Mathematical Sciences ***Vol. 4, No. 2 (2010) 135-142***

Common fixed point theorem with w-distance

M. Boujary[1]

Department of Mathematics, Islamic Azad University, Shahrood Branch, Shahrood, Iran.

Abstract

In this paper, we prove the existence of common fixed point for mappings defined on complete metric spaces with w-distance p satisfying a general contractive inequality of type integral.

Keywords: Common fixed point, w-distance.

1 Introduction and Preliminaries

Jungck initiated a study of common fixed points of commuting maps. He proved the following common fixed point theorem in [3].

Theorem 1.1 *A continuous self map of a complete metric space (X, d) has a fixed point iff there exist $c \in (0, 1)$ and a mapping $g : X \to X$ which commute with f and satisfies :$g(X) \subset f(X)$ and $d(g(x), g(y)) \leq cd(f(x), f(y))$ for all x, y in X. In fact, f and g have a unique common fixed point.*

Than , he obtained the Banach contraction principle as a consequence of it. Further, Jungck [4] made generalization of commuting maps by introducing the notion of compatible mappings.

[1]E-mail Address: m.boujary@gmail.com

In other hand, Kada and et al in [5] for first time introduced definition of w-distance and then give some Lemmas which are connected with w-distance.

Definition 1.2 Let X be a metric space with metric d. Then a function $p : X \times X \longrightarrow [0, \infty)$ is called w-distance on X if the following satisfy:
(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$.
(2) $p(x, .)$ is lower semi-continuous , i.e. if $x \in X$ and $y_n \to y$ on X then $p(x, y) \leq \liminf_n p(x, y_n)$.
(3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

For example the metric d is w-distance in every metric space (X, d).

Example 1.3 *Let X be a normed linear space with norm $\| . \|$. Then a function $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = \| y \| \qquad for\, every \qquad x, y \in X$$

is a w-distance on X.

Lemma 1.4 *(See [5]) Let (X, d) be a metric space and p be a w-distance on X. If $\{x_n\}$ is a sequence in X such that $\lim_n p(x_n, x) = \lim_n p(x_n, y) = 0$ then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$ then $x = y$.*

Lemma 1.5 *(See [5]) Let p be a w-distance on metric space (X, d) and $\{x_n\}$ be a sequence in X such that for each $\varepsilon > 0$, there exist $n \in N$ such that $m > n > N$ implies $p(x_n, x_m) < \varepsilon$,then $\{x_n\}$ is a Cauchy sequence .*

Also, Branciari in [1] established a fixed point result for an integral-type inequality, that is generalization Banach's contraction principle. Baraciari in [1] proved the following fixed point theorem.

Theorem 1.6 *Let (X, d) be a complete metric space,$c \in]0, 1[$, and let $f : X \to X$ a mapping such that for each $x, y \in X$,*

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt.$$

where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable on each compact subset of R^+ ,nonnegative, and for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0.$$

then f has a unique fixed point $a \in X$ such that $lim_{n\to\infty} f^n(x) = a$, for each $x \in X$.

In this paper, we prove a common fixed point theorem which generalizations of results in [1] and [3] by w- distance. First, we prove main theorem. Then, we discuss the relation between it and Branciari's Theorem and Jungck's common fixed point Theorem .

2 Main Results

Let N represent the set of natural numbers , R the set of real numbers , and R^+ the set of nonnegative real numbers.

The proof of the following theorem is based on an argument similar to the one used by Baraciari[1]

Theorem 2.1 *Let (X, d) be a complete metric space, let p be a w-distance on X and let $f : X \to X$ a mapping . Then f has a fixed point in X iff there exists $c \in]0, 1[$ and mapping $g : X \to X$ which commutes with f such that $g(X) \subset f(X)$ and for each $x, y \in X$, satisfies*

$$\int_0^{p(g(x),g(y))} \varphi(t)dt \leq c \int_0^{p(f(x),f(y))} \varphi(t)dt. \quad (1)$$

where $\varphi : R^+ \to R^+$ *is a Lebesgue-integrable mapping which is summable on each compact subset of* R^+*, nonnegative, and for each* $\epsilon > 0, \int_0^{\epsilon} \varphi(t)dt > 0$*. Indeed,* f *and* g *have a unique common fixed point if* (1) *holds.*

