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Solving a Volterra integral equation with weakly singular kernel in the reproducing kernel space

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Abstract

In this paper, we will present a new method for a Volterra integral equation with weakly singular kernel in the reproducing kernel space. Firstly the equation is transformed into a new equivalent equation. Its exact solution is represented in the form of series in the reproducing kernel space. In the mean time, the n-term approximation $u_n(t)$ to the exact solution $u(t)$ is obtained. Some numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Keywords: Exact solution; Integral equation; Weakly singular kernel; Reproducing kernel

1 Introduction

In this paper, we consider the following second kind Volterra integral equation with weakly singular kernel in the reproducing kernel space

$$u(t) - \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)ds = f(t), t \in [0, T] \tag{1.1}$$

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where $\mu > 0$, $u(t) \in W_2^2[0,T]$, $f(t)$ is a given function and $f(t) \in W_2^2[0,T]$. Equations of this type arise from certain heat conduction problems(see Ref.[1-2]). The distinctive feature of the equation is the presence of a singularity at $t = 0$ for all values of $\mu > 0$ and at $s = 0$ for all values of $t > 0$ for $0 < \mu < 1$. This means that conventional analytical and numerical theory does not apply. In fact, this equation has been the subject of previous analysis in [3-9]. In [3], the authors presented some results on existence, uniqueness and smoothness. In [4], the author gave a spectral approach to an integral equation. In [5], [8], the authors presented product integral methods and Hermite-type collocation method for equation of this type. In [6], [7], extrapolation methods for the equation of this type was presented. In [9], the authors also gave the numerical solution of the equation of this type. It is of interest because of the rather unusual singularity. For values of $\mu > 1$, the singularity at $t = 0$ does not persist for $t > 0$. Thus the solution is quite well-behaved. However, for $0 < \mu < 1$, there are infinitely many solutions to $Eq.(1.1)$. In [5], it was proved that $Eq.(1.1)$ has a unique solution in the continuity class $C^m[0,T]$ if $f(t)$ is in $C^m[0,T]$ and $\mu > 1$. However, if $0 < \mu \leq 1$, then $Eq.(1.1)$ has a family of solutions in $C[0,T]$, of which only one has C^1 continuity(see Ref.[3]). Therefore, in the reproducing kernel space $W_2^2[0,T]$, $Eq.(1.1)$ has a unique solution for $\mu > 0$.

Reproducing kernel theory has important application in numerical analysis, differential equation, probability and statistics and so on [10-19]. Recently, using the RKM, Geng and Cui [14-19] discussed two-point boundary value problems, periodic boundary value problems. For Volterra integral equation with weakly singular kernel, however, this method has not yet been applied. The aim of this paper is to fill this gap.

In this paper, we will give the representation of exact solution to $Eq.(1.1)$ and approximate solution in the reproducing kernel space. The approach is simple and effective. We shall consider the condition of $0 < \mu < 1$. When $\mu \geq 1$, it is easier to solve $Eq.(1.1)$ by similar method.

Multiplication of both sides of $Eq.(1.1)$ by t^μ yields

$$t^\mu u(t) - \int_0^t s^{\mu-1} u(s)ds = t^\mu f(t)$$

Hence differentiation with respect to t gives

$$t^\mu u'(t) + \mu t^{\mu-1} u(t) - \frac{1}{t^{1-\mu}} u(t) = \mu t^{\mu-1} f(t) + t^\mu f'(t),$$

and multiplication by $t^{1-\mu}$ gives

$$tu'(t) + (\mu - 1)u(t) = \mu f(t) + tf'(t)$$

Remark 1.1 Note that $\lim\limits_{t\to 0} \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)ds = \frac{u(0)}{\mu}$. Therefore, if $u(0) \neq 0$ we have $u(0) \neq f(0)$, more precisely $u(0) = \frac{\mu}{\mu-1} f(0)$.

