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Best Approximation of Multivariate Functions in $\mathcal{L}_1$ and $\mathcal{L}_2$ by Optimization

Majid Amirfakhrian [1]

Department of Mathematics, Islamic Azad University, Central Tehran Branch, Payambar Complex, Shahrak Gharb, Tehran, Iran.

Abstract

In this paper we consider the problem of the best approximation of a continuous multivariate function in two spaces $\mathcal{L}_1$ and $\mathcal{L}_2$ by optimization.

Keywords: Best Approximation, Multivariate Polynomials, Linear Programming, Quadratic Programming.

1 Introduction

Approximation is the problem of constructing a function P belonging to a finite dimensional linear space from a set of given data. Usually the approximation obtains by simplifying another more difficult function f. In this situation P, approximates f. Approximation of univariate polynomials is a classical work. But approximation by multivariate polynomials is more complicated and is an active subject to research. There are some new works on multivariate approximations [8, 10, 11].

[1] E-mail Address: m_amirfakhrian@iauctb.ac.ir, majiamir@yahoo.com

We first introduce the basic concepts of approximation, and a polynomial $P_n(x)$ from degree n in which $x \in \mathbb{R}^d$. In Section 3 we represent some structure of points. We try to find the best polynomial approximation of a function in $\mathcal{L}_1$ by linear programming, and we find the relation between number of points, dimension of space and degree of approximating polynomial, in Section 4. In Section 5 we try to find the best polynomial approximation of a function in $\mathcal{L}_2$ by quadratic programming, and we try to check the sufficient condition for this problem.

2 Basic concepts

Consider a functions $f : \mathbb{R}^d \to \mathbb{R}$ which maps $x = (\gamma_1, \gamma_2, \ldots, \gamma_d)^T$ to a real number, where $\gamma_i \in \mathbb{R}$. We consider the set of all nonnegative integers by $\mathbb{Z}_+$, and $\mathbb{Z}_+^d$ is the set of all d-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ where $\alpha_i \in \mathbb{Z}_+$ for $i = 1, \cdots, d$. Such a d-indices is named a multiindices.

Definition 2.1. *For a multiindices $\alpha \in \mathbb{Z}_+^d$ and $x = (\gamma_1, \gamma_2, \ldots, \gamma_d)^T$, we define*

$$| \alpha | := \alpha_1 + \alpha_2 + \ldots + \alpha_d = \sum_{i=1}^{d} \alpha_i \tag{1}$$

and

$$x^\alpha := \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \ldots \gamma_d^{\alpha_d} = \prod_{i=1}^{d} \gamma_i^{\alpha_i}. \tag{2}$$

For $i = 1, \cdots, d$, the degree of mononomial x^α respect to γ_i is α_i; and $| \alpha |$ is the total degree of x^α. Let $N_0^{d,n} := \{\alpha \in \mathbb{Z}_+^d ; | \alpha | \leq n\}$ and we use the symbol # for the cardinal of a set. Let $\prod^d$ be the set of all polynomials from $\mathbb{R}^d$ to $\mathbb{R}$ and we use $\prod_n^d$ for the set of all polynomials belonging to $\prod^d$ where total degree of them does not exceed n.

Let $X_N = \{x_1, x_2, \ldots, x_N\}$ be a sequence of distinct points in $\mathbb{R}^d$, for a positive integer N, which we name it the set of approximation nodes (for interpolation, the set of interpolation nodes). Consider the set of nodes X_N and a subset of real points $Y_N = \{y_1, y_2, \ldots, y_N\}$.

Definition 2.2. *The approximation problem in* $\prod_n^d$ *with respect to* X_N *is said to be poised, if there exists a unique polynomial* $P \in \prod_n^d$ *such that with a good approximation we have:*

$$P(x_i) \approx f(x_i) \quad , \quad 1 \leq i \leq N$$

3 Multivariate polynomials

A multivariate polynomial is a function from $\mathbb{R}^d$ to $\mathbb{R}$ with the following form:

$$P(x) = \sum c_\alpha x^\alpha \quad , \quad x \in \mathbb{R}^d$$

which sum is finite and for $\alpha \in \mathbb{Z}_+^d$, the coefficients c_α are fixed real numbers. Degree of polynomial P is defined by $\max\{| \alpha |: c_\alpha \neq 0\}$ and a d-variables polynomial from degree at most n, is defined by:

$$P(x) = \sum_{\alpha \in N_0^{d,n}} c_\alpha x^\alpha \tag{3}$$

The set of mononomials $\{x \to x^\alpha :| \alpha |\leq n\}$ is a basis for $\prod_n^d$ [2].

