



Numerical Solution of Weakly Singular Integral Equations by Using Taylor Series and Legendre Polynomials

A. Arzhang ¹

Department of Mathematics, Faculty of Science, Islamic Azad University, Karaj Branch, Karaj, Iran.

Abstract

In this paper, we use Taylor series and Legendre functions of the second kind to remove singularity of the weakly singular Fredholm integral equations of the second kind with the kernel $k(x, y) = \frac{1}{(x-y)^\alpha}$, $0 < \alpha \leq 1$. Legendre polynomials are used as a basis and some integrals that appear in this method are computed with Cauchy principal value sense without using any numerical quadrature. Three examples are given to show the efficiency of the method.

Keywords: Cauchy kernel; Weakly Singular; Taylor-series; Galerkin method; Legendre functions.

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1 Introduction

Weakly singular integral equations arise in some problems of mathematical physics. It is difficult to solve these equations analytically and analytical solutions in some special cases can be found in [8, 11, 18, 19] hence, the numerical solutions are required. Recently, numerical solutions for these equations have been developed by many authors and researchers I.K. Lifanov in [17] introduced hypersingular integral equations with applications and numerical solution for a class of these equations of Prandtl's type is given by B.N. Mandal in [18]. Numerical solutions for Cauchy and Abel type of weakly

¹E-mail Address: arzhang@kiaau.ac.ir

singular integral equations are discussed in [1, 5 – 7, 9, 14 – 22]. Consider the following integral equation

$$\mu(x)\phi(x) + \lambda(x) \int_{-1}^1 \frac{\phi(y)}{(y-x)^\alpha} dy = f(x), \quad |x| < 1, \quad 0 < \alpha \leq 1, \quad (1.1)$$

where $\mu(x) \neq 0, \lambda(x) \neq 0$ and $\mu(x), \lambda(x), f(x) \in L^2[-1, 1]$ are given functions and $\phi(x)$ is the unknown function to be determined. In [1, 5, 7], Eq. (1.1) has been considered but $\mu(x)$ and $\lambda(x)$ taken to be constants. With $\alpha = 1$ we have Cauchy type singular integral equation and in this case we suppose the integral in Eq. (1.1) exists in Cauchy principal value sense and

$$p.v. \int_{-1}^1 \frac{dy}{y-x} = \ln \left(\frac{1-x}{1+x} \right), \quad |x| < 1.$$

(see[11]). In Eq. (1.1) the kernel

$$k(x, y) = \frac{1}{(x-y)^\alpha}, \quad 0 < \alpha \leq 1.$$

is the polar kernel that has been introduced in [16, 20]. Here we consider integral equation given in the relation (1.1) and assume that $\phi(y)$ has Taylor series expansion of any order on the interval $(-1, 1)$.

2 Numerical Solution

Consider the integral equation with the given conditions in the relation (1.1). With the Taylor series expansion of $\phi(y)$ based on expanding about the given point x belong to the interval $I = (-1, 1)$ we have the Taylor series approximation of $\phi(y)$ in the following form

$$\phi(y) = \sum_{k=0}^n \frac{(y-x)^k}{k!} \phi^{(k)}(x) + H_n(x),$$

where

$$H_n(x) = \frac{(y-x)^{n+1}}{(n+1)!} \phi^{(n+1)}(\zeta_{x,y}), \quad (2.1)$$

where $\zeta_{x,y}$ is between x and y . By substituting the relation (2.1) into the Eq. (1.1) we have

$$\mu(x)\phi(x) + \lambda(x) \sum_{k=0}^n \frac{\phi^{(k)}(x)}{k!} \int_{-1}^1 (y-x)^{k-\alpha} dy + E_n(x) = f(x), \quad (2.2)$$

where

$$\phi^{(0)}(x) = \phi(x), \quad E_n(x) = \frac{1}{(n+1)!} \int_{-1}^1 (y-x)^{n+1-\alpha} \phi^{(n+1)}(\zeta_{x,y}) dy.$$

Alternatively, we use truncated Taylor series of $\phi(y)$ and solve the following equation

$$\mu(x)\phi(x) + \lambda(x) \sum_{k=0}^n \frac{\phi^{(k)}(x)}{k!} I_{\alpha,k}(x) dy = f(x), \quad (2.3)$$

where

$$I_{\alpha,k}(x) = \int_{-1}^1 (y-x)^{k-\alpha} dy, \quad |x| < 1, \quad 0 < \alpha \leq 1, \quad k = 0, 1, \dots, n.$$

