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Some results about the index of matrix and Drazin inverse

M. Nikuie[a,1], M.K. Mirnia[b], Y. Mahmoudi[b]

[a] Member of Young Researchers Club , Islamic Azad University-Tabriz Branch

[b] Islamic Azad University-Tabriz Branch

Abstract

In this paper , some results about the index of matrix and Drazin inverse are given. We then explain applications of them in solving singular linear system of equations.

Keywords: Index , Drazin inverse, Singular linear system, Jordan normal form.

1 Introduction

Let $C^{n\times n}$ denote the space comprising those matrices of order n , over the complex field. Any matrix $A \in C^{n\times n}$ have an index that we get Drazin inverse of A to make use of it. On the contrary to our expectation, for any matrix $A \in C^{n\times n}$, even singular matrices, index and Drazin inverse of matrix A exists and is unique[10]. According to this truth, the aim of this paper is to give the new results of them. The Drazin inverse has various applications in the theory of finite Markov chains, the study of singular differentail and difference equations, the investigation of Cesaro-Neumann iterations, cryptograph, iterative methods in numerical analysis, multibody system dynamics and others[4,8,9]. Computing the Drazin inverse is a current issue in recent years[3,10]. Section 2 provides preliminaries for index of matrix and Drazin inverse. Applications

[1]Corresponding Author. E-mail Address: nikoie_m@yahoo.com

of them in solving singular linear system of equations is also explained. Some new results on the index of matrix and Drazin inverse are given in section 3. Numerical examples to illustrate previous sections are given in section 4.

2 Preliminaries and Basic Definitions

We first present some definitions and theorems about the index of matrix and Drazin inverse which are needed in this paper.

Definition 2.1 *Let $A \in C^{n\times n}$. We say the nonnegative integer number k to be the index of matrix A , if k is the smallest nonnegative integer number such that*

$$rank(A^{k+1}) = rank(A^k) \tag{1}$$

It is equivalent to the dimension of largest Jordan block corresponding to the zero eigenvalue of A [9]. The index of matrix A , is denoted by $ind(A)$. For any matrix $A \in C^{n\times n}$ the unique Jordan normal form of a matrix A can be built [6]. Therefore for any matrix $A \in C^{n\times n}$ the index of A exists and is unique.

Definition 2.2 *A number $\lambda \in C$, is called an eigenvalue of the matrix A if there is a vector $x \neq 0$ such that $Ax = \lambda x$. Any such vector is called an eigenvector of A associated to the eigenvalue λ .*

The set $L(\lambda) = \{x \mid (A - \lambda I)x = 0\}$ forms a linear subspace of C^n , of dimension

$$\rho(\lambda) = n - rank(A - \lambda I)$$

The integer $\rho(\lambda) = dimL(\lambda)$ specifies the maximum number of linearly independent eigenvectors associated with the eigenvalue λ . It is easily seen that $\varphi(\mu) = det(A - \mu I)$ is a nth-degree polynomial of the form

$$\varphi(\mu) = (-1)^n(\mu^n + \alpha_{n-1}\mu^{n-1} + \cdots + \alpha_0) \tag{2}$$

It is called the characteristic polynomial of the matrix A . Its zeros are the eigenvalues of A . If $\lambda_1, \cdots, \lambda_k$ are the distinct zeros of $\varphi(\mu)$, then φ can be represented in the form

$$\varphi(\mu) = (-1)^n(\mu - \lambda_1)^{\sigma_1}(\mu - \lambda_2)^{\sigma_2} \cdots (\mu - \lambda_k)^{\sigma_k}$$

The integer σ_i , which we also denote by $\sigma(\lambda_i) = \sigma_i$, is called the multiplicity of the eigenvalue λ_i.

Theorem 2.3 . *If λ be an eigenvalue of matrix $A \in C^{n\times n}$, then*

$$1 \leq \rho(\lambda) \leq \sigma(\lambda) \leq n \tag{3}$$

Proof See [6].

