



## Variational iteration method for a class of singular boundary value problems

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### Abstract

In this paper, the variational iteration method (VIM) is used to solve a class of singular boundary value problems. Three numerical examples are selected to illustrate the effectiveness and simplicity of the method. Numerical solutions obtained by the method are of high accuracy. Numerical results show that the method is a satisfactory method.

**Keywords:** Singularity, Boundary value problems, Variational iteration method.

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## 1 Introduction

In this paper, we tend to extend the use of variational iteration method to the following singular boundary value problems:

$$\begin{cases} u'' + \frac{k}{x}u' + q(x)u(x) = f(x), & 0 < x < 1, \\ u'(0) = \alpha, u(1) = \beta \text{ or } u(0) = \alpha, u(1) = \beta, \end{cases} \quad (1.1)$$

where  $k$  is a real number, which occur very frequently in the theory of thermal explosions and in the study of Electro-hydrodynamics. Such type of problems also arise in

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the study of generalized axially symmetric potentials after separation of variables has been employed. Therefore, the problems have attracted much attention and has been studied by many authors. There is considerable interest in numerical methods on singular boundary value problems. In general, classical numerical methods fail to produce good approximations for the equations. Hence one has to go for non-classical method. The usual classical three-point finite difference discretization for singular boundary value problems has been studied by Russell and Shampine [16]. Iyengar and Jain [8] have discussed the spline function and the finite difference methods for singular boundary value problems. Eriksson and Thomee [2] have studied the Galerkin type piece wise polynomial procedure for these types problems. Kadalbajoo [9] treated the homogeneous equation via Chebyshev polynomial and B-spline. Ravi Kanth and Reddy [12-15] studied singular boundary value problems by applying higher order finite difference method and cubic spline method. Mohanty and his co-workers [10] considered such an singular boundary value problem  $u''(x) + \frac{a}{x}u'(x) + \frac{a}{x^2}u(x) = r(x)$  using a four order accurate cubic spline method. Cui and Geng [1] presented a reproducing kernel Hilbert space method for singular boundary value problems.

In this paper, we solve (1.1) by using the VIM and obtain a highly accurate numerical solution. The advantages of variational iteration method over the existing methods for solving this problem are its simplicity and fast convergence. The method can lead to a fast convergent solution, only a few number of iteration steps can be used for numerical purpose with a high accuracy. So, this can avoid the unnecessary calculation arise in the solution procedure and increase the computation speed. The variational iteration method, which was proposed originally by He [4-7], has been proved by many authors to be a powerful mathematical tool for various kinds of linear and nonlinear problems [3,11,17-19]. The reliability of the method and the reduction in the size of computation gave this method a wider applications.

The rest of the paper is organized as follows. In the next section, The VIM is introduced. The VIM is applied to singular boundary value problem (1.1) in section 3.

The numerical examples are presented in section 4. Section 5 ends this paper with a brief conclusion.

## 2 Analysis of the variational iteration method

Consider the differential equation

$$Lu + Nu = g(x), \quad (2.1)$$

where  $L$  and  $N$  are linear and nonlinear operators, respectively, and  $g(x)$  is the source inhomogeneous term. In [4-7], the VIM was introduced by He where a correct functional equation for (2.1) can be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{Lu_n(t) + N\tilde{u}_n(t) - g(t)\} dt, \quad (2.2)$$

where  $\lambda$  is a general Lagrangian multiplier [5], which can be identified optimally via variational theory, and  $\tilde{u}_n$  is a restricted variation which means  $\delta\tilde{u}_n = 0$ . By this method, it is required first to determine the Lagrangian multiplier  $\lambda$  that will be identified optimally. The successive approximates  $u_{n+1}$ ,  $n \geq 0$ , of the solution  $u$  will be readily obtained upon using the determined Lagrangian multiplier and any selective function  $u_0$ . Consequently, the solution is given by

$$u = \lim_{n \rightarrow \infty} u_n.$$

In fact, the solution of problem (2.1) is considered as the fixed point of the following functional under the suitable choice of the initial term  $u_0(x)$

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{Lu_n(t) + Nu_n(t) - g(t)\} dt. \quad (2.3)$$

As a well known powerful tool, we have

**Theorem 2.1.** (*Banach's fixed point theorem*), Assume that  $X$  be a Banach space and

$$A : X \rightarrow X$$

is a nonlinear mapping, and suppose that

$$\| A[u] - A[v] \| \leq \alpha \| u - v \|, \quad u, v \in X$$

for some constant  $\alpha < 1$ . Then  $A$  has a unique fixed point. Furthermore, the sequence

$$u_{n+1} = A[u_n],$$

with an arbitrary choice of  $u_0 \in X$ , converges to the fixed point of  $A$ .

