



Classification the integral curves of a second degree homogeneous ODE

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Abstract

This paper classifies the integral curves of the $y' = (ax^2 + bxy + cy^2)/(dx^2 + exy + fy^2)$. A new approach has been given, instead of which, that has been described by J. Argemi, L. Lyagin, K.S. Sibirsky et al.

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1 Introduction

The study of homogeneous polynomial vector fields started in 1960 with a paper of Markus [9], where he classified the quadratic homogeneous polynomial vector fields $X = P \partial_x + Q \partial_y$ such that P and Q have no common factor.

Later in 1968 Argem [1] completed the classification of Markus. Moreover, he furnished the classification of the cubic vector fields that have no common factor. At the same time, he obtained upper and lower bounds for the number of phase portraits of the planar homogeneous polynomial vector fields of degree m which have no common factor.

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Subsequent results, relative to an algebraic classification of quadratic homogeneous polynomial vector fields can be found in the paper of Date [5] in 1979. There, the author also gives the classification of quadratic vector fields with common factors. This algebraic classification has also been made in a different way by Sibirsky [10] using algebraic invariants.

In 1990 Cima and Llibre [3] obtained a topological classification of the cubic homogeneous polynomial vector fields with or without common factors and they present an algorithm for studying the phase portraits of homogeneous polynomial vector fields of arbitrary degree. In that paper, one can also find an algebraic classification of the cubic homogeneous polynomial vector fields which was extended by Collins [4] to arbitrary degree.

In this paper we classify the integral curves of the $y' = (ax^2 + bxy + cy^2)/(dx^2 + exy + fy^2)$, by a different approach.

2 General properties of homogeneous systems

Consider a homogeneous ODE of the form

$$\frac{dy}{dx} = \frac{A_m(x, y)}{B_m(x, y)} = \frac{a_0y^m + a_1y^{m-1}x + \dots + a_mx^m}{b_0y^m + b_1y^{m-1}x + \dots + b_mx^m}, \quad (1)$$

where $A_m(x, y)$ and $B_m(x, y)$ have no any real linear factors in common, and $A_m/B_m = y/x$ does not hold identically in x and y .

Introducing polar coordinates $x = r \cos \varphi$ and $y = r \sin \varphi$, and assume

$$\begin{aligned} \tau &= \tan \varphi, \\ A_m(\varphi) &= a_0 \sin^m \varphi + a_1 \sin^{m-1} \varphi \cos \varphi + \dots + a_m \cos^m \varphi, \\ B_m(\varphi) &= b_0 \sin^m \varphi + b_1 \sin^{m-1} \varphi \cos \varphi + \dots + b_m \cos^m \varphi, \\ Z(\varphi) &= A_m(\varphi) \sin \varphi + B_m(\varphi) \cos \varphi, \\ N(\varphi) &= B_m(\varphi) \sin \varphi - A_m(\varphi) \cos \varphi = N_1(\tau) \cos^{m+1} \varphi. \end{aligned} \quad (2)$$

Then, writing the equation in polar form, we have

$$\frac{dr}{d\varphi} = -r \frac{Z(\varphi)}{N(\varphi)}. \quad (3)$$

Therefore,

$$r(\varphi) = r_0 \exp\left(-\int_{\varphi_0}^{\varphi} \frac{Z(\varphi)}{N(\varphi)} d\varphi\right). \quad (4)$$

To describe the behavior of trajectories near the origin we determine first the straight-line trajectories which adhere to the origin. In view of (3) these trajectories lie on the rays (integral rays) determined by the real roots of the equation

$$N(\varphi) = 0 \quad (5)$$

If $\varphi = \varphi_1$ is a real root of (5), then $\varphi = \varphi_1 + \pi$ is also a real root and the resulting pair of corresponding rays lies on the same straight line. Since $N(\varphi)$ is a homogeneous polynomial in $\sin \varphi$, $\cos \varphi$ of degree $m+1$ and is not identically zero by hypothesis there are at most $m+1$ lines through the origin determined by the above pairs of straight-line trajectories.