A d-variables polynomial P from degree at most n has the following form:

$$P_n(x) = a_0 + \sum_{j=1}^{n} \left\{ \sum_{i_1=1}^{d} \sum_{i_2=i_1}^{d} \cdots \sum_{i_j=i_{j-1}}^{d} a_{i_1,i_2,\ldots,i_j} \gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_j} \right\} \tag{4}$$

In this formula for $j = 1, 2, \ldots, n$ and $1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq d$; $a_{i_1,i_2,\ldots,i_j}$ is the coefficient of $\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_j}$. For $j = 1, 2, \ldots, n$ define $I_j := (i_1, i_2, \ldots, i_j)$ such that $1 \leq i_1 \leq d$, and $i_{k-1} \leq i_k \leq d$ for $k = 2, 3, \ldots, j$. By using these definitions we have:

$$P_n(x) = a_0 + \sum_{j=1}^{n} \sum_{I_j} a_{I_j}\gamma_{I_j} \tag{5}$$

We define the sets E_j^d, for $j = 1, 2, \ldots, n$, and $E^{d,n}$ as follows:

$$E_j^d := \{a_{I_j} : I_j := (i_1, i_2, \ldots, i_j) \;\; \& \;\; 1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq d\} \tag{6}$$

and

$$E^{d,n} := \bigcup_{j=1}^{n} E_j^d. \tag{7}$$

Let

$$\begin{cases} r_j^d := \#E_j^d \quad , \quad j = 1, 2, \dots, n \\ s_n^d := \#E^{n,d} \end{cases} \tag{8}$$

Similarly we define $r_0^d := 1$; $(r_0^d = \#E_0^d = \#\{a_0\} = 1)$.

Lemma 3.1. *r_j^d satisfies in the following recursive formula:*

$$\begin{cases} r_{j+1}^d = \sum\limits_{i=1}^{d} r_j^i, \quad j \geq 0 \\ r_0^d = 1 \end{cases} \tag{9}$$

Proof. For $j = 1$ we know that the number of mononomials γ_{i_1} is d.

Suppose that the formula is true for j. In the next step we try to find the number of mononomials $\gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_{j+1}}$ for $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{j+1} \leq d$.

If $i_1 = 1$ then $1 \leq i_2 \leq \cdots \leq i_{j+1} \leq d$; and according to assumption of induction the number of mononnomials when $i_1 = 1$ is r_j^d.

Now suppose i_1 be an arbitrary index, then we have $1 \leq i_2 - i_1 + 1 \leq i_3 - i_1 + 1 \leq \cdots \leq i_{j+1} - i_1 + 1 \leq d - i_1 + 1$. Thus we have $1 \leq i'_2 \leq i'_3 \leq \cdots \leq i'_{j+1} \leq d - i_1 + 1$. It means that the number of mononomials , is $r_j^{d-i_1+1}$. Therefore the number of all mononomials $\gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_{j+1}}$ is: $r_{j+1}^d = \sum\limits_{k=1}^{d} r_j^{d-k+1} = \sum\limits_{i=1}^{d} r_j^i$. □

By using the Lemma 3.1 we have $s_n^d = \sum_{j=1}^{n} r_j^d$. Consequently the number of all coefficients in $P_n(x)$ is $t_n^d = s_n^d + 1$. (We know that this number is the dimension of space $\prod_n^d$.)