For $k = 0$ and $0 < \alpha < 1$ we have

$$I_{\alpha,0}(x) = \int_{-1}^1 (y-x)^{-\alpha} dy = \lim_{\eta \rightarrow 0^+} \int_{-1}^{x-\eta} (y-x)^{-\alpha} dy + \lim_{\epsilon \rightarrow 0^+} \int_{x+\epsilon}^1 (y-x)^{-\alpha} dy =$$

$$\frac{1}{1-\alpha} \left[(1-x)^{(1-\alpha)} - (-1-x)^{(1-\alpha)} \right]. \quad (2.4)$$

For $\alpha = 1$ and $k = 0$

$$I_{1,0}(x) = p.v. \int_{-1}^1 \frac{dy}{y-x} = \ln \left(\frac{1-x}{1+x} \right), \quad |x| < 1.$$

For $k = 1, 2, \dots, n$, $I_{\alpha,k}(x)$ is computed as follows

$$I_{\alpha,k}(x) = \frac{1}{k-\alpha+1} \left\{ (1-x)^{(k-\alpha+1)} - (-1-x)^{(k-\alpha+1)} \right\}. \quad (2.5)$$

When $0 < \alpha < 1$ for the two cases $k = 0$ and $k > 0$, $I_{\alpha,k}(x)$ is computed from the same formula (2.5). Hence, for $k = 0, 1, \dots, n$ we have

$$I_{\alpha,k}(x) = \frac{1}{k-\alpha+1} \left\{ (1-x)^{(k-\alpha+1)} + (-1)^{(k-\alpha)} (1+x)^{(k-\alpha+1)} \right\} =$$

$$\begin{aligned} & \frac{1}{k - \alpha + 1} \left\{ (1 - x)^{(k-\alpha+1)} + (-1)^k (\cos(\alpha\pi) - i \sin(\alpha\pi))(1 + x)^{(k-\alpha+1)} \right\} = \\ & \frac{1}{k - \alpha + 1} \left\{ (1 - x)^{(k-\alpha+1)} + (-1)^k \cos(\alpha\pi)(1 + x)^{(k-\alpha+1)} \right\} - \\ & \frac{i(-1)^k}{k - \alpha + 1} \sin(\alpha\pi)(1 + x)^{(k-\alpha+1)}. \end{aligned}$$

Now we solve Eq. (2.3) numerically. Suppose the sequence $\{\phi_n(x)\}$ be a complete orthogonal basis with respect to the weight function $w(x)$ for the space $L^2[-1, 1]$ and $\phi(x)$ is expanded based on this basis as follows

$$\phi(x) \simeq \phi_N(x) = \sum_{j=0}^N \psi_j \phi_j(x). \tag{2.6}$$

where the coefficients ψ_0, \dots, ψ_N are unknowns that must be determined. By substituting $\phi_N(x)$ from Eq. (2.6) into the Eq. (2.3) we get

$$\mu(x) \sum_{j=0}^N \psi_j \phi_j(x) + \lambda(x) \sum_{k=0}^n \frac{I_{\alpha,k}(x)}{k!} \sum_{j=0}^N \psi_j \phi_j^{(k)}(x) = f(x), \tag{2.7}$$

or

$$\sum_{j=0}^N \psi_j r_j(x) = f(x), \tag{2.8}$$

where

$$r_j(x) = \mu(x) \phi_j(x) + \lambda(x) \sum_{k=0}^n \frac{I_{\alpha,k}(x)}{k!} \phi_j^{(k)}(x).$$

For determining coefficients ψ_j we define residual function

$$R_N(x) = f(x) - \sum_{j=0}^N \psi_j r_j(x).$$

By using Galerkin method we put

$$\left(R_N(x), \phi_i(x) \right) = 0, \quad i = 0, \dots, N, \tag{2.9}$$

where $(R_N(x), \phi_i(x))$ is inner product of two functions $R_N(x)$ and $\phi_i(x)$ with respect to the weight function $w(x)$, i.e.