Proof. Suppose that $f(a) = a$ for some $a \in X$. Define $g : X \to X$ by $g(x) = a$ for $x \in X$. Then $g(f(x)) = a$ and $f(g(x)) = f(a) = a$, so $g(f(x)) = f(g(x))$ for all $x \in X$ and g commutes with f. Moreover, $g(x) = a = f(a)$ for all $x \in X$ so that $g(X) \subset f(X)$. Also (1) is holds.
On the other hand, suppose there is a mapping $g : X \to X$ which commutes with f and for which (1) holds. We will show, this condition is sufficient to ensure that f and g have a unique common fixed point. Let $x_0 \in X$ and let x_1 be such that $f(x_1) = g(x_0)$. In general, choose x_n so that

$$f(x_n) = g(x_{n-1}). \quad (2)$$

This is true because $g(X) \subset f(X)$. By (1) and (2.3) , we have

$$\int_0^{p(f(x_n),f(x_{n+1}))} \varphi(t)dt\varphi(t)dt \leq \cdots \leq c^{n-1} \int_0^{p(f(x_0),f(x_1))} \varphi(t)dt. \quad (3)$$

Then we have

$$\lim_{n\to\infty} \int_0^{p(f(x_n),f(x_{n+1}))} \varphi(t)dt = 0 \quad (4)$$

which (1) implies that

$$\lim_{n\to\infty} p(f(x_n), f(x_{n+1})) = 0 \quad (5)$$

Now, we show that $\{f(x_n)\}$ is Cauchy. Suppose that $\{x_n\}$ is not p- Cauchy , that is ,

$$\exists \epsilon > 0, \forall N_0, \exists m_\epsilon, n_\epsilon \in N\ (m_\epsilon > n_\epsilon > N_0\ ,\ p(f(x_m), f(x_n)) \geq \epsilon.)$$

We choose the sequences $\{m_k\}_{k\in N}, \{n_k\}_{k\in N}$ such that for $k \in N$, m_k is minimal in the sense that $p(f(x_{m_k}), f(x_{n_k})) \geq \epsilon$, but $p(f(x_i), f(x_{n_k})) < \epsilon$ for each $i \in \{n_k+1, \cdots, m_k-$

1}.We have $p(f(x_{m_k}),f(x_{n_k}))\longrightarrow \epsilon+$ as $k\longrightarrow +\infty$,in fact by the triangular inequality and (5)

$$\begin{aligned}\epsilon \leq\ & p(f(x_{m_k}),f(x_{n_k}))\\ \leq\ & p(f(x_{m_k}),f(x_{m_k-1}))+p(f(x_{m_k-1}),f(x_{n_k}))\\ \leq\ & p(f(x_{m_k}),f(x_{m_k-1}))+\epsilon\longrightarrow \epsilon+\end{aligned}\tag{6}$$

as $k\longrightarrow\infty$. Further, there exists $\mu\in N$ such that for each natural number $k>\mu$, one has $p(f(x_{m_k+1}),f(x_{n_k+1}))<\epsilon$; because, if exists a subsequence $\{k_j\}_{j\in N}\subseteq N$ such that $p(f(x_{m_{k_j}+1}),f(x_{n_{k_j}+1}))\geq\epsilon$, than

$$\begin{aligned}\epsilon \leq\ & p(f(x_{m_{k_j}+1}),f(x_{n_{k_j}+1}))\\ \leq\ & p(f(x_{m_{k_j}+1}),f(x_{m_{k_j}}))+p(f(x_{m_{k_j}}),f(x_{n_{k_j}}))\\ +\ & p(f(x_{n_{k_j}}),f(x_{n_{k_j}+1}))\longrightarrow\epsilon\end{aligned}\tag{7}$$

as $j\longrightarrow\infty$. We have from (1),

$$\int_0^{p(f(x_{m_{k_j}+1}),f(x_{n_{k_j}+1}))}\varphi(t)dt\leq c\int_0^{p(f(x_{m_{k_j}}),f(x_{n_{k_j}}))}\varphi(t)dt.\tag{8}$$

