



Estimate of Extinction Probability of Bisexual Galton-Watson Branching Process

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Abstract

In this paper a bisexual Galton-Watson branching process is studied. Monte Carlo method is purposed to calculate the extinction probability. For certain class of processes $\{Z_n\}$ extinction probability is calculated and simulated, when initially population size (Z_0) has a different value, then results of two methods are compared.

Keywords: Bisexual Galton-Watson process, extinction probability, simulation, Monte Carlo method.

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1 Introduction

The bisexual Galton-Watson branching process (BGWBP) initially introduced by Daley [8]. The BGWBP is a two-type branching model with F_n females and M_n males in the n th generation, $n = 0, 1, \dots$ which form $Z_n = L(F_n, M_n)$ mating units, where $L(x, y)$ is a non-negative integer-valued function which is non-decreasing in both $x, y \in N$. These mating units reproduce the next generation independently through the same offspring probability distribution (p_n) in each generation. Each offspring is female with

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probability α ($0 < \alpha < 1$). Daley argues that $\{Z_n\}$ is Markov chain with the non-negative integers as states. Daley *et. al.* [5] also obtained a necessary and sufficient condition to extinction with probability one. Hull [7], Bruss [6] and Alsmeyer and Rosler [3] obtain extinction probability for some class of this process. Xing Yongsheng and Wang Xueqiang [1] studied extinction of population-size-depending. Farnoosh and Zarabi Zadeh [2] estimate extinction probability based on Monte Carlo method.

Usually the calculus of extinction probability is so difficult. In this paper Monte Carlo method is purposed to calculate the extinction probability. For certain class of this processes extinction probability is studied when initially population size (Z_0) has a different value. In section 2 the bisexual Galton-Watson branching process is introduced. In section 3 the extinction probability is estimated, and confidence interval for extinction probability is obtained. For certain class these processes are simulated, and results of two method are compared. Finally extinction probability is estimated when indicate of distribution of Z_n is so difficult and then we can't calculate q_j .

2 The bisexual Galton-Watson branching process

In this section the bisexual Galton-Watson branching process is introduced.

The bisexual Galton-Watson branching process are identified by the following four parameters:

- 1) The number of mating units in the initial generation, (Z_0).
- 2) The mating function (L).
- 3) The offspring probability low ($\{p_n\}_{n=0}^{\infty}$).
- 4) The probability that any individual offspring will be female (α).

A bisexual Galton-Watson branching process can be described as follows:

Let $\{(F_{n,i}, M_{n,i}); n = 0, 1, 2, \dots, i = 1, 2, \dots\}$ be a family of integer-value, independent and identically distributed bivariate random variables, the mating function $L : Z^+ \times$

$Z^+ \rightarrow Z^+$ is a function with following characteristics, monotonic non-decreasing in each argument and integer valued for integer-valued arguments and such that $L(x, y) \leq xy$. We define processes $\{Z_n\}_n$ and $\{(F_n, M_n)\}_n$ by the iterative relation:

$$Z_0 = N \geq 1, \quad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (F_{n,i}, M_{n,i}), \quad Z_{n+1} = L(F_{n+1}, M_{n+1}),$$

for $n = 0, 1, 2, \dots$ and N is a positive integer, when $Z_n = 0$ then $(F_{n+1}, M_{n+1}) = (0, 0)$. In this model, $F_{n,i}$ and $M_{n,i}$ represent the number of females and males produced by i th mating unit in the n th generation, respectively. F_n and M_n are the number of females and males in the n th generation, respectively, which form, $Z_n = L(F_n, M_n)$ mating unit. These mating units reproduce the next generation independently through the same offspring probability distribution in each generation. Daley argues that $\{Z_n\}, \{(F_n, M_n)\}$ are Markov chain with the non-negative integers as states. The state has stationary one step transition probabilities and 0 and $(0, 0)$ states are absorbing. Let $T_{n,i} = F_{n,i} + M_{n,i}$ for $i = 1, 2, \dots, Z_n$, the term $T_{n,i}$ denotes the total number of offspring produced by the i th mating unit in the n th generation. $\{T_{n,i}\}_{n,i}$ is a double array of integer-valued, independent and identically distributed random variables, each offspring is female with probability α ($0 < \alpha < 1$) or male with probability $(1 - \alpha)$.