Definition 2.4 *Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order n is the Drazin inverse of A ,denoted by A^D , if X satisfies the following conditions*

$$AX = XA, XAX = X, A^kXA = A^k \tag{4}$$

When $ind(A) = 1$, A^D is called the group inverse of A , and denoted by A_g . For any matrix $A \in C^{n\times n}$, the Drazin inverse A^D of A exists and is unique [3,10].

Theorem 2.5 *Let $A \in C^{n\times n}$, with $ind(A) = k$, $rank(A^k) = r$. We may assume that the Jordan normal form of A has the form as follows*

$$A = P\begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix}P^{-1}$$

where P is a nonsingular matrix, D is a nonsingular matrix of order r , and N is a nilpotent matrix that $N^k = \bar{o}$. Then we can write the Drazin inverse of A in the form

$$A^D = P\begin{pmatrix} D^{-1} & 0 \\ 0 & N \end{pmatrix}P^{-1}$$

When $ind(A) = 1$, it is obvious that $N = \bar{o}$ [3,10].

Corollary 2.6 . *Let $ind(A) = 1$ and $D = I$. It is clear that $D^{-1} = I$, thus $A = A_g$.*

Theorem 2.7 *If* $Ax = b$ *be a consistent or inconsistent singular linear system where* $ind(A) = k$*,then the linear system of equations*

$$A^k Ax = A^k b \tag{5}$$

is consistent.

Proof The linear system $Ax = b$ has solution if and only if $rank(A) = rank[A \mid b]$ [8]. From (1) we have $rank(A^{k+1}) = rank[A^{k+1} \mid A^k b]$. Therefore (5) is consistent.

According to[5]and properties of the Drazin inverse , in order to obtain the Drazin inverse the projection method solves consistent or inconsistent singular linear system $Ax = b$ where $ind(A) = k$ through solving the consistent singular linear system (5)[9].

Theorem 2.8 $A^D b$ *is a solution of*

$$Ax = b, k = ind(A) \tag{6}$$

if and only if $b \in R(A^k)$ *, and* $A^D b$ *is an unique solution of (6) provided that* $x \in R(A^k)$*.*

We now explain applications of the index of matrix and Drazin inverse in solving singular linear system of equations.

Singular linear system of equations arise in many different scientific applications. Notably, partial differential equations discretized with finite difference or finite element methods yield singular linear systems of equations. Large singular linear systems can be solved with either sparse decompositions techniques or with iterative methods. For consistent singular linear systems, these two approaches can be also combined into a method that uses approximate decomposition preconditioning for an iterative method. However, we cannot use preconditioned iterative method for inconsistent singular linear systems[9]. A cramer rule for $A^D b$ was given in [7].

Singular linear systems with index one arises in many applications, such as Markov chain modelling and numerical experiment on the perturbed Navier-Stokes equation. Therefore, for the singular linear system with unit index, we must solve the system

$$AAx = Ab \tag{7}$$

which is consistent. So we can choose all kinds of preconditioned methods, such as PCG , PGMRES, etc., to solve system (7). In [9] Wei proposed a two step algorithm for solving singular linear system with index one. In an implementation we first solve the system $Ay = Ab$, the inner iteration, and then we solve $Ax = y$, the outer iteration.

3 Some Results

In this section, we give some results about the index of matrix and Drazin inverse.

Theorem 3.1 *If $A \in C^{n\times n}$ be a nonsingular matrix, then $ind(A) = 0$.*

Proof . Let $A \in C^{n\times n}$ be nonsingular matrix, we know that $rank(A) = rank(I_n) = n$. Thus $ind(A) = 0$. Moreover the eigenvalues of A are nonzero.
Therefore if A be a nonsingular matrix then $ind(A) = 0$ and $A^D = A^{-1}$, which satisfies the conditions (4).