According to Theorem 2.1, for the nonlinear mapping

$$A[u(x)] = u(x) + \int_0^x \lambda \{ Lu(t) + Nu(t) - g(t) \} dt,$$

a sufficient condition for convergence of the variational iteration method is strictly contraction of  $A$ . Furthermore, the sequence (2.3) converges to the fixed point of  $A$  which is also the solution of problem (2.1).

The efficiency of variational iteration method depends on not only the identification parameter, but also the initial approximation. For linear problems, under the suitable choice of the initial approximations with some unknown parameters, their exact solutions can be obtained by only one iteration step due to the fact that the Lagrangian multiplier can be identified exactly. For nonlinear problems, the lagrange multiplier is difficult to be identified exactly. To overcome the difficulty, we apply restricted variations to nonlinear term. Due to the approximate identification of the Lagrangian multiplier, the approximate solutions converge to their exact solutions relatively slowly. It should be specially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to their exact solutions.

### 3 Analysis of singular boundary value problem (1.1)

For (1.1), according to the VIM, we construct the correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t) \left\{ u_n''(t) + \frac{2}{t} u_n'(t) + \frac{k-2}{t} \tilde{u}_n'(t) + q(t) \tilde{u}_n(t) - f(t) \right\} dt, \quad (3.1)$$

where  $\lambda$  is a general Lagrangian multiplier [5], which can be identified optimally via variational theory, and  $\tilde{u}_n$  is a restricted variation which means  $\delta\tilde{u}_n = 0$ .

The Lagrange multiplier can be identified via variational theory, i.e. the multiplier should be such chosen that the correction functional equation is stationary, i.e.  $\delta u_{n+1}(x) = 0$ .

Making the above correct functional equation stationary with respect to  $u_n$ , noticing that  $\delta\tilde{u}_n = 0$ ,

$$\begin{aligned}
 \delta u_{n+1}(x) &= \delta u_n(x) + \delta \int_0^x \lambda(t) \{u_n''(t) + \frac{2}{t}u_n'(t) + \frac{k-2}{t}\tilde{u}_n'(t) + q(t)\tilde{u}_n(t) - f(t)\} dt \\
 &= \delta u_n(x) + \delta \int_0^x \lambda(t) \{u_n''(t) + \frac{2}{t}u_n'(t)\} dt \\
 &= \delta u_n(x) + \int_0^x \delta \{ \lambda(t) [u_n''(t) + \frac{2}{t}u_n'(t)] \} dt \\
 &= \delta u_n(x) + \int_0^x [\lambda(t)\delta u_n''(t) + \frac{2\lambda(t)}{2}\delta u_n'(t)] dt \\
 &= \delta u_n(x) + \int_0^x [\lambda(t)\frac{d^2}{dt^2}(\delta u_n) + \frac{2\lambda(t)}{2}\frac{d}{dt}(\delta u_n)] dt \\
 &= \delta u_n(x) + \lambda(t)\delta u_n'(t)|_0^x - \lambda'(t)\delta u_n(t)|_0^x + \frac{2\lambda(t)}{t}\delta u_n(t)|_0^x + \\
 &\int_0^x [\lambda''(t)\delta u_n(t) - (\frac{2\lambda(t)}{t})'\delta u_n(t)] dt \\
 &= [1 - \lambda'(x) + \frac{2}{x}\lambda(x)]\delta u_n(x) + \lambda(x)\delta u_n'(x) + \int_0^x \delta u_n(x) [\lambda''(t) - 2\frac{t\lambda'(t)-\lambda(t)}{t^2}] dt \\
 &= 0.
 \end{aligned}$$

We, therefore, have the following stationary conditions:

$$\begin{cases} \lambda''(t) - 2\frac{t\lambda'(t)-\lambda(t)}{t^2} = 0, \\ 1 - \lambda'(x) + \frac{2}{x}\lambda(x) = 0, \\ \lambda(x) = 0. \end{cases} \quad (3.2)$$

The Lagrangian multiplier can be identified in the form

$$\lambda(t) = \frac{t^2}{x} - t. \quad (3.3)$$

So we can obtain the following iteration formula:

$$\begin{aligned}
 u_{n+1}(x) &= u_n(x) + \int_0^x (\frac{t^2}{x} - t) \{u_n''(t) + \frac{k}{t}u_n'(t) + q(t)u_n(t) - f(t)\} dt, \\
 &= u_n(x) + \int_0^x (\frac{t}{x} - 1) \{tu_n''(t) + ku_n'(t) + tq(t)u_n(t) - tf(t)\} dt.
 \end{aligned} \quad (3.4)$$

Now we assume that an initial approximation has the form

$$u_0(x) = \alpha_0 + \alpha_1 x, \quad (3.5)$$

where  $\alpha_0, \alpha_1$  are unknown constants to be further determined.