Corollary 3.2. *By the above notations we have $r_j^d = \binom{d+j-1}{j}$ for $j = 0, 1, 2, \dots, n$; also we have $t_n^d = \binom{n+d}{d}$.*

4 Structure of approximation points

Poised in approximation of a multivariate function is dependent on geometrical structure of the nodes, Thus one must recognize the points and space of polynomials. In this section we focus on various presented methods to choose the points $x_1, x_2, \ldots, x_N$ in $\mathbb{R}^d$, which the approximation problem with respect to these points can be poised in $\prod_n^d$. Let $N = \dim \prod_n^d$.

The first and the most natural approach to choose such approximation nodes is triangular grid of the simplex formed by the points in $\frac{1}{n}N_0^{d,n}$. In bivariate case, we have the more general case of arrays formed by points (x_i, y_i) , $0 \leq i+j \leq n$, where $\{x_i\}$, $\{y_i\}$, $i, j = 0, 1, \ldots, n$, are two set of $n+1$ distinct points [6, 7]. Suppose $X_N = \{x_1, x_2, \ldots, x_N\}$ is the set of $N = \binom{n+d}{d}$ points in $\mathbb{R}^d$, we say X satisfies the GC condition (Geometric Characterization), if for each points x_i there exist hyperplanes H_{il} , $l = 1, 2, \ldots, n$, such that x_i is not on any of these hyperplanes, and all points of X lies on at least one of them. Equivalently, we have

$$x_j \in \bigcup_{i=1}^{n} H_{il} \Longleftrightarrow j \neq i \quad , \quad i, j = 1, 2, \ldots, N \tag{10}$$

If X satisfies the GC condition, arbitrary data at the nodes can be approximated [2]. In general, it is difficult to recognizing whether a set of nodes satisfies the GC condition or not, even for $\mathbb{R}^2$. However, there are some special methods. For example let $r_0, r_1, \ldots, r_{n+1}$, be $n+2$ straight lines in $\mathbb{R}^2$ such that any two of them r_i, r_j intersect at exactly one points x_{ij} and these points have the property that $x_{ij} \neq x_{kl} \Longleftrightarrow \{i,j\} \neq \{k,l\}$. Then the set $X = \{x_{ij} : 0 \leq i < j \leq n+1\}$ satisfies the GC condition. And the set X is called a natural lattice of order n.

A *pencil of order* n in $\mathbb{R}^d$ is a family of $n+1$ hyperplanes which either all intersect in an affine subspace of codimension 2 or are all parallel. The intersection (in the projective sense) of the hyperplanes of a pencil is called its *center*. We consider $d+1$ pencils of order n in $\mathbb{R}^d$ with centers $C_1, C_2, \ldots, C_{d+1}$, not contained in a hyperplane of the d-dimensional projective space ,with the additional condition that there exist

$N = \binom{n+d}{d}$ points, each of them lying precisely on $d+1$ hyperplanes, one from each pencil [9].

In [2], Cheney and Light proved that if the set of nodes $\{x_1, x_2, \ldots, x_{10}\}$ satisfies the conditions (a)-(d); then approximation of arbitrary data at the nodes is possible by unique polynomial in $\prod_3^2$:

a. x_7, x_8, x_9, x_{10} are on a line L_1

b. x_4, x_5, x_6, x_7 are on a line L_2

c. $L_1 \neq L_2$

d. x_1, x_2, x_3 are not colinear and are not in $L_1 \cup L_2$

Also in [5], Gasca and Maeztu have shown the following Theorem:

Theorem 4.1. *Let X be a set of $\frac{1}{2}(n+1)(n+2)$ nodes in $\mathbb{R}^d$, where $n \geq 2$. Suppose that there exist hyperplanes $H_0, H_1, \ldots, H_n$ in $\mathbb{R}^d$, If $X \subset H_0 \cup H_1 \cup \cdots \cup H_n$ and $\#(X \cap H_i) = i+1$, for $i = 0, \ldots, n$; then arbitrary data on X can be interpolated by $\prod_n^d$.*

5 Best approximation in $\mathcal{L}_1$

In this section we try to find the coefficients a_{I_j} for $j = 0, 1, \ldots, n$, such that

$$\sum_{i=1}^{N} |f(x_i) - P_n^*(x_i)| = \min_{P_n \in \prod_n^d} \left(\sum_{i=1}^{N} |f(x_i) - P_n(x_i)|\right)$$