$$(R_N(x), \phi_i(x)) = \int_{-1}^1 R_N(x)\phi_i(x)w(x) dx.$$

The relation (2.9) leads to the linear system of equations with matrix representation

$$A\Psi = b,$$

or

$$\sum_{j=0}^N A_{ij}\psi_j = b_i, \quad i = 0, \dots, N.$$

where

$$A_{ij} = \int_{-1}^1 r_j(x)\phi_i(x)w(x) dx, \quad i, j = 0, \dots, N,$$

$$b_i = \int_{-1}^1 f(x)\phi_i(x)w(x) dx, \quad i = 0, \dots, N,$$

$$\Psi = [\psi_0, \dots, \psi_N]^T.$$

Since $(-1)^\alpha$ is a complex number we solve the following linear system of equations

$$\Re(A)\Psi = \Re(b).$$

When $\alpha = 1$ the above method holds and in the following section we use Legendre polynomials as a basis. Alternatively, for $\alpha = 1$ by using Legendre function of the second kind we establish the following method.

3 Numerical Discussion with Legendre Polynomials

In this section we use Legendre polynomials as a basis and in the next section error analysis based on these polynomials is given. Now suppose $\phi_n(x) = p_n(x)$ where $p_n(x)$

is Legendre polynomial of degree n . These polynomials are orthogonal over the interval $[-1, 1]$ with respect to the weight function $w(x) = 1$ and

$$\int_{-1}^1 p_n(x)p_m(x) dx = \frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} is Kronecker delta [10]

$$\delta_{mn} = \begin{cases} 1 & m = n, \\ 0 & m \neq n. \end{cases}$$

When $\alpha = 1$ the Eq. (1.1) turns into the following Cauchy type integral equation

$$\mu(x)\phi(x) + \lambda(x) \int_{-1}^1 \frac{\phi(y)}{y-x} dy = f(x), \quad |x| < 1, \tag{3.1}$$

in this case we use the famous relation [2 – 5, 13]

$$Q_n(x) = \frac{-1}{2} p.v. \int_{-1}^1 \frac{p_n(y)}{y-x} dy,$$

where $Q_n(x)$ is Legendre function of the second kind. By substituting $\phi(x) \simeq \phi_N(x) = \sum_{j=0}^N \psi_j p_j(x)$ into the Eq. (3.1) we have

$$\mu(x) \sum_{j=0}^N \psi_j p_j(x) + \lambda(x) \sum_{j=0}^N \psi_j \int_{-1}^1 \frac{p_j(y)}{y-x} dy = f(x), \quad |x| < 1,$$

or

$$\sum_{j=0}^N \psi_j \left(\mu(x)p_j(x) - 2\lambda(x)Q_j(x) \right) = f(x). \tag{3.2}$$

For determining coefficients ψ_j we multiply both sides of the Eq. (3.2) by $p_i(x)$, $i = 0 \dots, N$ and integrate from -1 to 1 . This process leads to

$$\sum_{j=0}^N A_{ij} \psi_j = b_i, \quad i = 0 \dots, N,$$

where

$$A_{ij} = \int_{-1}^1 \left(\mu(x)p_j(x) - 2\lambda(x)Q_j(x) \right) p_i(x) dx, \quad i, j = 0 \dots, N.$$

$$b_i = \int_{-1}^1 f(x)p_i(x) dx, \quad i = 0, \dots, N.$$

In the both cases $\alpha = 1$ and $0 < \alpha < 1$ for computing A_{ij} and b_i elements of matrix A and b we use Gauss-Legendre quadrature. In some cases, when $\lambda(x)$ and $\mu(x)$ are constants or polynomials or if we have Taylor series of $\lambda(x)$ and $\mu(x)$ then we don't need any quadrature. Furthermore, if we have Fourier series of $\lambda(x)$ and $\mu(x)$ based on Legendre polynomials

$$\lambda(x) = \sum_{l=0}^{\infty} \lambda_l p_l(x), \quad \mu(x) = \sum_{l=0}^{\infty} \mu_l p_l(x),$$

or in the form of the power series

$$\lambda(x) = \sum_{l=0}^{\infty} \lambda_l x^l, \quad \mu(x) = \sum_{l=0}^{\infty} \mu_l x^l,$$

then for $\alpha = 1$, $i, j = 0, \dots, N$,

$$A_{ij} = \sum_{l=0}^{\infty} \mu_l \int_{-1}^1 p_i(x)p_j(x)p_l(x) dx - 2 \sum_{l=0}^{\infty} \lambda_l \int_{-1}^1 p_i(x)p_l(x)Q_j(x) dx. \quad (3.3)$$