Theorem 3.2 *If $A \in C^{n\times n}$ be a matrix with index one, then $rank(A) = rank(A_g)$.*

Proof . From $rank(A_g) = rank(A_gAA_g) \leq rank(AA_g) \leq rank(A)$ we have

$$rank(A_g) \leq rank(A) \tag{8}$$

Since $rank(A) = rank(AA_gA) \leq rank(A_gA) \leq rank(A_g)$

$$rank(A) \leq rank(A_g) \tag{9}$$

Thus of (8) , (9) we have $rank(A) = rank(A_g)$.

Corollary 3.3 *Let $A \in C^{n\times n}$ where $ind(A) > 1$. We have $rank(A^D) < rank(A)$.*

Theorem 3.4 *Let $A \in C^{n\times n}$,then A_g exists if and only if $rank(A) = rank(A^2)$[1].*

Corollary 3.5 *The index of any idempotent matrix equal one. The converse of this statement is not true always.*

Theorem 3.6 *If $A \in C^{n\times n}$ be a symmetric matrix with index one, then $A_g = (A_g)^T$.*

Proof Since A_g be group inverse of A we have

$$AA_g = A_gA, \quad A_gAA_g = A_g, \quad AA_gA = A$$

From $A = A^T$ we have

$$(A_g)^TA = A(A_g)^T, \quad (A_g)^TA(A_g)^T = (A_g)^T, \quad A(A_g)^TA = A \tag{10}$$

From (10) by Definition (2.4), $(A_g)^T$ is group inverse of A . The group inverse of matrix A is unique. Therefore $A_g = (A_g)^T$.

Corollary 3.7 *For any matrix $A \in C^{n\times n}$ with index one , $(A_g)_g = A$.*

Theorem 3.8 *. If $A \in C^{n\times n}$ where $ind(A) = k$,then $ind(A) = ind(A^T)$.*

Proof Since $ind(A) = k$, by Definition(2.1) we have $rank(A^{k+1}) = rank(A^k)$. From $(A^n)^T = (A^T)^n$. We have

$$(A^{k+1})^T = (A^T)^{k+1}, (A^k)^T = (A^T)^k$$

Since $rank(A) = rank(A^T)$ we have

$$rank(A^{k+1}) = rank((A^T)^{k+1}), rank(A^k) = rank((A^T)^k)$$

Thus, we have $rank((A^T)^{k+1}) = rank((A^T)^k)$. Therefore $ind(A) = ind(A^T)$.

Theorem 3.9 *. If λ be an eigenvalue of $A \in C^{n\times n}$, then $\frac{1}{\lambda}$ is an eigenvalue of A^D.*

Proof From $Ax = \lambda x, (x \neq 0)$ we have

$$AA^Dx = \lambda A^Dx$$

$$A^DAA^Dx = \lambda A^DA^Dx$$

From (4) we can get $A^Dx = \lambda A^DA^Dx$. Now if we set $A^Dx = y$ we have $\frac{1}{\lambda}y = A^Dy$. Therefore $\frac{1}{\lambda}$ is an eigenvalue of A^D.

Theorem 3.10 . *If* $A \in C^{n\times n}$ *be a nilpotent matrix that* $A^k = \bar{o}$, *then* $ind(A) = k$.

Proof From $A^k = \bar{o}$ we have

$$rank(A^k) = rank(\bar{o})$$

From $A^{k+n} = \bar{o}, n \in N$ we can get $rank(A^{k+1}) = rank(A^k)$. Therefore $ind(A) = k$.

Corollary 3.11 *Let* $A \in C^{n\times n}$ *be a nilpotent matrix of index* n , *so* $ind(A) = n$ *and* $rank(A^n) = 0$. *Thus in Theorem 1.2* , $D = \bar{o}$. *Therefore* $A^D = \bar{o}$.

Theorem 3.12 . *If* $A \in C^{n\times n}$, *then* $0 \leq ind(A) \leq n$.