In terms of (3.4), one can obtain the  $n$ -order approximation  $u_n(x)$ . Incorporating the boundary conditions of (1.1) into  $u_n(x)$ , we have a system with respect to  $\alpha_0, \alpha_1$ . Solving the above system, we obtain  $\alpha_0, \alpha_1$ , and therefore, we obtain the  $n$ -order approximation  $u_n(x)$ .

#### 4 Numerical examples

Now we apply the variational iteration method to solve three singular boundary value problems. Numerical results show that the VIM is very effective.

##### Example 4.1

Consider the following singular boundary value problem [1,5]:

$$\begin{cases} u''(x) + \frac{2}{x}u'(x) - 4u(x) = -2, & 0 < x < 1, \\ u'(0) = 0, u(1) = 5.5. \end{cases} \quad (4.1)$$

The exact solution of this problem is  $u(x) = 0.5 + \frac{5 \sinh 2x}{x \sinh 2}$ .

According to (3.4), we have the following iteration formulae:

$$u_{n+1}(x) = u_n(x) + \int_0^x \left(\frac{t^2}{x} - t\right) \{u_n''(t) + \frac{2}{t}u_n'(t) - 4u_n(t) + 2\} dt. \quad (4.2)$$

Now we assume that an initial approximation has the form

$$u_0(x) = \alpha_0 + \alpha_1 x, \quad (4.3)$$

where  $\alpha_0, \alpha_1$  are unknown constants to be further determined.

In terms of (4.2), one can obtain the 5-order approximation  $u_5(x)$ . Incorporating the boundary conditions of (4.1) into  $u_5(x)$ , we have

$$\begin{cases} \frac{-190251 + 848277\alpha_0 + 646043\alpha_1}{467775} = \frac{11}{2}, \\ \alpha_1 = 0. \end{cases} \quad (4.4)$$

Solving the above system, we obtain

$$\alpha_0 = \frac{614003}{188506}, \alpha_1 = 0,$$

and therefore, we obtain the 5-order approximation

$$u_5(x) = 3.25721 + 1.83814x^2 + 0.367628x^4 + 0.0350121x^6 + 0.00194512x^8 + 0.0000707316x^{10}. \quad (4.5)$$

In a similar way, we can obtain the 10-order approximation  $u_{10}(x)$ . The numerical results are shown in Figure 1.

#### Example 4.2

Consider the following singular boundary value problem [2,5]:

$$\begin{cases} u''(x) + \frac{2}{x}u'(x) + (x^2 - 1)u(x) = -x^4 + 2x^2 - 7, & 0 < x < 1, \\ u'(0) = 0, u(1) = 0. \end{cases} \quad (4.6)$$

The exact solution of this problem is  $u(x) = 1 - x^2$ .

According to (3.4), we have the following iteration formulae:

$$u_{n+1}(x) = u_n(x) + \int_0^x \left(\frac{t^2}{x} - t\right) \{u_n''(t) + \frac{2}{t}u_n'(t) + (t^2 - 1)u_n(t) + t^4 - 2t^2 + 7\} dt \quad (4.7)$$

Now we assume that an initial approximation has the form

$$u_0(x) = \beta_0 + \beta_1 x, \quad (4.8)$$

where  $\beta_0, \beta_1$  are unknown constants to be further determined.

In terms of (4.7), one can obtain the 3-order approximation  $u_3(x)$ . Incorporating the boundary conditions of (4.6) into  $u_3(x)$ , we have

$$\begin{cases} \frac{-2035235646 + 2035241677\beta_0 + 1908981109\beta_1}{1816214400} = 0, \\ \beta_1 = 0. \end{cases} \quad (4.9)$$

Solving the above system, we obtain

$$\beta_0 = \frac{2035235646}{2035241677}, \beta_1 = 0,$$

and therefore, we obtain the 3-order approximation

$$u_3(x) = 0.999997 - x^2 + 1.2347 \cdot 10^{-7} x^4 + 1.46988 \cdot 10^{-8} x^6 + 0.0000165329 x^8 - 0.0000201421 x^{10} + 7.95888 \cdot 10^{-6} x^{12} - 1.03072 \cdot 10^{-6} x^{14}. \quad (4.10)$$

In a similar way, we can obtain the 8-order approximation  $u_8(x)$ . The numerical results are shown in Figure 2.