1.1. Theorem. *When $N(\varphi) = 0$ has no real roots, we have*

- 1) *If $\int_0^{2\pi} \frac{Z(\varphi)}{N(\varphi)} d\varphi = 0$, then all the trajectories are closed and the origin is a center.*
- 2) *If $\int_0^{2\pi} \frac{Z(\varphi)}{N(\varphi)} d\varphi > 0$, then $\lim_{\varphi \rightarrow +\infty} r(\varphi) = 0$, the trajectories are spirals, and the origin is a stable focus.*
- 3) *If $\int_0^{2\pi} \frac{Z(\varphi)}{N(\varphi)} d\varphi < 0$, then $\lim_{\varphi \rightarrow +\infty} r(\varphi) = 0$, the trajectories are spirals, and the origin is a unstable focus.*

We consider next the case when (5) has real roots. Let $\varphi = \varphi_1$ and $\varphi = \varphi_2$, $\varphi_1 < \varphi_2$, be two consecutive straight-line trajectories (integral rays). The trajectory $r = r(\varphi)$ through a point of sector $r > 0$, $\varphi_1 < \varphi < \varphi_2$ is well defined for $\varphi_1 < \varphi < \varphi_2$ and as φ tends to one of φ_1 or φ_2 , $r(\varphi)$ tends either to 0 or to $+\infty$.

We say the sector $r > 0$, $\varphi_1 < \varphi < \varphi_2$, elliptic sector as φ tends to φ_1 and φ_2 , $r(\varphi) \rightarrow 0$, hyperbolic sector as φ tends to φ_1 and φ_2 , $r(\varphi) \rightarrow +\infty$ or parabolic sector as φ tends to φ_1 and φ_2 , $r(\varphi)$ tends to 0 and $+\infty$ or vice versa.

Let φ_j be a real root of (5), of multiplicity ν_j , i.e. let $\tau_j = \tan \varphi_j$ be such a root of $N_1(\tau)$. Since $N(\varphi)$ is homogeneous in $\sin \varphi$ and $\cos \varphi$ we may write

$$N(\varphi) = \sin^{\nu_j}(\varphi - \varphi_j) \cdot Q_j(\varphi), \quad (6)$$

where $Q_j(\varphi)$ is a homogeneous polynomial of degree $m - \nu_j$ in $\sin \varphi$, $\cos \varphi$ and $Q_j(\varphi_j) \neq 0$. Also $Z(\varphi_j)/Q(\varphi_j) \neq 0$.

The behavior of trajectories in the neighborhood of integral ray $\varphi = \varphi_j$ is described by

1.2. Theorem. *Consider an integral ray corresponding to a real root $N(\varphi) = 0$ of multiplicity ν_j .*

1) *If ν_j is odd, then the trajectories have a similar behavior on both side of the ray.*

a) *In case $Z(\varphi_j)/Q_j(\varphi_j) > 0$, on both sides of our the trajectories recede from the origin as $\varphi \rightarrow \varphi_j$. (Isolated ray)*

b) *In case $Z(\varphi_j)/Q_j(\varphi_j) < 0$, on both sides of our the trajectories tend toward the origin as $\varphi \rightarrow \varphi_j$. (Nodal ray)*

2) *If ν_j is even then the trajectories do not behave alike on both sides of the ray.*

a) *In case $Z(\varphi_j)/Q_j(\varphi_j) < 0$ we have: $r(\varphi) \rightarrow 0$ as $\varphi \rightarrow \varphi_j^+$ and $r(\varphi) \rightarrow +\infty$ as $\varphi \rightarrow \varphi_j^-$*

b) *In case $Z(\varphi_j)/Q_j(\varphi_j) > 0$ we have: $r(\varphi) \rightarrow +\infty$ as $\varphi \rightarrow \varphi_j^+$ and $r(\varphi) \rightarrow 0$ as $\varphi \rightarrow \varphi_j^-$*

In all cases the trajectories which tend to origin are tangent to the given ray at the origin. i.e. not only dose $\varphi(t)$ tend toward φ_j but the slope of tangent to $(r(t), \varphi(t))$ at t tends to the slope of to polar ray φ_j .

1.3 Corollary *Consider an integral ray $\varphi = \varphi_j$ corresponding to a real root $N(\varphi) = 0$ of even (odd) multiplicity. If $\varphi = \varphi_j$ is the boundary of two consecutive elliptic or hyperbolic sector, these sector can not be (must be) both elliptic or both hyperbolic.*

3 The index of Poincaré

We denote by $\vec{V}(M)$ the vector whose initial point is $M(x, y)$ and whose components are $P(x, y)$ and $Q(x, y)$. the totality of such vectors is the vector field defined by system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (7)$$

and $M(x, y)$ is a singular point if and only if $\vec{V}(M) = \mathbf{0}$.

Consider next a closed curve k not passing through a singular point. Take a point M on k . As M describe k once, say in the positive sense, the vector $\vec{V}(M)$ may go through a number of complete revolutions some in the positive and some in the negative sense. In the process the angle which $\vec{V}(M)$ makes with a fixed vector changes by an integral multiple $2\pi J_k$ of 2π . The integral (positive, zero, or negative) J_k is called the index of k relative to to vector field $V = \{\vec{V}(M)\}$. the index J_k has the following property which is useful for what follows.