Proof By Theorem 3.1 , $ind(A) = 0$ for a nonsingular matrix A. By Theorem 3.10 $ind(A) = n$ for a nilpotent matrix A such that $A^n = \bar{o}$.
Let $\lambda_1, \cdots, \lambda_k$ are the distinct zeros of the characteristic polynomial of the matrix $A \in C^{n\times n}$, and $\lambda_1 = 0$ with multiplicity $\sigma(\lambda_1) = k < n$. From (2) any matrix $A \in C^{n\times n}$ has exactly n eigenvalue. By Theorem 2.3 $\rho(\lambda_1) \leq \sigma(\lambda_1) < n$. From Definition 2.1 we can get $0 < ind(A) < n$. Therefore $0 \leq ind(A) \leq n$.

4 Numerical Examples

We now give the following examples to explain the present results.

Example 4.1 *Determine the index and Drazin inverse of the following matrix*

$$A = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$$

It is obvious that $rank(A) = rank(A^2)$, *so* $ind(A) = 1$. *Moreover the matrix* A *has the eigenvalues* $\lambda_1 = 0, \lambda_2 = 1$ *with multiplicity*

$$\sigma(\lambda_1) = 1, \rho(\lambda_1) = 1$$

$$\sigma(\lambda_2) = 2, \rho(\lambda_2) = 2$$

Thus Jordan normal form of matrix A has the following form

$$J = PAP^{-1} = \begin{pmatrix} [1] & 0 & 0 \\ 0 & [1] & 0 \\ 0 & 0 & [0] \end{pmatrix}, P = \begin{pmatrix} -1 & 3 & 4 \\ -2 & 3 & 5 \\ -1 & 3 & 5 \end{pmatrix} \tag{11}$$

P is a nonsingular matrix. The dimension of largest Jordan block corresponding to the zero eigenvalue of (11) is equal to one. Moreover $A = A^2$,then A is an idempotent matrix, by Corollary 3.5 $ind(A) = 1$. From Theorem 2.5 we have

$$A = P^{-1} \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} P = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$$

wherein $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is a nonsingular matrix of order 2 , and $N = \bar{o}$, is a nilpotent matrix. Therefore the group inverse of A is

$$A_g = P^{-1} \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} P = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$$

Moreover $D = I$, so of Corollary 2.6, we have $A = A_g$. A ,A_g satisfies the conditions (4), thus A_g is group inverse of A. The system

$$\begin{cases} 2x_1 - x_2 + x_3 & = 1 \\ -3x_1 + 4x_2 - 3x_3 & = 1 \\ -5x_1 + 5x_2 - 4x_3 & = 2 \end{cases}$$

is inconsistent. $Ind(A^T) = 1$, it is clear that the following system is consistent.

$$\begin{cases} 2x_1 - x_2 + x_3 & = 3 \\ -3x_1 + 4x_2 - 3x_3 & = -5 \\ -5x_1 + 5x_2 - 4x_3 & = -8 \end{cases}$$

Example 4.2 *Determine the index and Drazin inverse of the following matrix*

$$B = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

It is obvious that $rank(B^3) = rank(B^4)$ *, so* $ind(B) = 3$ *. Moreover the matrix* B *has the eigenvalues* $\lambda = 0$ *with multiplicity*

$$\sigma(\lambda) = 3, \rho(\lambda) = 1$$

So Jordan normal form of matrix B *has the following form*

$$J = PBP^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, P = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & 3 \end{pmatrix} \tag{12}$$

P *is a nonsingular matrix. The dimension of largest Jordan block corresponding to the zero eigenvalue of (12) equal to 3. Moreover* $B^3 = \bar{o}$ *,then* $ind(B) = 3$ *By Theorem 2.5 we have*

$$B = P^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

Therefore the Drazin inverse of B *is*

$$B^D = P^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P = 0$$

Moreover $B \in C^{3\times 3}$ *is a nilpotent matrix of index 3, so of Corollary 3.11 we have* $B^D = \bar{o}$*.* B *,* B^D *fulfills the conditions (4) , thus* B^D *is Drazin inverse of* B*.*