letting now $j\longrightarrow\infty$ in both sides of (8), we have $\int_0^\epsilon\varphi(t)dt\leq c\int_0^\epsilon\varphi(t)dt$ which is a contradiction being $c\in]0,1[$ and the integral being positive . Therefore for a certain $\mu\in N$ one has $p(f(x_{m_k}),f(x_{n_k}))<\epsilon$ for all $k>\mu$. Finally , we prove the stronger property that there that there exist a $h_\epsilon\in]0,\epsilon[$ and a N_ϵ such that for each $k>N_\epsilon$ we have $p(f(x_{m_k+1}),f(x_{n_k+1}))<\epsilon-h_\epsilon$; suppose the existence of a subsequence $\{k_j\}_{j\in N}\subseteq N$ such that $p(f(x_{m_{k_j}+1}),f(x_{n_{k_j}+1}))\longrightarrow\epsilon-$ as letting now $j\longrightarrow\infty$, then from

$$\int_0^{p(f(x_{m_{k_j}+1}),f(x_{n_{k_j}+1}))}\varphi(t)dt\leq c\int_0^{p(f(x_{m_{k_j}}),f(x_{n_{k_j}}))}\varphi(t)dt.\tag{9}$$

Again,letting $j\longrightarrow\infty$ in both sides of (9), we have the contradiction that $\int_0^\epsilon\varphi(t)dt\leq c\int_0^\epsilon\varphi(t)dt$. In conclusion , we can prove the Caushy character of $\{f(x_n)\}$. For each $k>N_\epsilon$ (N_ϵ as above)

$$\begin{aligned}\epsilon\leq\ & p(f(x_{m_k}),f(x_{n_k}))\leq p(f(x_{m_k}),f(x_{m_k+1}))+p(f(x_{m_k+1}),f(x_{n_k+1}))\\ +\ & p(f(x_{n_k+1}),f(x_{n_k}))\leq p(f(x_{m_k}),f(x_{m_k+1}))\\ +\ & (\epsilon-h_\epsilon)+p(f(x_{n_k+1}),f(x_{n_k+1}))\longrightarrow\epsilon-h_\epsilon\end{aligned}\tag{10}$$

thus $\epsilon \leq \epsilon - h_\epsilon$ which is a contradiction .This proves that $\{f(x_n)\}$ is p-Cauchy ,so Lemma 1.5 imply that it is Cauchy. Since (X,d) is a complete metric space, there exists a point $a \in X$ such that $a = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n-1})$. For each $\epsilon > 0$ there exist $N_\epsilon \in N$ such that $n > N_\epsilon$ implies $p(f(x_{N_\epsilon}), f(x_n)) < \epsilon$, but $a = \lim_{n\to\infty} f(x_n)$ and $p(x,.)$ is lower semi continuous thus

$$p(f(x_{N_\epsilon}), a) \leq \liminf_{n\to\infty} p(f(x_{N_\epsilon}), f(x_n)) \leq \epsilon$$

there for $p(f(x_{N_\epsilon}), a) < \epsilon$, we put $\epsilon = 1/k, N_\epsilon = n_k$ and we have

$$\lim_{k\to\infty} p(f(x_{n_k}), a) = 0. \tag{11}$$

In other hand, suppose $p(f(x_{n_k}), f(a))$ does not to 0 as $k \to \infty$,then there exist a subsequence $\{x_{n_{k_j+1}}\} \subseteq \{x_k + 1\}$ such that $p(f(x_{n_{k_j+1}}), f(a)) \geq \epsilon$ for a certain $\epsilon > 0$; thus we have the following contradiction

$$0 < \int_0^\epsilon \varphi(t)dt \leq \int_0^{p(f(x_{n_{k_j+1}}), f(a))} \varphi(t)dt \leq c \int_0^{p(f(x_{n_{k_j}}), a)} \varphi(t)dt \longrightarrow 0$$