Thus we want to find

$$\min_{P_n \in \prod_n^d} \left(\sum_{i=1}^{N} |f(x_i) - P_n(x_i)|\right). \tag{11}$$

For $i = 1, 2, \ldots, N$, we use two definitions for the error in the node x_i as:

$$e_i = f(x_i) - P_n^*(x_i) \tag{12}$$

and

$$e_i = u_i - v_i \tag{13}$$

where u_i and v_i are defined as:

$$\begin{cases} u_i = e_i \ , \ v_i = 0; & e_i \geq 0 \\ u_i = 0 \ , \ v_i = -e_i; & e_i < 0 \end{cases} \tag{14}$$

Now we define β and λ as follows

$$\begin{aligned} &\beta := \min(a_{I_n}, a_{I_{n-1}}, \ldots, a_{I_1}, a_0) \\ &\lambda := \max(0, \beta) \end{aligned} \tag{15}$$

For $j = 1, 2, \ldots, n$; we define $\overline{a}_0$ and $\overline{a}_{I_j}$ as follows:

$$\begin{cases} \overline{a}_{I_j} := a_{I_j} + \lambda, & j = 1, 2, \ldots, n \\ \overline{a}_0 := a_0 + \lambda. \end{cases}$$

By these definitions $P_n^*(x)$ has the following form

$$P_n^*(x) = \overline{a}_0 + \sum_{j=1}^{n} \sum_{I_j} \overline{a}_{I_j} \gamma_{I_j} - \lambda(1 + \sum_{j=1}^{n} \sum_{I_j} \gamma_{I_j}). \tag{16}$$

From (12) and (13), we have

$$u_i - v_i + P_n^*(x) = f(x_i) \quad , \quad i = 1, 2, \ldots, N.$$

Thus one must solve the following linear programming problem:

$$(LP1) \qquad \min \ \sum_{i=1}^{N} (u_i + v_i) \tag{17}$$

with the following constraints:

$$\begin{cases} u_i - v_i + \overline{a}_0 + \sum_{j=1}^{n} \sum_{I_j} \overline{a}_{I_j} \gamma_{(i,I_j)} - \lambda(1 + \sum_{j=1}^{n} \sum_{I_j} \gamma_{(i,I_j)}) = f(x_i), & i = 1, 2, \ldots, N \\ u_i \geq 0, \ , \ v_i \geq 0 \qquad i = 1, 2, \ldots, N \\ \overline{a}_{I_j} \geq 0, \qquad j = 1, 2, \ldots, n \\ \overline{a}_0 \geq 0 \\ \lambda \geq 0 \end{cases} \tag{18}$$

We know that the number of all variables in $(LP1)$ is $2N + t_n^d + 1$, and the number of constraints is N.

Lemma 5.1. *In optimal solution of the problem* $(LP1)$ *we have*

$$u_i v_i = 0 \quad , \quad i = 1, 2, \ldots, N. \tag{19}$$

Proof. Let $F := \{i \ : \ u_i v_i > 0\}$. Let $m_i := \min(u_i, v_i)$, for $i \in F$. Thus we have $u_i - m_i \geq 0$ and $v_i - m_i \geq 0$. In $(LP1)$ for $i \in F$, we put $u_i - m_i$ and $v_i - m_i$ instead of u_i and v_i, respectively. It can be shown that constraints of $(LP1)$ are unchanged, but in objective function we have

$$\sum_{i=1}^{N}(u_i + v_i) = \sum_{i \notin F}(u_i + v_i) + \sum_{i \in F}(u_i + v_i)$$

or equivalently

$$\sum_{i \notin F}(u_i + v_i) + \sum_{i \in F}(u_i + v_i - 2m_i) = \sum_{i=1}^{N}(u_i + v_i) - 2\sum_{i \in F} m_i.$$

Now we have a new problem that its constraints are (18) but its objective function is

$$(LP1^*) \qquad \min \ \{\sum_{i=1}^{N}(u_i + v_i) - 2\sum_{i \in F}\min(u_i, v_i)\}. \tag{20}$$

