta \subset \delta \circ \rho$. Similarly, $\delta \circ \rho \subseteq \rho \circ \delta$. Now the result follows from above corollary.

Example 3.6 *If M, N are normal subgroups of G, then*

$$\rho_M \circ \rho_N = \rho_{MN} = (\rho_N \circ \rho_M) \circ \rho_M \circ (\rho_N \circ \rho_M) = \rho_{(NM)M(NM)}.$$

Where $\rho_N = \{(a,\ b) \in G \times G \mid ab^{-1} \in N\}$.

Proposition 3.7 *Let κbe a sublattice of the lattice $(\zeta(S), \subseteq, \cup, \cap)$ of congruences of a semigroup S, and suppose that $\rho \circ \delta = \delta \circ \rho$ for all ρ, δ in κ. Then κ is a commuting regular lattice.*

Corollary 3.8 *The lattice of congruences on a group is commuting regular lattice.*

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