as $j \to \infty$.Thus $\lim_{k\to\infty} p(f(x_{n_k}), f(a)) = 0$, but we have

$$p(f(x_{n_k}), f(a) \leq p(f(x_{n_k}), f(x_{n_k+1})) + p(f(x_{n_k+1}, f(a))$$

thus

$$\lim_{k\to\infty} p(f(x_{n_k}), f(a)) = 0. \tag{12}$$

Now (11), (12) and Lemma 1.4 implies $f(a) = a$. In this way, we have $g(a) = a$. Thus a is a common fixed point of f and g. Also, (1) implies that f and g can have only one common fixed point. Suppose there are two distinct fixed points $a, b \in X$ such that $f(a) = a$ and $f(b) = b$, then by (1) we have the following contradiction

$$0 < \int_0^{p(a,b)} \varphi(t)dt = \int_0^{p(g(a),g(b))} \varphi(t)dt \leq c \int_0^{p(f(a),f(b))} \varphi(t)dt = c \int_0^{p(a,b)} \varphi(t)dt.$$

Then $p(a,b) = 0$.In this way, we have $p(b,a) = 0$, so $a = b$. $\diamond$

Corollary 2.2 *Theorem (2.1)is a generalization Theorem (1.6), let be* $p = d$ *and* $f(x) = x$ *.*

But the converse (2.2) is not true that show its the following example.

Example 2.3 *Let be* $X = \{\frac{1}{n}|n \in N\} \bigcup \{0\}$ *, let be for each* $x, y \in X$*,* $d(x, y) = x+y$ *if* $x \neq y$*, and* $d(x, y) = 0$ *if* $x = y$*.* (X, d) *is a complete metric space. Also, we define* w*-distance* $p(x, y) = y$ *on* (X, d)*. Since for every* $x, y \in X(y \neq 0), p(x, y) = y = d(0, y)$ *thus every Branciari contraction map* f *is w-Branciari contraction , that is,*

$$\int_0^{p(f(x),f(y))} \varphi(t)dt \leq c \int_0^{p(x,y)} \varphi(t)dt \quad for\, every \quad x, y \in X$$

but its inverse is not true . Let $g(x) = \frac{1}{2}x$ *,*$f(x) = x$ *are maps on* X *and let be* $\varphi(t) = 1$ *,if* $0 \leq t \leq \frac{1}{2}, \varphi(t) = 0,$ *if* $t > \frac{1}{2}$ *,then* g *is w-Branciari contraction by* $c = \frac{3}{4}$ *but is not Branciari contraction since if* $y = \frac{1}{n}$ *then*

$$\begin{aligned} \int_0^{p(g(x),g(y))} \varphi(t)dt = & \int_0^{g(y)} \varphi(t)dt = \int_0^{\frac{1}{2n}} dt = \frac{1}{2n} \\ \leq \frac{3}{n4} = & \frac{3}{4} \int_0^{\frac{1}{n}} \varphi(t)dt = \int_0^{p(x,y)} \varphi(t)dt. \end{aligned} \tag{13}$$

But for $n \neq 1$ *, then*

$$\begin{aligned} \int_0^{d(g(\frac{1}{n}),g(1))} \varphi(t)dt = & \int_0^{\frac{1}{2n}+\frac{1}{2}} \varphi(t)dt = \frac{1}{2} > \\ \frac{3}{8} = & \frac{3}{4} \int_0^{\frac{1}{n}+1} \varphi(t)dt = \int_0^{d(\frac{1}{n},1)} \varphi(t)dt. \end{aligned} \tag{14}$$

Corollary 2.4 *Theorem (2.1)is a generalization Theorem (1.1), let be* $p = d$ *and let be* $\varphi(t) = 1$ *.*

Acknowledgment

The author want to thank Islamic Azad University, Shahrood Branch, for its financial support and the referee(s) for of suggestions on a previous version of this paper.

References

[1] Banach S. (1922) "Sur les operations dans les ensembles abstraits et leur application aux equations integrales," Fund. Math, 3, 133-181.

[2] Branciari A. (2002) "A fixed point theorem for mapping satisfying a general contractive condition of integral type," International Journal of Mathematics and Mathematical Sciences, 10, 531-536.

[3] Jungck G. (1976) "Commuting mappings and fixed points," Amer. Math. Monthly, 83, 261-263.

[4] Jungck G. (1986) "Compatible mappings and common fixed points," Int. J. Math. Math. Sci, 9, 771-779.

[5] Kada O., Suzuki T., Takahashi W. (1996) "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," Math. Japonica, 44, 381-591.