A branching process is said to be superadditive when its mating function L is superadditive *i.e.* for any x_1, x_2, y_1, y_2 in Z^+ :

$$L(x_1 + x_2, y_1 + y_2) \geq L(x_1, y_1) + L(x_2, y_2).$$

The extinction probability define as a following:

$$q_j = \lim_{n \rightarrow \infty} P(Z_n = 0 \mid Z_0 = j)$$

The extinction problem has been studied by Daley, Hull, Bruss, Alsmeyer and Rosler and Xing Yong Sheng. The main result proved based on the concept of mean growth rate by Daley *et. al.*.

Let $r_k = k^{-1}E(Z_{n+1}|Z_n = k), k = 1, 2, \dots$ (mean growth rates), Daley et. al. show $\lim_{k \rightarrow \infty} r_k$ exists and:

$$r = \lim_{k \rightarrow \infty} r_k = \sup_{k > 0} r_k \quad (2.1)$$

moreover

$$q_j = 1, j = 1, 2, \dots \Leftrightarrow r \leq 1.$$

3 Estimation of extinction probability

In this section an estimator for extinction probability and confidence interval for extinction probability are introduced and for certain class these processes are simulated and extinctions probability are estimated.

Define random variable $(X_{n,j})$, when $Z_0 = j$ as a following:

$$X_{n,j} = I_{\{Z_n=0\}}.$$

$X_{n,j}$ has the Bernolli(1, q_j) distribution. When $X_{n_1,j}, \dots, X_{n_N,j}$ be *iid* then an estimator for the extinction probability is given by:

$$\hat{q}_j = \frac{\sum_{i=1}^N X_{n_i,j}}{N}.$$

This estimator is a UMVU estimator of q_j (Lehmann [4]), then \hat{q}_j is Monte Carlo estimator of q_j , also when n is large standard error of \hat{q}_j is obtained as following:

$$\frac{S}{\sqrt{N}} = \sqrt{\frac{\hat{q}_j(1 - \hat{q}_j)}{N}},$$

when n is large $1 - \alpha$ confidence interval for q_j is given by:

$$\{q : \hat{q}_j - z_{1-\alpha/2} \sqrt{\frac{\hat{q}_j(1 - \hat{q}_j)}{N}} < q < \hat{q}_j + z_{1-\alpha/2} \sqrt{\frac{\hat{q}_j(1 - \hat{q}_j)}{N}}\}.$$

Here Monte Carlo simulation of this process is purposed to estimate q_j .

To estimate q_j we should identified four parameters of this process then in each generation for each mating unit we simulate the number of offspring (m). Since number of female in each mating unit has the Bernolli(m, α) we can simulate number of female and male in each mating unit ($F_{n,i}, M_{n,i}$) and finally we can obtain number of female and male in n th generation (F_n, M_n). Then we use mating function (L) and obtain number of mating unit in n th generation (Z_{n+1}).

Example 3.1. *A bisexual Galton-Watson branching process with four following parameters is considered:*

$$Z_0 = j, \quad P(T_{n,i} = 1) = 1, \quad L(x, y) = \begin{cases} x + y + 1 & x \geq 1, y \geq 1, \\ 0 & x = 0, y = 0. \end{cases} \quad (3.2)$$

Also each offspring is female with probability α . This implies $Z_{n+1} = 0$ or $Z_{n+1} = Z_n + 1$ moreover, $Z_n > 0$ iff $X_n \geq 1, Y_n \geq 1$. Then we can obtain:

$$\begin{aligned} q_j &= \lim_{n \rightarrow \infty} P(Z_n = 0 \mid Z_0 = j) \\ &= 1 - \lim_{n \rightarrow \infty} P(Z_n > 0 \mid Z_0 = j) \\ &= 1 - \lim_{n \rightarrow \infty} P(Z_n > 0 \mid Z_{n-1} > 0)P(Z_{n-1} > 0 \mid Z_0 = j) \\ &= 1 - \lim_{n \rightarrow \infty} \prod_{i=1}^n P(Z_i > 0 \mid Z_{i-1} > 0) \\ &= 1 - \lim_{n \rightarrow \infty} \prod_{i=1}^n (1 - P(X_i = 0 \text{ or } Y_i = 0 \mid Z_{i-1} = Z_0 + i - 1)). \end{aligned}$$

When $\alpha = 0.5$ we have

$$q_j = 1 - \lim_{n \rightarrow \infty} \prod_{i=1}^n (1 - (0.5)^{Z_0+i-2}). \quad (3.3)$$

When $Z_0 = 3$ convergence of \hat{q}_j is given in Figure1. For different value of Z_0 we calculate q_j , also we simulate \hat{q}_j and compare q_j and \hat{q}_j , standard error and 0.99 confidence interval are calculated. Results is shown in Table 1.