For $0 < \alpha < 1$, $i, j = 0, \dots, N$,

$$A_{ij} = \sum_{l=0}^{\infty} \mu_l \int_{-1}^1 p_i(x)p_j(x)p_l(x) dx + \sum_{l=0}^{\infty} \lambda_l \sum_{k=0}^n \frac{1}{k!} \int_{-1}^1 I_{\alpha,k}(x)p_i(x)p_l(x)p_j^{(k)}(x) dx. \quad (3.4)$$

By using the famous formula of $p_n(x)$, i.e.

$$p_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \theta_{k,n} x^{n-2k}, \quad (3.5)$$

where

$$\theta_{k,n} = \frac{(-1)^k (2n - 2k)!}{2^n k! (n - k)! (n - 2k)!}, \quad k = 0, \dots, \lfloor \frac{n}{2} \rfloor,$$

and substituting $p_l(x)$ from the above relation into the Eq. (3.3) and Eq. (3.4), we need to compute the following integrals

$$I_m^n = \int_{-1}^1 x^m p_n(x) dx.$$

$$I_{i,j,l}^p = \int_{-1}^1 p_i(x)p_j(x)p_l(x) dx.$$

$$I_{m,n}^{p,Q} = p.v. \int_{-1}^1 p_m(x)Q_n(x) dx.$$

$$I_{m,i,j}^{p,Q} = p.v. \int_{-1}^1 x^m p_i(x)Q_j(x) dx.$$

$$I_{m,l}^{\alpha,j} = \int_{-1}^1 x^m p_l(x)I_{\alpha,j}(x) dx, \quad 0 < \alpha < 1.$$

If $m < n$ then from the orthogonality of $p_n(x)$, $I_m^n = 0$. If $m \geq n$ and $m - n$ is odd then $I_m^n = 0$ else, for even $m - n$ we use the following lemma.

Lemma 3.1. *Let $g(x)$ be n times continuously differentiable on the interval $[-1, 1]$ then*

$$\int_{-1}^1 g(x)p_n(x) dx = \frac{(-1)^n}{2^n n!} \int_{-1}^1 (x^2 - 1)^n g^{(n)}(x) dx.$$

Proof. By using Rodrigues' formula of $p_n(x)$, i.e.

$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

and integration by parts we can obtain the result. □

In the above lemma for every $m \geq n$ if we take $g(x) = x^m$ then

$$g^{(n)}(x) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n},$$

and

$$I_m^n = \int_{-1}^1 x^m p_n(x) dx = \frac{\Gamma(m+1) \Gamma\left(\frac{m-n+1}{2}\right)}{2^n \Gamma(m-n+1) \Gamma\left(\frac{m+n+3}{2}\right)}, \tag{3.6}$$

for $m = n$

$$I_n^n = \int_{-1}^1 x^n p_n(x) dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}. \tag{3.7}$$

Now we have a tool to compute the above integrals. For this purpose let

$$I_{i,j}^{p,l} = \int_{-1}^1 x^i p_j(x) p_l(x) dx.$$

From the Rodrigues' formula of $p_j(x)$ and relations (3.6) and (3.7) we have

$$I_{i,j}^{p,l} = \int_{-1}^1 x^i p_j(x) p_l(x) dx = \sum_{k=0}^{[\frac{j}{2}]} \theta_{k,j} \int_{-1}^1 x^{j-2k+i} p_l(x) dx = \sum_{k=0}^{[\frac{j}{2}]} \theta_{k,j} I_{j-2k+i}^l.$$

Therefore

$$I_{i,j,l}^p = \int_{-1}^1 p_i(x) p_j(x) p_l(x) dx = \sum_{\nu=0}^{[\frac{i}{2}]} \theta_{\nu,i} \int_{-1}^1 x^{i-2\nu} p_j(x) p_l(x) dx = \sum_{\nu=0}^{[\frac{i}{2}]} \theta_{\nu,i} I_{i-2\nu,j}^{p,l}.$$

For computing $I_{m,n}^{p,Q}$ from [2, 3, 13] we have

$$\int_{-1}^1 Q_n^m(x) p_k^m(x) dx = (-1)^m \frac{(1 - (-1)^{n+k})(n+m)!}{(k-n)(k+n+1)(n-m)!}. \tag{3.8}$$

where $p_k^m(x)$ and $Q_n^m(x)$ are associated Legendre functions of the first and second kind.