### Example 4.3

Consider the Bessel's equation of zero order [3,5]:

$$\begin{cases} u''(x) + \frac{1}{x}u'(x) + u(x) = 0, & 0 < x < 1, \\ u'(0) = 0, u(1) = 1. \end{cases} \quad (4.11)$$

The exact solution of this problem is  $u(x) = \frac{J_0(x)}{J_0(1)}$ , where  $J_0(x)$  is the Bessel function of the first kind.

According to (3.4), we have the following iteration formulae:

$$u_{n+1}(x) = u_n(x) + \int_0^x \left(\frac{t^2}{x} - t\right) \left\{ u_n''(t) + \frac{1}{t}u_n'(t) + u_n(t) \right\} dt. \quad (4.12)$$

Now we assume that an initial approximation has the form

$$u_0(x) = \gamma_0 + \gamma_1 x, \quad (4.13)$$

where  $\gamma_0, \gamma_1$  are unknown constants to be further determined.

In terms of (4.12), one can obtain the 5-order approximation  $u_5(x)$ . Incorporating the boundary conditions of (4.11) into  $u_5(x)$ , we have

$$\begin{cases} \frac{5392970460368\gamma_0 + 12425036079855\gamma_1}{7041323520000} = 1, \\ \frac{63\gamma_1}{32} = 0. \end{cases} \quad (4.14)$$

Solving the above system, we obtain

$$\gamma_0 = \frac{440082720000}{337060653773}, \gamma_1 = 0,$$

and therefore, we obtain the 5-order approximation

$$u_5(x) = 1.30565 - 0.325069 x^2 + 0.0199297 x^4 - 0.000515671 x^6 + 6.43074 \cdot 10^{-6} x^8 - 3.27092 \cdot 10^{-8} x^{10}. \quad (4.15)$$



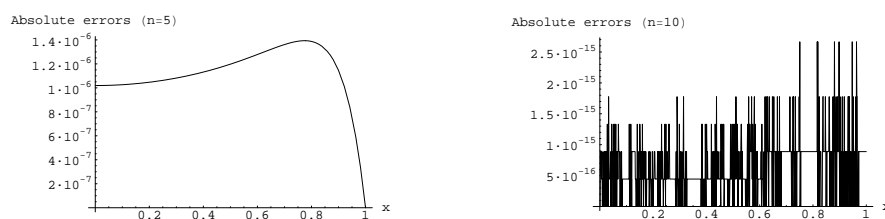


Figure 1: Figures of absolute errors  $|u - u_5|$  and  $|u - u_{10}|$  on  $[0, 1]$  for Example 4.1

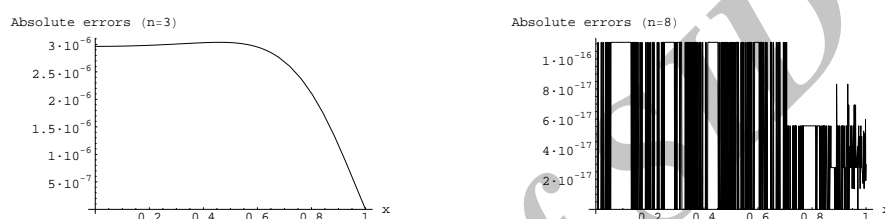


Figure 2: Figures of absolute errors  $|u - u_3|$  and  $|u - u_8|$  on  $[0, 1]$  for Example 4.2

In a similar way, we can obtain the 30-order approximation  $u_{30}(x)$ . The numerical results are shown in Figure 3.

**Remark:** When we solve Example 4.1, 4.2, 4.3 using higher order finite difference method presented in [5], the maximum absolute error is  $10^{-5}$ , while the maximum absolute error can be less than  $10^{-15}$  using the present method. It is clear that the present method is much more accurate and efficient.

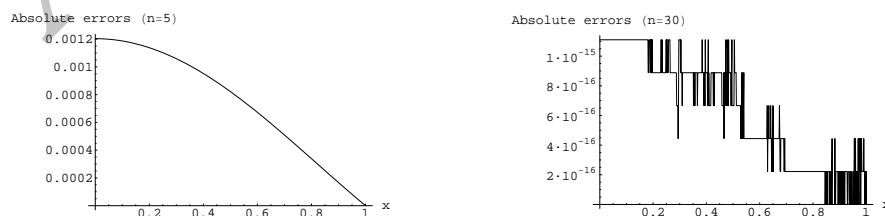


Figure 3: Figures of absolute errors  $|u - u_5|$  and  $|u - u_{30}|$  on  $[0, 1]$  for Example 4.3

## 5 Conclusion

In this paper, the VIM is used to solve a class of singular boundary value problems. Comparing with other methods, the results of numerical example demonstrate that this method is more accurate than the existing methods. Therefore, our conclusion is that the VIM is a satisfactory method for solving the class of singular boundary value problems.

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