Hence $Eq.(1.1)$ can be converted into the following equivalent form

$$\begin{cases} tu'(t) + (\mu - 1)u(t) = \mu f(t) + tf'(t), 0 < t \leq T \\ u(0) = \frac{\mu}{\mu-1} f(0), \end{cases} \tag{1.2}$$

Using change of variable $\overline{u}(t) = u(t) - \frac{\mu}{\mu-1} f(0)$, put $L\overline{u}(t) = t\overline{u}'(t) + (\mu - 1)\overline{u}(t)$ and rewrite $g(t) = \mu f(t) + tf'(t) - \mu f(0)$ simply, then $Eq.(1.1)$ can further be converted into the following form

$$\begin{cases} L\overline{u}(t) = g(t), 0 < t \leq T \\ \overline{u}(0) = 0, \end{cases} \tag{1.3}$$

where $g \in W_2^1[0, T], \overline{u} \in W_2^{2,0}[0, T]$. From the uniqueness of solution to $Eq.(1.1)$ in the space $W_2^2[0, T]$, it is easy to see $Eq.(1.3)$ has a unique solution in the space $W_2^{2,0}[0, T]$. $W_2^1[0, T]$, $W_2^{2,0}[0, T]$ and $W_2^2[0, T]$ are defined in the following section.

2 Several Reproducing Kernel Spaces

1 The reproducing kernel space $W_2^{2,0}[0, T]$

The inner product space $W_2^{2,0}[0, T]$ is defined as $W_2^{2,0}[0, T] = \{u(x) \mid u, u'$ are absolutely continuous real valued functions, $u, u', u'' \in L^2[0,T], u(0) = 0\}$. The inner product in $W_2^{2,0}[0,T]$ is given by

$$(u(y), v(y))_{W_2^{2,0}} = \int_0^1 (4uv + 5u'v' + u''v'')dy, \qquad (2.1)$$

and the norm $\| u \|_{W_2^{2,0}}$ is denoted by $\| u \|_{W_2^{2,0}} = \sqrt{(u,u)_{W_2^{2,0}}}$, where $u, v \in W_2^{2,0}[0,T]$.

Theorem 2.1. *The space $W_2^{2,0}[0,T]$ is a reproducing kernel space. That is, for any $u(y) \in W_2^{2,0}[0,T]$ and each fixed $x \in [0,T]$, there exists $R_x(y) \in W_2^{2,0}[0,T]$, $y \in [0,T]$, such that $(u(y), R_x(y))_{W_2^{2,0}} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} c_1e^y + c_2e^{-y} + c_3e^{2y} + c_4e^{-2y}, & y \le x, \\ d_1e^y + d_2e^{-y} + d_3e^{2y} + d_4e^{-2y}, & y > x. \end{cases} \qquad (2.2)$$

The coefficients of the reproducing kernel $R_x(y)$ and the proof of Theorem 2.1 are given in appendix A, B.

2 The reproducing kernel space $W_2^2[0,T]$

The inner product space $W_2^2[0,T]$ is defined as $W_2^2[0,T] = \{u(x) \mid u, u'$ are absolutely continuous real valued functions, $u, u', u'' \in L^2[0,T]\}$. The inner product in $W_2^2[0,T]$ is given by

$$(u(y), v(y))_{W_2^2} = \int_0^1 (4uv + 5u'v' + u''v'')dy, \qquad (2.3)$$

and the norm $\| u \|_{W_2^2}$ is denoted by $\| u \|_{W_2^2} = \sqrt{(u,u)_{W_2^2}}$, where $u, v \in W_2^2[0,T]$.

Similarly, we can prove that $W_2^2[0,T]$ is a reproducing kernel space and obtain its reproducing kernel.

3 The reproducing kernel space $W_2^1[0,T]$

The inner product space $W_2^1[0,T]$ is defined by $W_2^1[0,T] = \{u(x) \mid u$ is absolutely continuous real value function, $u, u' \in L^2[0,T]\}$. The inner product and norm in $W_2^1[0,T]$ are given respectively by

$$(u(x), v(x))_{W_2^1} = \int_0^T (uv + u'v')dx, \ \| u \|_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

where $u(x), v(x) \in W_2^1[0,T]$. In Ref.[10], the authors proved that $W_2^1[0,T]$ is a complete reproducing kernel space and its reproducing kernel is

$$\overline{R}_x(y) = \frac{1}{2\sinh(T)}[\cosh(x+y-T) + \cosh(|x-y|-T)].$$

3 The solution of Eq.(1.3)

In this section, the solution of $Eq.(1.3)$ is given in the reproducing kernel space $W_2^{2,0}[0,T]$.