For $k = n$ and $m = 0$, Eq. (3.8) takes a simple form. So, we use the following interesting relation from [13]

$$Q_n(x) = \frac{-1}{2} p_n(x) \ln \left(\frac{1-x}{1+x} \right) - \sum_{k=1}^n \frac{1}{k} p_{k-1}(x) p_{n-k}(x). \tag{3.9}$$

Since

$$p_n^2(x) \ln \left(\frac{1-x}{1+x} \right),$$

is an odd function in the interval $[-1, 1]$ then

$$p.v. \int_{-1}^1 p_n^2(x) \ln \left(\frac{1-x}{1+x} \right) dx = \lim_{\epsilon \rightarrow 0^+} \int_{-1+\epsilon}^{1-\epsilon} p_n^2(x) \ln \left(\frac{1-x}{1+x} \right) dx = 0.$$

So

$$\begin{aligned} p.v. \int_{-1}^1 Q_n(x) p_n(x) dx &= \frac{-1}{2} \left(p.v. \int_{-1}^1 p_n^2(x) \ln \left(\frac{1-x}{1+x} \right) dx \right) + \\ &\sum_{k=1}^n \frac{1}{k} \int_{-1}^1 p_n(x) p_{k-1}(x) p_{n-k}(x) dx = 0, \end{aligned}$$

and

$$I_{m,n}^{p,Q} = \begin{cases} 0 & m+n \text{ even,} \\ \frac{2}{(m-n)(m+n+1)} & m+n \text{ odd.} \end{cases}$$

For even $i + j + m$, $I_{m,i,j}^{p,Q} = 0$, for odd $i + j + m$ we have

$$I_{m,i,j}^{p,Q} = p.v. \int_{-1}^1 x^m p_i(x) Q_j(x) dx = \sum_{\nu=0}^{\lfloor \frac{j}{2} \rfloor} \theta_{\nu,i} (p.v. \int_{-1}^1 x^{i-2\nu+m} Q_j(x) dx).$$

For computing $I_{m,i,j}^{p,Q}$ by using Eq. (3.9) we need to compute Cauchy principal value integrals in the following form

$$I_n^Q = p.v. \int_{-1}^1 x^n \ln \left(\frac{1-x}{1+x} \right) dx.$$

Lemma 3.2. For every odd n

$$p.v. \int_{-1}^1 x^n \ln \left(\frac{1-x}{1+x} \right) dx = -4 \sum_{k=0}^{\infty} \frac{1}{(2k+1)(n+2k+2)} = \frac{-4}{n+1} \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{2k+1}.$$

Proof. From [4]

$$\ln \left(\frac{1-x}{1+x} \right) = -2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}, \quad |x| < 1,$$

so

$$\lim_{\epsilon \rightarrow 0^+} \left[\int_{-1+\epsilon}^{1-\epsilon} x^n \ln \left(\frac{1-x}{1+x} \right) dx \right] = -4 \sum_{k=0}^{\infty} \frac{1}{(2k+1)(n+2k+2)}.$$

On the other hand

$$\frac{1}{(2k+1)(n+2k+2)} = \frac{1}{n+1} \left\{ \frac{1}{2k+1} - \frac{1}{n+2k+2} \right\},$$

so

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(n+2k+2)} = \frac{1}{n+1} \sum_{k=0}^{\infty} \left\{ \frac{1}{2k+1} - \frac{1}{n+2k+2} \right\} =$$

$$\frac{1}{n+1} \left\{ \sum_{k=0}^{\infty} \frac{1}{2k+1} - \sum_{k=0}^{\infty} \frac{1}{n+2k+2} \right\} = \frac{1}{n+1} \left\{ \sum_{k=0}^{\infty} \frac{1}{2k+1} - \sum_{k=\frac{n+1}{2}}^{\infty} \frac{1}{2k+1} \right\} =$$