Let z^* and z^{**} be the optimum values of $(LP1)$ and $(LP1^*)$, respectively. Since two problems have the same constraints we have:

$$z^{**} = z^* - 2\sum_{i \in F}\min(u_i, v_i)\}. \tag{21}$$

It means that $z^{**} < z^*$. Consequently F is empty. □

Lemma 5.2. *The problem* $(LP1)$ *is always feasible, and has a finite solution.*

Theorem 5.3. *For finding the best polynomial approximation of degree* n *for an arbitrary function, the number of points should be greater than or equal to* t_n^d*. i.e.*

$$N \geq t_n^d. \tag{22}$$

Proof. Let A be a $t \times s$ matrix. Also $X, C \in \mathbb{R}^s$, $b \in \mathbb{R}^t$.

Let the following problem (LP)

$$\begin{cases} \min \ C^T X \\ s.t. \\ \ AX = b \\ \quad X \geq 0 \end{cases}$$

has an optimum solution $X^* = (\gamma_1^*, \gamma_2^*, \dots, \gamma_s^*)^T$. Now define $U := \{i \ , \ \gamma_i^* \neq 0 \ , \ 1 \leq i \leq s\}$. Thus we have

$$\#U \leq t \tag{23}$$

Now suppose the problem of finding the best approximating polynomial of a polynomial $f(x) = a_0 + \sum_{j=1}^n \sum_{I_j} a_{I_j} \gamma_{I_j}$, where $a_{I_j} \neq 0$ for $j = 1, 2, \dots, n$; and also $a_0 \neq 0$. Thus we should have $\#U = t_n^d$ and consequently we should have $t_n^d \leq N$. □

6 Best approximation in $\mathcal{L}_2$

In this section we try to find a_{I_j} for $j = 0, 1, \dots, n$, such that

$$\sum_{i=1}^N [f(x_i) - P_n^*(x_i)]^2 = \min_{P_n \in \prod_n^d} (\sum_{i=1}^N [f(x_i) - P_n(x_i)]^2).$$

Thus we want to find

$$\min_{P_n \in \prod_n^d} (\sum_{i=1}^N [f(x_i) - P_n(x_i)]^2). \tag{24}$$

Let $y \in \mathbb{R}^{t_n^d}$ be in the form of $y^T := ({y^n}^T, {y^{n-1}}^T, \dots, {y^1}^T, 1)$, such that

$${y^j}^T = (\gamma_{I_j}) \quad ; \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq d \ \ , \ \ j = 1, 2, \dots, n.$$

Also consider the elements of y by w_i. i.e.

$$(w_1, w_2, \dots, w_{t_n^d})^T = \left((\gamma_d)^n, (\gamma_{d-1})^{n-1}\gamma_d, \dots, 1\right)^T.$$

For y_i that is dependence to x_i, we have

$$y_i{}^T = (y_i^{nT}, y_i^{n-1T}, \ldots, y_i^{1T}, 1),$$

where $y_i^{jT} = (\gamma_{(i,I_j)}) \quad ; \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq d \quad , \quad j = 1, 2, \ldots, n.$ and for $i = 1, 2, \ldots, N$; $\Big(w_{i,1}, w_{i,2}, \ldots, w_{i,t_n^d}\Big) = \Big((\gamma_{i,d})^n, (\gamma_{i,d-1})^{n-1}\gamma_{i,d}, \ldots, 1\Big)$.

Suppose that for $j = 1, 2, \ldots, n$; a_{I_j} be a vector, we define

$$\Phi^T := (a_{I_n}{}^T, a_{I_{n-1}}{}^T, \ldots, a_{I_1}{}^T, a_0).$$

Also suppose that we show the elements of Φ by ϕ_i. i.e.

$$\Big(\ \phi_1, \phi_2, \ldots, \phi_{t_n^d}\ \Big) = \Big(\ a_{(n,n,\ldots,n,n)}, a_{(n-1,n-1,\ldots,n-1,n)}, \ldots, a_0\ \Big).$$

By these definitions we have

$$P_n(x) = \Phi^T y. \tag{25}$$

Thus we want to find Φ^* such that $P_n^*(x) = \Phi^{*T} y$, is the best approximating polynomial of $f(x)$ from degree at most n on X_N.