In $Eq.(1.3)$, it is clear that $L : W_2^{2,0}[0,T] \to W_2^1[0,T]$ is a bounded linear operator. Put $\varphi_i(t) = \overline{R}_{t_i}(t)$ and $\psi_i(t) = L^*\varphi_i(t)$ where L^* is the adjoint operator of L . The orthonormal system $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ of $W_2^{2,0}[0,T]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(t)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(t), (\beta_{ii} > 0, i = 1, 2, ...). \tag{3.1}$$

Theorem 3.1. *For $Eq.(1.3)$, if $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,T]$, then $\{\psi_i(t)\}_{i=1}^{\infty}$ is the complete system of $W_2^{2,0}[0,T]$ and $\psi_i(t) = L_y R_t(y)|_{y=t_i}$.*

Proof. We have

$$\begin{aligned}\psi_i(t) &= (L^*\varphi_i)(t) = ((L^*\varphi_i)(y), R_t(y)) \\ &= (\varphi_i(y), L_y R_t(y)) = L_y R_t(y)|_{y=t_i}.\end{aligned}$$

The subscript y by the operator L indicates that the operator L applies to the function of y.

Clearly, $\psi_i(t) \in W_2^{2,0}[0,T]$.

For each fixed $u(t) \in W_2^{2,0}[0,T]$, let $(u(t), \psi_i(t)) = 0, (i = 1, 2, ...)$, which means that,

$$(u(t), (L^*\varphi_i)(t)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(t_i) = 0. \tag{3.2}$$

Note that $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,T]$, hence, $(Lu)(t) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of the Theorem 3.1 is complete. □

Theorem 3.2. *If $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,T]$, then the solution of Eq.(1.3) is*

$$\overline{u}(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(t_k)\overline{\psi}_i(t). \tag{3.3}$$

Proof. Applying Theorem 3.1, it is easy to know that $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^{2,0}[0,T]$.

Note that $(v(t),\varphi_i(t)) = v(t_i)$ for each $v(t)\in W_2^1[0,T]$, hence we have

$$\begin{aligned}\overline{u}(t) &= \sum_{i=1}^{\infty}(\overline{u}(t),\overline{\psi}_i(t))\overline{\psi}_i(t)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(\overline{u}(t),L^*\varphi_k(t))\overline{\psi}_i(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(L\overline{u}(t),\varphi_k(t))\overline{\psi}_i(t)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(g(t),\varphi_k(t))\overline{\psi}_i(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}g(x_k)\overline{\psi}_i(t)\end{aligned} \tag{3.4}$$

and the proof of the theorem is complete. □

Therefore, the solution to $Eq.(1.1)$ in the space $W_2^2[0,T]$ is

$$u(t) = \overline{u}(t) + \frac{\mu}{\mu-1}f(0). \tag{3.5}$$

Now, the approximate solution $u_n(t)$ to $Eq.(1.1)$ can be obtained by the n-term intercept of the exact solution $u(t)$ and

$$u_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}g(t_k)\overline{\psi}_i(t) + \frac{\mu}{\mu-1}f(0). \tag{3.6}$$

Theorem 3.3. *Assume $u(t)$ is the solution of Eq.(1.1) and $r_n(t)$ is the error between the approximate $u_n(t)$ and the exact solution $u(t)$. Then the error $r_n(t)$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^2}$.*

Proof. From (3.5), (3.6), it follows that

$$\begin{aligned}\| r_n \|_{W_2^2} &= \| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(t_k)\overline{\psi}_i(t) \|_{W_2^2}\\ &= \sum_{i=n+1}^{\infty}(\sum_{k=1}^{i}\beta_{ik}f(t_k))^2.\end{aligned} \tag{3.7}$$

(3.7) shows that the error r_n is monotone decreasing in the sense of $\|\cdot\|_{W_2^2}$.

The proof is complete. □

4 Numerical example

In this section, some numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using Mathematica 4.2. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 1

Considering equation

$$u(t) - \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)ds = f(t), t \in [0,1]$$

where $f(t) = t+1$. For any $0 < \mu < 1$, we can find the true solution is $\frac{\mu}{\mu-1}+\frac{\mu+1}{\mu}t+\alpha t^{1-\mu}$, where α is a arbitrary constant. However, the true solution that is in $W_2^2[0,1]$ is $\frac{\mu}{\mu-1} + \frac{\mu+1}{\mu}t$. For $\mu = 0.5$, using our method, we choose 100 points and 200 points on $[0,1]$ respectively. The numerical results are given in the following table 1, 2. For $\mu = 0.4$, using our method, we choose 100 points and 200 points on $[0,1]$ respectively. The numerical results are given in the following table 3, 4.