$$\frac{1}{n+1} \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{2k+1}.$$

□

For computing $I_{m,l}^{\alpha,j}$ with $0 < \alpha < 1$ let

$$I_m^{\alpha,j} = \int_{-1}^1 x^m I_{\alpha,j}(x) dx,$$

where $I_{\alpha,j}(x)$ is given by Eq. (2.5) so

$$I_m^{\alpha,j} = \int_{-1}^1 x^m I_{\alpha,j}(x) dx = \int_{-1}^0 x^m I_{\alpha,j}(x) dx + \int_0^1 x^m I_{\alpha,j}(x) dx$$

$$= \frac{1}{j-\alpha+1} (I_1 + I_2 + I_3 + I_4) = \left(\frac{1 + (-1)^{j-\alpha+m}}{j-\alpha+1} \right) \times$$

$$\left\{ \beta(m+1, j-\alpha+2) + \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{2^{j+k-\alpha+2} - 1}{j+k-\alpha+2} \right\},$$

where

$$I_1 = \int_{-1}^0 x^m (1-x)^{j-\alpha+1} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{2^{j+k-\alpha+2} - 1}{j+k-\alpha+2}.$$

$$I_2 = (-1)^{j-\alpha} \int_{-1}^0 x^m (1+x)^{j-\alpha+1} dx = (-1)^{j-\alpha+m} \beta(m+1, j-\alpha+2).$$

$$I_3 = \int_0^1 x^m (1-x)^{j-\alpha+1} dx = \beta(m+1, j-\alpha+2).$$

$$I_4 = (-1)^{j-\alpha} \int_0^1 x^m (1+x)^{j-\alpha+1} dx = (-1)^{j-\alpha+m} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{2^{j+k-\alpha+2} - 1}{j+k-\alpha+2}.$$

and

$$I_{m,l}^{\alpha,j} = \int_{-1}^1 x^m p_l(x) I_{\alpha,j}(x) dx = \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \theta_{\nu,l} \int_{-1}^1 x^{l+m-2\nu} I_{\alpha,j}(x) dx$$

$$= \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \theta_{\nu,l} I_{l+m-2\nu}^{\alpha,j}.$$

4 Numerical Examples

In this section three examples with the exact solutions are given. Since for two natural numbers $N < n$ and for every polynomial $p_N(x)$ of degree N , $p_N^{(n)}(x) = 0$. So, in these examples we take $N = n$ where n is the number of the terms of the Taylor series and N is the number of the terms of the Fourier series of the unknown function $\phi(x)$.

Example 1.

Consider the following integral equation

$$(1 + x^2)\phi(x) + (1 + x^2) \int_{-1}^1 \frac{\phi(y)}{y - x} dy = f(x), \quad |x| < 1,$$

where

$$f(x) = x^3 \ln\left(\frac{1-x}{1+x}\right) + x^3 + 2x^2 + \frac{4-\pi}{2},$$

and the exact solution is

$$\phi(x) = \frac{x^3}{1+x^2}.$$

Let $N = n$ and

$$\phi_N(x) = \sum_{k=0}^N \psi_k p_k(x),$$

$$Er_N(x) = |\phi(x) - \phi_N(x)|.$$

The approximate solutions with $N = 5, 7, 9$ are shown in table 2.1.

$$\begin{aligned} \phi_5(x) = & 0.0002461225391 + 0.02705643090x - 0.001630559812x^2 + 0.7473264464x^3 + \\ & 0.002364374459x^4 - 0.2783418124x^5, \end{aligned}$$

$$\phi_7(x) = 0.7733275480 \times 10^{-5} + 0.005975855 x - 0.0005819087019 x^2 + 0.9119489551 x^3 + 0.001916337658 x^4 - 0.6077258238 x^5 - 0.001577719858 x^6 + 0.1905577137 x^7,$$

$$\phi_9(x) = 0.7102369987 \times 10^{-5} + 0.001252967713 x - 0.0001632349973 x^2 + 0.9730666526 x^3 + 0.0009853442092 x^4 - 0.8259127393 x^5 - 0.001847815222 x^6 + 0.4819827683 x^7 + 0.001071171327 x^8 - 0.1305310512 x^9.$$

Table 1 shows that the approximate solution is very close to the exact solution.

x	$Er_5(x)$	$Er_7(x)$	$Er_9(x)$
0.0	0.0002461225	$0.7733275E - 5$	$0.710236E - 5$
0.2	0.0037932038	$0.5938439E - 3$	$0.866635E - 4$
0.4	0.0006745919	$0.3710175E - 3$	$0.975018E - 4$
0.6	0.0028454738	$0.2064124E - 3$	$0.724724E - 4$
0.8	0.0010451303	$0.3333319E - 3$	$0.566285E - 4$
1.0	0.0029789979	$0.5211424E - 3$	$0.888348E - 4$

Table 1: Errors for Example 1.