Example 4.3 *Consider the following symmetric matrix*

$$C = \begin{pmatrix} -1 & -1 & -1 \\ -1 & \frac{1}{3} & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

The matrix C *has the eigenvalues* $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -\frac{8}{3}$. *The index of matrix* C *is equal to one, because* $rank(C) = rank(C^2)$. *So Jordan normal form of matrix* C *has the following form*

$$J = PCP^{-1} = \begin{pmatrix} [1] & 0 & 0 \\ 0 & [-\frac{8}{3}] & 0 \\ 0 & 0 & [0] \end{pmatrix}, P = \begin{pmatrix} \frac{1}{11} & -\frac{3}{11} & \frac{1}{11} \\ -\frac{9}{22} & -\frac{3}{11} & -\frac{9}{22} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \tag{13}$$

P *is a nonsingular matrix. The dimension of largest Jordan block corresponding to the zero eigenvalue of (13) is equal to one. By Theorem 2.5 we have*

$$C_g = P^{-1}JP = \frac{1}{16}\begin{pmatrix} -1 & -6 & -1 \\ -6 & 12 & -6 \\ -1 & -6 & -1 \end{pmatrix}$$

C , C_g *satisfies the conditions (4) thus* C_g *is group inverse of* C . *It is clear that* $rank(C) = rank(C_g)$ *and* $(C_g)_g = C$. *Consider the following singular linear system of equation with index one*

$$\begin{cases} -x_1 - x_2 - x_3 & = 3 \\ -x_1 + \frac{1}{3}x_2 - x_3 & = 1 \\ -x_1 - x_2 - x_3 & = 3 \end{cases} \tag{14}$$

Since $b = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \in R(C)$, *the solution of (14) is*

$$x = C_g b = \begin{pmatrix} -\frac{1}{16} & -\frac{6}{16} & -\frac{1}{16} \\ -\frac{6}{16} & \frac{12}{16} & -\frac{6}{16} \\ -\frac{1}{16} & -\frac{6}{16} & -\frac{1}{16} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} \\ -\frac{3}{2} \\ -\frac{3}{4} \end{pmatrix}$$

5 Conclusions

In this paper, we prove some properties of index of matrix and Drazin inverse and give some examples.

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References

[1] Bhaskara K., Meyer C., The Theory of Generalized inverse over commutative Rings, London and New York, 2002.

[2] Campbell S.L., Meyer C., Generalized inverse of Linear Transformations , Pitman London , 1979.

[3] Chen J., Xu Z. (2008) "Comment on condition number of Drazin inverse and their condition number of singular systems," Applied Mathematics and Computation, 199, 512-526.

[4] Chen J., Xu Z. (2008) "Comment on condition number of Drazin inverse and their condition number of singular systems," Applied Mathematics and Computation, 199, 512-526.

[5] Lin L., Wei Y., Zhang N. (2008) "Convergence and quotient convergence of iterative methods for solving singular linear equations with index one," Linear Algebra and its Applications, 203, 35-45.

[6] Stoer J., Bulirsch R., Introduction to Numerical Analysis, Springer, 1992.

[7] Wang G. (1989) "A cramer rule for finding the solution of a class of singular equations," Linear Algebra and its Applications, 116, 27-34.

[8] Wei Y. (1998) "Index splitting for the Drazin inverse and the singular linear system," Applied Mathematics and Computation, 95, 115-124.

[9] Wei Y., Zhou J. (2006) "A two-step algorithm for solving singular linear systems with index one," Applied Mathematics and Computation, 175, 472-485.

[10] Zheng L. (2001) "A characterization of the Drazin inverse," Linear Algebra and its Applications, 335, 183-188.