Let

$$h(\Phi) := \sum_{i=1}^{N} [f(x_i) - \Phi^T y_i]^2 \tag{26}$$

and also $f_i = f(x_i) \quad , \quad i = 1, 2, \ldots, N$. Therefore

$$\begin{aligned} h(\Phi) &= \sum_{i=1}^{N} [f_i - \Phi^T y_i]^2 \\ &= \sum_{i=1}^{N} [f_i - \sum_{j=1}^{t_n^d} \phi_j w_{i,j}]^2 \\ &= \sum_{i=1}^{N} f_i^2 - 2 \sum_{j=1}^{t_n^d} \phi_j (\textstyle\sum_{i=1}^{N} f_i w_{i,j}) + (\sum_{j=1}^{t_n^d} \sum_{k=1}^{t_n^d} \phi_j \phi_k \sum_{i=1}^{N} w_{i,j} w_{i,k}). \end{aligned}$$

By defining

$$\begin{cases} q_{j,k} := \sum_{i=1}^{N} w_{i,j} w_{i,k} & j,k = 1,2,\ldots,t_n^d \\ \\ b_j := \sum_{i=1}^{N} f_i w_{i,j} & j = 1,2,\ldots,t_n^d \\ \\ c := \sum_{i=1}^{N} f_i^2 \end{cases}$$

and

$$b^T := (b_1, b_2, \ldots, b_{t_n^d}) \quad , \quad Q := \left(\; q_{i,j} \; \right)_{i,j}$$

We have

$$h(\Phi) = c - 2\sum_{j=1}^{t_n^d} \phi_j b_j + \sum_{j=1}^{t_n^d} \sum_{k=1}^{t_n^d} \phi_j \phi_k q_{j,k}$$

or equivalently

$$h(\Phi) = c - 2b^T\Phi + \Phi^T Q \Phi. \tag{27}$$

Therefore we want to solve the following problem

$$\min_{\Phi} h(\Phi) = \min_{\Phi} c - 2b^T\Phi + \Phi^T Q \Phi \tag{28}$$

which is an unconstrained quadratic programming problem. It can be shown that Q is a symmetric and positive semi definite matrix, and we have

$$\nabla h(\Phi) = 2Q\Phi - 2b.$$

The first necessary condition for minimizer of $h(\Phi)$ is that $Q\Phi = b$. We know that x_i's are distinct, then y_i's are distinct too; thus Q has a full rank. If $N \geq t_n^d$, then Q is positive definite and Φ is a minimizer of $h(\Phi)$.

7 Numerical Examples

In this section we present some examples to show the efficiency of this method.

Example 7.1.

Considering $f(x,y) = \sin(x-y),\ d = 2,\ n = 1$ and $X = \{(0,0.1),(0.1,0),(0.1,0.1))\}$ in $\mathcal{L}_1$ we have $P(x,y) = 0.998334166468281836\,x - 0.998334166468281125\,y$. The exact and approximating functions are shown in Figure 1:

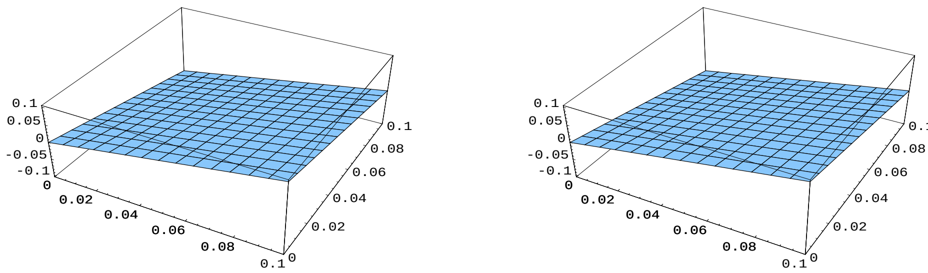

Figure 1 : The exact function in left and the approximating function in right

Example 7.2.