Example 2.

Consider the following integral equation

$$e^{-x} \cos(x)\phi(x) + e^x \int_{-1}^1 \frac{\phi(y)}{(y-x)^{\frac{1}{3}}} dy = f(x), \quad |x| < 1,$$

where

$$f(x) = xe^{-2x} \cos(x) + e^{x+1} (-1-x)^{\frac{2}{3}} - e^{x-1} (1-x)^{\frac{2}{3}} + \left(\frac{2}{3} + x\right) \left(\Gamma\left(\frac{2}{3}, -1-x\right) - \Gamma\left(\frac{2}{3}, 1-x\right)\right),$$

and the exact solution is

$$\phi(x) = xe^{-x}.$$

Let $n = N = 7$,

$$\phi(x) \simeq \phi_7(x) = \sum_{k=0}^7 \psi_k p_k(x).$$

and

$$Er_7(x) = | xe^{-x} - \phi_7(x) | .$$

x	$Er_7(x)$	x	$Er_7(x)$
-1.0	$0.3926E - 5$	0.2	$0.4687650966E - 6$
-0.8	$0.56699E - 7$	0.4	$0.1017243986E - 5$
-0.6	$0.63006364E - 6$	0.6	$0.111989356E - 5$
-0.4	$0.1055530434E - 5$	0.8	$0.359511E - 6$
-0.2	$0.128636815E - 6$	1.0	$0.4855E - 5$
0.0	$0.10589E - 5$		

Table 2: Errors for Example 2.

Table 2 shows that the approximate solution is very close to the exact solution.

Example 3.

Consider the following equation

$$\mu(x)\phi'(x) + \lambda(x) \int_{-1}^1 \frac{\phi(y)}{y-x} dy = f(x), \quad |x| < 1, \quad \phi(1) = 0,$$

where $\mu(x) = 2$, $\lambda(x) = 1$ and $f(x) = -\frac{x}{2}$. This equation has been solved by Mandal in [7] and by Frankel in [12]. Here we solve this equation by the method of this paper and compare our results with Mandal and Frankel results. The approximate solutions with $N = n = 4, 6, 8$ are as follows

$$\phi_4(x) = 0.06923671996 - 0.05569881381 x^2 - 0.01353790613 x^4.$$

$$\phi_6(x) = 0.06956265318 - 0.06060540191 x^2 - 0.001851739438 x^4 - 0.007105511831 x^6.$$

$$\phi_8(x) = 0.06949010241 - 0.05850969139 x^2 - 0.01139736658 x^4 + 0.00702059165 x^6 - 0.00660363609 x^8.$$

In the following table for a comparison between the proposed method of this paper and that of the methods used in [7, 12] we give values of $\phi(x)$ at the points $x = (0.2)k$, $k = 0, \dots, 5$.

x		0	0.2	0.4	0.6	0.8	1
present method	$\phi_4(x)$	0.06923	0.06698	0.05997	0.04743	0.02804	0
	$\phi_6(x)$	0.06956	0.06713	0.05978	0.04717	0.02815	0
	$\phi_8(x)$	0.06949	0.06713	0.05986	0.04716	0.02810	0
Frankel's method	$\phi(x)$	0.06950	0.06712	0.05984	0.04718	0.02891	0
Mandal's method $n = 13$	$\phi(x)$	0.06950	0.06717	0.05981	0.04723	0.02805	0

Table 3: Results for Example 3.

5 Conclusion

In this paper, we established a method to find numerical solution of weakly singular Fredholm integral and integro-differential equations. We used Taylor series and Legendre functions of the second kind to remove singularity. Legendre polynomials were used as a basis. Numerical examples show that the accuracy of the method is very high.

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