The results for 300000 simulations are reasonable.

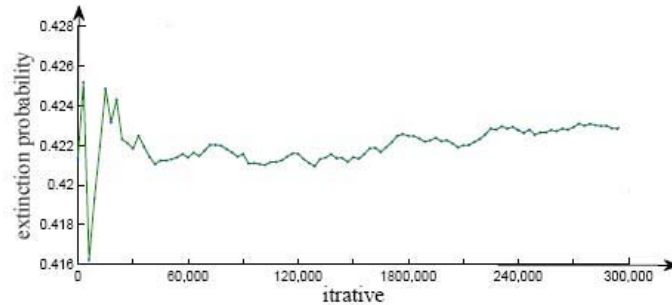


Figure 1: Convergence of \hat{q} (Example 3.1, $Z_0 = 3$, $\alpha = 0.5$)

Example 3.2. A bisexual Galton-Watson branching process with four following parameters is considered:

$$Z_0 = j, \quad P(T_{n,i} = 3) = 1, \quad L(x, y) = \begin{cases} x + y - 1 & x \geq 1, y \geq 1, \\ 0 & x < 1, y < 1. \end{cases} \quad (3.4)$$

Also each offspring is female with probability α . This implies $Z_{n+1} = 0$ or $Z_{n+1} = 3Z_n - 1 = 3^{n+1}Z_0 + \frac{1-3^{n+1}}{2}$ moreover, $Z_n > 0$ iff $X_n \geq 1, Y_n \geq 1$. Then we can obtain:

$$\begin{aligned} q_j &= \lim_{n \rightarrow \infty} P(Z_n = 0 \mid Z_0 = j) \\ &= 1 - \lim_{n \rightarrow \infty} \prod_{i=1}^n \left(1 - P(X_i < 1 \text{ or } Y_i < 1 \mid Z_{i-1} = 3^{i-1}Z_0 + \frac{1-3^{i-1}}{2})\right). \end{aligned}$$

When $\alpha = 0.5$ we have:

$$q_j = 1 - \lim_{n \rightarrow \infty} \prod_{i=1}^n \left(1 - (0.5)^{\frac{(2j-1)3^i+1}{2}}\right). \quad (3.5)$$

When Z_0 convergence of \hat{q}_j is given in Figure 2. For different value of Z_0 we calculate q_j , also we simulate \hat{q}_j and compare q_j and \hat{q}_j , standard error and 0.99 confidence interval are calculated. Results is shown in Table 2.

The results for 300000 simulations are reasonable.

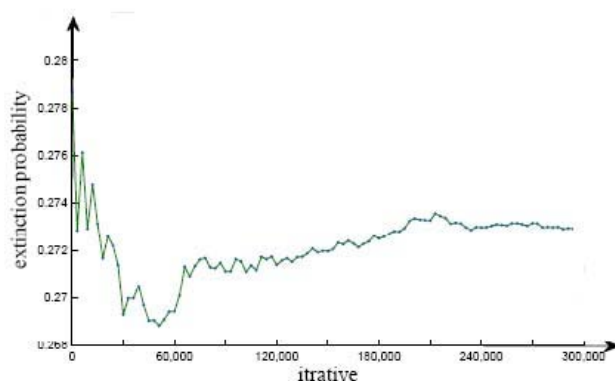


Figure 2: Convergence of \hat{q} (Example 3.2, $Z_0 = 1$, $\alpha = 0.5$)

Example 3.3. A bisexual Galton-Watson branching process with four following parameters is considered:

$$\text{label } Z_0 = j, \quad P(T_{n,i} = 3) = 1, \quad L(x, y) = x \min\{1, y\}. \quad (3.6)$$

Also each offspring is female with probability α .