The index of J_k does not change as k varies continuously without crossing singular points.

Consider now a point M and choose a circular neighborhood $S(M, \varepsilon)$ containing no singular points with the possible exception of M itself. Let k be the circumference of S . then J_k is called the index of M . In this definition the circumference k may be replaced by any other simply-closed Jordan curve k_1 containing in it's interior our point M but no singular points with the possible exception of M . We write J_M for the index of the point M .

4 The Bendixson formula

Now we state the Bendixson formula which is about the relationship between the index, elliptic sectors and hyperbolic sectors.

3.1 Theorem *If (x_0, y_0) is an isolated equilibrium point of a two dimensional system*

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (8)$$

then the index of (x_0, y_0) is $1 + \frac{1}{2}(N_E - N_H)$ where N_E is the number of elliptic sectors and N_H is the number of hyperbolic sectors.

5 Index of homogeneous fields

In the following by index we mean the index of the origin. Let P, Q are polynomials in x and y of degrees p and q respectively. And have no common real linear factors. Now we write P and Q in factored form, i.e.

$$P(x, y) = K_1 \prod_{i=1}^m (y - a_i x)^{p_i}, \quad Q(x, y) = K_2 \prod_{j=1}^m (y - b_j x)^{q_j}. \quad (9)$$

Where K_1 and K_2 are constants. (We include the possibility that some $a_i = \infty$ or some $b_j = \infty$; equivalently, that the factor $y - a_i x$ is equal to $-x$ or the factor $y - b_j x$ is equal to $-x$)

If K_1 and K_2 are omitted the value of index is multiplied by sign $(K_1 K_2)$. also the value of index is not affected if the following factors in the products on the right-hand side of (9) are omitted.

- 1) Pairs of factors $(y - a_{i_1} x), (y - a_{i_2} x)$ in which a_{i_1} and a_{i_2} are complex conjugates.
- 2) Factors $(y - a_{i_1} x)^{p_i}$ where a_{i_1} is real and p_i is even.
- 3) Pairs of factors $(y - a_{i_1} x), (y - a_{i_2} x)$ such that $a_{i_1} < a_{i_2}$ and there is no b_j or a_{i_3} such that $a_{i_1} < b_j < a_{i_2}$ or $a_{i_1} < a_{i_3} < a_{i_2}$.

- 4) Pairs of factors $(y - a_{i_1}x)$, $(y - a_{i_2}x)$ in which a_{i_1} and a_{i_2} are the smallest and largest of all the numbers $a_1, \dots, a_m, b_1, \dots, b_n$.
- 5) the factors listed above with the a'_i s replaced by b'_i s.

Therefore, either all the factors are omitted and the index is zero or we have a mapping such that

$$a_1 < b_1 < a_2 < \dots < a_m < b_m, \quad \text{or} \quad b_1 < a_1 < b_2 < \dots < b_m < a_m, \quad (10)$$

and with each factor $(y - a_i x)$ and $(y - b_i x)$ having exponent one, and the index is m or $-m$.

We consider the homogeneous system of the form

$$\begin{aligned} \frac{dx}{dt} &= b_0 y^m + b_1 y^{m-1} x + \dots + b_m x^m, \\ \frac{dy}{dt} &= a_0 y^m + a_1 y^{m-1} x + \dots + a_m x^m. \end{aligned} \quad (11)$$

Let the characteristic function:

$$\varphi(\lambda) = \frac{a_0 \lambda^m + a_1 \lambda^{m-1} + \dots + a_m}{b_0 \lambda^m + b_1 \lambda^{m-1} + \dots + b_m}, \quad a_0^2 + b_0^2 \neq 0. \quad (12)$$

Now we state a theorem which is very important and practical.

4.1 Theorem *If the denominator of characteristic function can be factored by m distinct linear factor, and $\varphi'(\lambda)$ is positive (negative) the index of the equation is $m(-m)$. If $\varphi'(\lambda) = 0$ has a root, the index J of the equation satisfies, $2 - m \leq J \leq m - 2$.*

4.2 Theorem *If the denominator of characteristic function can be factored by m distinct linear factor, and $\varphi'(\lambda) < 0$ the origin is a saddle point with $2(m+1)$ hyperbolic sectors.*

Proof: it can be shown easily if the denominator can be factored by m distinct linear factors and $\varphi'(\lambda) < 0$ the equation $\varphi(\lambda) = \lambda$ has exactly $m + 1$ roots and there are

$m + 1$ straight line trajectories through the origin and $2(m + 1)$ sectors. Also the index of the equation is $-m$ then by the Bendixson formula all of them are hyperbolic sectors. Therefore, origin is a saddle point.

