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An Optimality Conditione for Locally Lipschitz semi-infinite Problems

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Abstract

This paper is devoted to the study of locally Lipschitz semi-infinite programming problems in which the index set of the inequality constraints is assumed to be arbitrary. We introduce an analogous of the Arrow-Hurwitcz-Uzawa constraint qualification which is based on the Clarke subdifferential. Then, we derive a Karush-Kuhn-Tucker type necessary condition . Finally, interrelations between the new and the Slater constraint qualifications are investigated.

Keywords: Optimality conditions, Semi-infinite problem, Nonsmooth analysis, Constraint qualification.

1 Introduction

A semi-infinite problem with inequality constraints is an optimization problem with finitely many variables $x = (x_1, x_2, \ldots, x_n) \in \Re^n$ on a feasible set described by –probably– infinitely many inequality constraints. In this paper we study the following semi-infinite programming problem (SIP, in brief)

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$$
\begin{array}{lll}
(SIP) & \inf f(x) & \\
s.t. & g_i(x) \leq 0 & i \in I, \\
 & x \in \Re^n, &
\end{array}
$$

where f and g_i, $i \in I$ are locally Lipschitz functions from $\Re^n$ to $\Re \cup \{+\infty\}$, and the index set I is an arbitrary set, not necessarily finite –but nonempty–. In the review papers [6, 10], as well as in [5], we will find many applications of SIP in different fields such as Chebyshev approximation, robotics, mathematical physics, engineering design, optimal control, transportation problems, fuzzy sets, robust optimization, etc.

If the set I is finite, necessary conditions of Karush-Kuhn-Tucker (KKT) type for optimality can be established under various constraint qualifications. In order to study and compare of these constraint qualifications in smooth and nonsmooth cases, see the book [3].

For linear semi-infinite systems, the "Farkas-Minkowski property" has been introduced by Goberna et al. in [4]. In [14], Puente et al. introduced the "locally Farkas-Minkowski (LFM) property" for linear SIPs and its role as constraint qualification was emphasized there. For an excellent study of linear SIPs, see the book [5], and the survey article [10].

Some constraint qualifications for semi-infinite systems with convex inequalities and linear inequalities are studied in [8]. There, characterizations of various constraint qualifications in terms of upper semicontinuity of certain multifunctions are given. In [9], López and Vercher have given optimality conditions for convex nondifferentiable semi-infinite programming problems which involves the notion of Lagrangian saddle point.

We point out most of the references cited above are restricted to differentiability or convexity assumptions, and equality constraints are not considered.

On the other hand, the classical Lagrange multiplier rule was generalized in the direction of replacing the usual gradient by certain generalized gradients such as in Clarke [2], Michael and Penot [11], Mordukhovich [12, 13] and Rockafellar [17].

Some constraint qualifications for nonconvex and nondifferentiable SIPs are introduced in [15, 16]; for instance Abadie, Basic, Zangwill, and Guignard constraint qualifications. There presented Fritz-John and Karush-Kuhn-Tucker type necessary and sufficient optimality conditions for these problem.

The aim of this paper is to provide the Karush-Kuhn-Tucker type condition for optimal solution of nonsmooth SIP, by using Arrow-Hurwicz-Uzawa constraint qualification, based on Clarke subdifferential.

We organize the paper as follows. In Section 2, basic notations and results of nonsmooth analysis are reviewed. In Section 3, we introduce the Arrow-Hurwicz-Uzawa constraint qualification for nonconvex SIPs, and investigate a necessary optimality condition of Karush-Kuhn-Tucker type. Then, there is devoted to the discussion of a new constraint qualification and its relation with the Slater constraint qualification.

2 Notations and Preliminaries

In this section we briefly overview some notions of variational analysis widely used in formulations and proofs of main results of the paper. For more details, discussion, and applications see [2, 7, 17].

Given a nonempty set $M \subseteq \Re^n$, we denote by $cl(M)$, $conv(M)$, and $cone(M)$, the closure of M, convex hull and convex cone (containing the origin) generated by M, respectively. The polar cone and strict polar cone of M are defined respectively by:

$$M^0 := \{d \in \Re^n \mid \langle x, d\rangle \leq 0, \quad \forall x \in M\}$$

$$M^- := \{d \in \Re^n \mid \langle x, d\rangle < 0, \qquad \forall x \in M\},$$

where $\langle .,. \rangle$ exhibits the standard inner product in $\Re^n$. Notice that M^0 is always closed convex cone. It is easy to show that if $M^- \neq \phi$ then $cl(M^-) = M^0$.