Considering

$$n = 3,\ d = 3,\ f(x,y) = e^{x^2 - y^2 + z^2}$$

and

i	1	2	3	4	5	6	7	8	9	10
x_i	0	0	0	0	0.1	0.1	0.1	0.1	0	0
y_i	0	0	0.1	0.1	0	0	0.1	0.1	0	−0.1
z_i	0	0.1	0	0.1	0	0.1	0	0.1	−0.1	0
i	11	12	13	14	15	16	17	18	19	20
x_i	0	−0.1	−0.1	−0.1	−0.1	0	0	0.1	−0.1	0.1
y_i	−0.1	0	0	−0.1	−0.1	0.1	−0.1	0	0	−0.1
z_i	−0.1	0	−0.1	0	−0.1	−0.1	0.1	−0.1	0.1	0.1

in $\mathcal{L}_1$ we have

$$P(x,y,z) = 1.+0.01106x+1.01512x^2-1.00502x^3+0.00905y-0.01010xy+0.10000x^2y - 1.00502y^2 - 0.10101xy^2 - 1.00502y^3 - 1.00502z - 7.32748\times10^{-15}xz - 1.82077\times10^{-14}x^2z - 3.77476\times10^{-15}yz + 0.00101xyz - 2.22045\times10^{-15}y^2z + 1.00502z^2 - 0.10101xz^2 + 0.100001yz^2 + 100.502z^3$$

with the maximum error 2.788573×10^{-13}.

Example 7.3.

Considering $n = 3,\ d = 2,\ f(x,y) = \sin(x+y)$ and

i	1	2	3	4	5	6	7	8	9	10
x_i	1	2	2	3	1	3	2	0	1	1
y_i	0	2	1	1	3	3	3	1	1	2

in $\mathcal{L}_2$ we have

$$P(x,y) = 0.687334 + 2.43507x - 1.02398x^2 + 0.11771x^3 + 2.595y - 2.1266xy + 0.252844x^2y - 1.1839y^2 + 0.412773xy^2 + 0.11771y^3$$

The exact and approximating functions are shown in Figure 2:

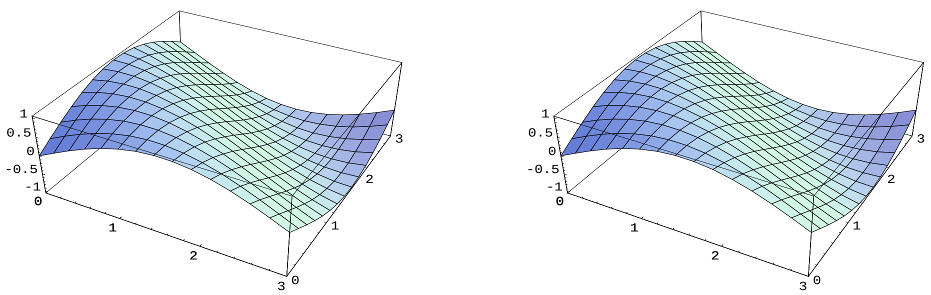

Figure 2 : The exact function in left and the approximating function in right

Example 7.4.

Considering

$$n = 3,\ d = 3,\ f(x, y) = e^{x^2 - y^2 + z^2}$$

with the points given in Example 7.2, in $\mathcal{L}_2$ we have

$$\begin{aligned}P(x, y, z) = 1. &+ 1.68349x + 1.01175x^2 - 168.349x^3 + 2.26782y - 0.01006xy + 0.10067x^2y \\ &- 1.00168y^2 - 0.10067xy^2 - 226.782y^3 - 1.58461z - 5.58347 \times 10^{-6}xz \\ &- 2.77972 \times 10^{-13}x^2z - 5.58347 \times 10^{-6}yz - 2.08623 \times 10^{-13}xyz \\ &- 2.31509 \times 10^{-13}y^2z + 1.00505z^2 + 1.93570 \times 10^{-14}xz^2 \\ &+ 5.15409 \times 10^{-13}yz^2 + 158.461z^3\end{aligned}$$

with the maximum error 1.116694×10^{-7}.

8 Conclusion

In this work we've extended the concept of best approximation of univariate function to the multivariate one and we gave some numerical examples.

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