4.3 Example In the case $m = 1$, we have the following equation

$$\frac{dx}{dt} = cx + dy, \quad \frac{dy}{dx} = ax + by, \quad (13)$$

and the characteristic function is $\varphi(\lambda) = (b\lambda + a)/(d\lambda + c)$ and its derivation is $\varphi'(\lambda) = (bc + ad)/(d\lambda + c)^2$. Therefore, $J = 1$ while $bc - ad > 0$ and $J = -1$ while $bc - ad < 0$.

4.4 Example In the case $m = 2$, we have the following equation

$$\frac{dx}{dt} = dx^2 + exy + fy^2, \quad \frac{dy}{dt} = ax^2 + bxy + cy^2. \quad (14)$$

The characteristic function is $\varphi(\lambda) = (c\lambda^2 + b\lambda + a)/(f\lambda^2 + e\lambda + d)$ with derivative

$$\varphi'(\lambda) = \frac{(ce - bf)\lambda^2 + 2(cd - af)\lambda + (bd - ae)}{(f\lambda^2 + e\lambda + d)^2}. \quad (15)$$

Let $A = ce - bf$, $B = cd - af$, and $C = bd - ae$; Then

- 1) If $B^2 - AC < 0$ and $A < 0$ then $\varphi'(\lambda) < 0$ and $f\lambda^2 + e\lambda + d = 0$ has two distinct real roots. Therefore, $J = -2$.
- 2) If $B^2 - AC < 0$ and $A > 0$ then $\varphi'(\lambda) > 0$ and $f\lambda^2 + e\lambda + d = 0$ has two distinct real roots. Therefore, $J = 2$.
- 3) If $B^2 - AC > 0$ then $\varphi'(\lambda) = 0$ has a root. Therefore, $-1 < J < 1$ or $J = 0$.

6 The main result

Now we are ready to state the main result and use example 4.4.

5.1 Theorem *Let N_L be the number of straight-line trajectories, N_H the number of hyperbolic sectors, then N_E the number of elliptic sectors and N_P be the number of parabolic sectors of the following equation (14). Then*

1) *If $B^2 - AC > 0$, and*

a) $N_L = 1$, then $(N_E = 0, N_H = 2, N_P = 0)$,

b) $N_L = 2$, then $(N_E = 0, N_H = 2, N_P = 2)$,

c) $N_L = 3$, then $(N_E = 0, N_H = 2, N_P = 4)$.

2) *If $B^2 - AC > 0$, $A > 0$, and*

a) $N_L = 1$, then $(N_E = 2, N_H = 0, N_P = 0)$,

b) $N_L = 2$, then $(N_E = 2, N_H = 0, N_P = 2)$,

c) $N_L = 3$, then $(N_E = 0, N_H = 0, N_P = 4)$.

3) *If $B^2 - AC < 0$, and $A < 0$, then $(N_E = 0, N_H = 6, N_P = 0)$.*

Proof: Because $N_1(\tau) = 0$ is of degree three, it has at least one real solution and always we have straight-line trajectories. Therefore, origin can not be center or focus.

Depending on the number of straight-line trajectories, we have two, four or six sectors. And because of symmetry in the equation, symmetric sectors are similar. Therefore, N_H and N_E are even numbers.

1) $B^2 - AC > 0$ implies the index is zero, then $N_E - N_H = -2$.

a) If we have two sectors, then $N_E + N_H \leq 2$ and $(N_E = 0, N_H = 2, N_P = 0)$.

b) If we have four sectors, then $N_E + N_H \leq 4$ and because N_H and N_E are even numbers. Therefore, we have only the case $(N_E = 0, N_H = 2, N_P = 2)$.

c) If we have six sectors, then $N_E + N_H \leq 6$ and because of corollary 1.3, we have only the case $(N_E = 0, N_H = 2, N_P = 4)$.

- 2) $B^2 - AC < 0$ and $A > 0$ implies the index is two. Therefore, $N_E - N_H = 2$.
- a) If we have two sectors, then $N_E + N_H \leq 2$ and $(N_E = 2, N_H = 0, N_P = 0)$.
 - b) If we have four sectors, then $N_E + N_H \leq 4$ and because N_H and N_E are even numbers. Therefore, we have only the case $(N_E = 2, N_H = 0, N_P = 2)$.
 - c) If we have six sectors, then $N_E + N_H \leq 6$ and because of corollary 1.3, we have only the case $(N_E = 2, N_H = 0, N_P = 4)$.
- 3) $B^2 - AC < 0$ and $A < 0$ implies the index is -2 , then $N_E - N_H = -6$ and we have six sectors. Therefore, we have only the case $(N_E = 0, N_H = 6, N_P = 0)$.

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