Table 1: Numerical results for $\mu = 0.5$ ($n = 100$).

t	True solution u(t)	Approximate solution u_{100}	Relative error
0.001	-0.997	-0.997001	1.1E-06
0.08	-0.76	-0.760166	2.1E-04
0.16	-0.52	-0.52028	5.3E-04
0.24	-0.28	-0.280376	1.3E-03
0.48	0.44	0.439391	1.3E-03
0.64	0.92	0.919261	8.0E-04
0.80	1.40	1.39914	6.1E-04
0.96	1.88	1.87904	5.1E-04
1.00	2.00	1.99901	4.9E-04

Table 2: Numerical results for $\mu = 0.5$ ($n = 200$).

t	True solution u(t)	Approximate solution u_{200}	Relative error
0.001	-0.997	-0.997001	9.4E-07
0.08	-0.76	-0.760051	6.7E-05
0.16	-0.52	-0.520083	1.6E-04
0.24	-0.28	-0.28011	3.9E-04
0.48	0.44	0.439824	3.9E-04
0.64	0.92	0.919788	2.3E-04
0.80	1.40	1.39976	1.7E-04
0.96	1.88	1.87973	1.4E-04
1.00	2.00	1.99972	1.4E-04

Table 3: Numerical results for $\mu = 0.4$ ($n = 100$).

t	True solution u(t)	Approximate solution u_{100}	Relative error
0.001	-0.663167	-0.663171	3.9E-06
0.08	-0.386667	-0.387055	1.0E-03
0.16	-0.106667	-0.107323	6.1E-03
0.24	0.173333	0.172451	5.0E-03
0.48	1.01333	1.01189	1.4E-03
0.64	1.57333	1.57158	1.1E-03
0.80	2.13333	2.13129	9.5E-04
0.96	2.69333	2.69102	8.5E-04
0.99	2.79833	2.79598	8.4E-04

Table 4: Numerical results for $\mu = 0.4$ ($n = 200$).

t	True solution u(t)	Approximate solution u_{200}	Relative error
0.001	-0.663167	-0.663168	1.3E-06
0.08	-0.386667	-0.386788	3.1E-04
0.16	-0.106667	-0.106868	1.8E-03
0.24	0.173333	0.173066	1.5E-03
0.48	1.01333	1.0129	4.2E-04
0.64	1.57333	1.57281	3.3E-04
0.80	2.13333	2.13272	2.8E-04
0.96	2.69333	2.69265	2.5E-04
0.99	2.79833	2.79763	2.5E-04

5 Appendix

A The proof of Theorem 2.1

Through several integrations by parts for (2.1), then

$$\begin{aligned}(u(y), R_x(y))_{W_2^{2,0}} = \; & \int_0^1 u(y)(4R_x(y) - 5R_x^{(2)}(y) + R_x^{(4)}(y))dy + u(y)(5R_x'(y) \\ & -3R_x^{(3)}(y)|_0^T + u'(y)R_x^{(2)}(y)|_0^T.\end{aligned} \tag{A.1}$$

Since $R_x(y) \in W_2^{2,0}[0,T]$, it follows that

$$R_x(0) = 0. \tag{A.2}$$

Since $u \in W_2^{2,0}[0,T]$, $u(0) = 0$. If

$$5R_x'(T) - 3R_x^{(3)}(T) = 0, R_x^{(2)}(0) = 0, R_x^{(2)}(T) = 0, \tag{A.3}$$

then $(A.1)$ implies that

$$(u(y), R_x(y))_{W_2^{2,0}} = \int_0^1 u(y)(4R_x(y) - 5R_x^{(2)}(y) + R_x^{(4)}(y))dy.$$