In this example indicate of distribution of Z_n is so difficult and then we can't calculate q_j . Daley et. al. [5] shows $r = \alpha g(s)$, so in this example $r = 3\alpha$. When $\alpha = 0.25$ we simulate this process and we obtained $\hat{q}_j = 0$. When $\alpha = 0.6$ for different value of Z_0 we simulate \hat{q}_j , also we obtained standard error and 0.99 confidence interval for q_j . We show this results in Table 3.

The results for 300000 simulations are reasonable.

4 Conclusion

In this article, we proposed Monte Carlo method for numerical solution of extinction probability. When we cant use classic method, results show Monte Carlo method is

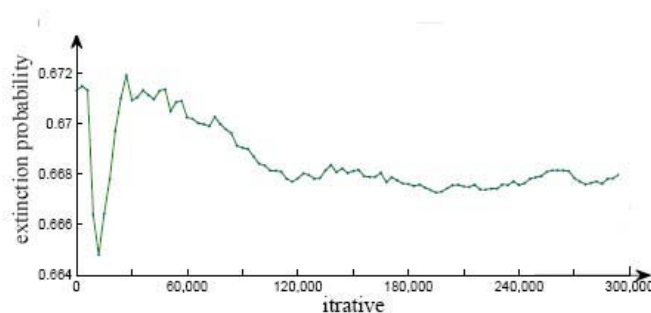


Figure 3: Convergence of \hat{q} (Example 3.3, $Z_0 = 1$, $\alpha = 0.6$)

reasonable.

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Table 1: Comparison of the q_j and \hat{q}_j (300000 simulations)

Z_0	q_j	\hat{q}_j	$ \hat{q}_j - q_j $	$S.E.$	0.99 Confidence interval
3	0.42224	0.42217	7.0×10^{-5}	9×10^{-4}	(0.41984 , 0.42440)
4	0.22990	0.22969	2.1×10^{-4}	8×10^{-4}	(0.22771 , 0.23167)
5	0.11988	0.11984	4.0×10^{-5}	6×10^{-4}	(0.11831 , 0.12137)
7	0.03093	0.03068	2.5×10^{-4}	3×10^{-4}	(0.02987 , 0.03149)
9	7.79×10^{-3}	7.72×10^{-3}	7.0×10^{-5}	2×10^{-4}	(7.31×10^{-3} , 8.13×10^{-3})
11	1.95×10^{-3}	1.79×10^{-3}	1.6×10^{-4}	8×10^{-5}	(1.59×10^{-3} , 1.99×10^{-3})
13	4.88×10^{-4}	4.72×10^{-4}	1.6×10^{-5}	4×10^{-5}	(3.70×10^{-4} , 5.74×10^{-4})

Table 2: Comparison of the q_j and \hat{q}_j (300000 simulations)

Z_0	q_j	\hat{q}_j	$ \hat{q}_j - q_j $	$S.E.$	0.99 Confidence interval
1	0.2735	0.27318	3.2×10^{-4}	8×10^{-4}	(0.27109 , 0.27527)
2	0.0313	0.03129	3.0×10^{-5}	3×10^{-4}	(0.3043 , 0.03211)
3	0.0039	0.00395	5.0×10^{-5}	1×10^{-4}	(0.00366 , 0.00424)
4	4.88×10^{-4}	4.67×10^{-4}	2.1×10^{-5}	4×10^{-5}	(3.85×10^{-4} , 5.92×10^{-4})
5	6.10×10^{-5}	7.67×10^{-5}	1.6×10^{-5}	2×10^{-5}	(3.56×10^{-5} , 1.18×10^{-4})
6	7.63×10^{-6}	3.3×10^{-6}	4.3×10^{-6}	3×10^{-6}	(-5.23×10^{-6} , 1.19×10^{-5})

Table 3: Simulation of \hat{q}_j (300000 simulations)

$Z_0 = j$	\hat{q}_j	$S.E.$	0.99 Confidence interval
1	0.67511	8.5×10^{-4}	(0.67291 , 0.67731)
2	0.36188	8.8×10^{-4}	(0.35962 , 0.36413)
3	0.19974	7.3×10^{-4}	(0.19786 , 0.20162)
4	0.11132	5.7×10^{-4}	(0.10984 , 0.11280)
5	0.06214	4.4×10^{-4}	(0.06101 , 0.06327)
6	0.03346	3.3×10^{-4}	(0.03262 , 0.03430)
7	0.01887	2.5×10^{-4}	(0.01823 , 0.01951)