For $\forall x \in [0, T]$, if $R_x(y)$ also satisfies

$$4R_x(y) - 5R_x^{(2)}(y) + R_x^{(4)}(y) = \delta(y - x), \tag{A.4}$$

then

$$(u(y), R_x(y))_{W_2^{2,0}} = u(x).$$

Characteristic equation of (A.4) is given by

$$\lambda^4 - 5\lambda^2 + 4 = 0,$$

then we can obtain characteristic values $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$, $\lambda_4 = -2$. So, let

$$R_x(y) = \begin{cases} c_1 e^y + c_2 e^{-y} + c_3 e^{2y} + c_4 e^{-2y}, & y \le x, \\ d_1 e^y + d_2 e^{-y} + d_3 e^{2y} + d_4 e^{-2y}, & y > x. \end{cases}$$

On the other hand, for (A.4), let $R_x(y)$ satisfy

$$R_x^{(k)}(x + 0) = R_x^{(k)}(x - 0), k = 0, 1, 2. \tag{A.5}$$

Integrating (A.4) from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $R_x^{(3)}(y)$ at $y = x$

$$R_x^{(3)}(x + 0) - R_x^{(3)}(x - 0) = 1. \tag{A.6}$$

Applying (A.2),(A.3), (A.5), (A.6), the unknown coefficients of (2.2) can be obtained.

B The coefficients of the reproducing kernel $R_x(y)$

$$c_1 = \frac{4\,e^{3\,T} - 7\,e^{3\,x} - 9\,e^{2\,T+x} + 7\,e^{6\,T+x} + 9\,e^{4\,T+3\,x} - 4\,e^{3\,T+4\,x}}{6\,e^{2\,x}\left(-7 - 9\,e^{2\,T} + 9\,e^{4\,T} + 7\,e^{6\,T}\right)}$$

$$c_2 = \frac{-4\,e^{3\,T} + 7\,e^{3\,x} + 9\,e^{2\,T+x} - 7\,e^{6\,T+x} - 9\,e^{4\,T+3\,x} + 4\,e^{3\,T+4\,x}}{6\,e^{2\,x}\left(-7 - 9\,e^{2\,T} + 9\,e^{4\,T} + 7\,e^{6\,T}\right)}$$

$$c_3 = \frac{-9\,e^{4\,T} - 7\,e^{6\,T} + 7\,e^{4\,x} - 8\,e^{3\,(T+x)} + 8\,e^{3\,T+x} + 9\,e^{2\,T+4\,x}}{12\,e^{2\,x}\left(-7 - 9\,e^{2\,T} + 9\,e^{4\,T} + 7\,e^{6\,T}\right)}$$

$$
\begin{aligned}
c_4 &= \frac{9\,e^{4\,T}+7\,e^{6\,T}-7\,e^{4\,x}+8\,e^{3\,(T+x)}-8\,e^{3\,T+x}-9\,e^{2\,T+4\,x}}{12\,e^{2\,x}\,(-7-9\,e^{2\,T}+9\,e^{4\,T}+7\,e^{6\,T})}\\
d_1 &= \frac{-\left(\left(-1+e^{2\,x}\right)\left(4\,e^{3\,T}+7\,e^{x}-9\,e^{4\,T+x}+4\,e^{3\,T+2\,x}\right)\right)}{6\,e^{2\,x}\,(-7-9\,e^{2\,T}+9\,e^{4\,T}+7\,e^{6\,T})}\\
d_2 &= \frac{e^{2\,T-2\,x}\left(-1+e^{2\,x}\right)\left(4\,e^{T}-9\,e^{x}+7\,e^{4\,T+x}+4\,e^{T+2\,x}\right)}{6\,(-7-9\,e^{2\,T}+9\,e^{4\,T}+7\,e^{6\,T})}\\
d_3 &= \frac{\left(-1+e^{2\,x}\right)\left(7+9\,e^{2\,T}+7\,e^{2\,x}+9\,e^{2\,(T+x)}-8\,e^{3\,T+x}\right)}{12\,e^{2\,x}\,(-7-9\,e^{2\,T}+9\,e^{4\,T}+7\,e^{6\,T})}\\
d_4 &= \frac{-\left(e^{3\,T-2\,x}\left(-1+e^{2\,x}\right)\left(9\,e^{T}+7\,e^{3\,T}-8\,e^{x}+9\,e^{T+2\,x}+7\,e^{3\,T+2\,x}\right)\right)}{12\,(-7-9\,e^{2\,T}+9\,e^{4\,T}+7\,e^{6\,T})}
\end{aligned}
$$

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