Definition 2.1 *Let $\varphi : \Re^n \to \Re$ be a locally Lipschitz function and $\hat{x} \in dom(f)$.*

I: *The generalized Clarke directional derivative of φ at $\hat{x}$ in the direction $d \in \Re^n$ is defined by*

$$\varphi^0(\hat{x}; d) := \limsup_{y \to \hat{x}, t \downarrow 0} \frac{\varphi(y + td) - \varphi(y)}{t}.$$

II: *The Clarke subdifferential of φ at $\hat{x}$ is defined by*

$$\partial_c \varphi(\hat{x}) := \{\xi \in \Re^n \mid \ \varphi^0(\hat{x}; d) \geq \langle \xi, d \rangle, \quad \forall d \in \Re^n\}.$$

Observe that the Clarke subdifferential of a locally Lipschitz function at an interior point of its domain is always nonempty, compact, and convex cone. The Clarke subdifferential reduce to the classical gradient for continuously differentiable functions and to the subdifferential of convex analysis for convex ones.

Let us recall the following theorems which will be used in the sequel.

Theorem 2.2 ([7]) *Let $\{M_\alpha | \alpha \in \Lambda\}$ be an arbitrary collection of nonempty convex sets in $\Re^n$. Then, every non-zero vector of $conv(\bigcup_{\alpha \in \Lambda} M_\alpha)$ can be expressed as a non-negative linear combination of n or fewer linearly independent vectors, each belonging to a different M_α.*

Theorem 2.3 ([7]) *Let M be a nonempty compact subset of $\Re^n$ such that $0 \notin conv(M)$. Then $cone(M)$ is a closed cone.*

Theorem 2.4 ([2]) *Let φ and ψ are locally Lipschitz from $\Re^n$ to $\Re$, and $\hat{x} \in dom(\varphi) \cap dom(\psi)$. Then, the following properties hold:*

a: $\varphi^0(\hat{x}; d) = \max\{\langle \xi, d \rangle \mid \ \xi \in \partial_c \varphi(\hat{x})\}, \qquad \forall d \in \Re^n.$

b: *$d \to \varphi^0(\hat{x}; d)$ is a convex function, and $\partial_c \varphi(x) = \partial \varphi^0(x; .)(0)$, where $\partial \varphi(\hat{x})$ denotes the subdifferential of convex function φ at $\hat{x}$.*

c: *$x \mapsto \varphi(x)$ is an upper semicontinuous set-valued function.*

d: $\partial_c(\varphi + \psi)(\bar{x}) \subseteq \partial_c\varphi(\bar{x}) + \partial_c\psi(\bar{x})$.

Furthermore, if φ and ψ are convex, then equality holds in above virtue.

e: *If $\hat{x}$ is a minimum point of φ over $\Re^n$, then $0 \in \partial_c\varphi(\hat{x})$.*

Definition 2.5 *Let $\varphi : \Re^n \to \Re$ be a locally Lipschitz function. φ is said to be pseudoconcave at $\hat{x}$ if for all $x \in \Re^n$,*

$$\varphi^0(\hat{x}; x - \hat{x}) \leq 0 \Rightarrow \varphi(x) \leq \varphi(\hat{x}).$$

3 Main Results

In this section we introduce a constraint qualification for a locally Lipschitz semi-infinite problem. Also, we shall obtain an Karush-Kuhn-Tucker (KKT, in brief) type necessary optimality conditions for it.

Let P denote the feasible solutions of SIP

$$P := \{x \in \Re^n \mid g_i(x) \leq 0, \quad \forall\, i \in I\}.$$

For a given $\hat{x} \in P$, let $I^{\hat{x}}$ denote the index set of all active constraints at $\hat{x}$; that is

$$I^{\hat{x}} := \{i \in I \mid g_i(\hat{x}) = 0\}.$$

Set

$$V := \{i \in I \mid g_i \textit{ is pseudoconcave at } \hat{x}\},$$
$$W := I \backslash V,$$
$$G(x) := \sup_{i \in W} g_i(x), \qquad \forall x \in P.$$

One reason for difficulty of extending the results from a finite inequality problem to SIP is that in the finite case $G(.)$ is locally Lipschitz and we have (see [2, Propisition 2.3.12])

$$\partial_c G(\hat{x}) \subseteq conv(\bigcup_{i \in W \cap I^{\hat{x}}} \partial_c g_i(\hat{x})), \qquad \forall x \in P, \tag{1}$$

but in general, (1) does not hold if I is infinite (see [2, Theorem 2.8.2]).

Remark 3.1 An interesting sufficient condition ensuring the Lipschitz property of $G(.)$ around $\hat{x}$, can found in [17, Theorem 9.2].

Let

$$E(\hat{x}) := \bigcup_{i \in W \cap I^{\hat{x}}} \partial_c g_i(\hat{x}),$$
$$F(\hat{x}) := \bigcup_{i \in V \cap I^{\hat{x}}} \partial_c g_i(\hat{x}),$$
$$K(\hat{x}) := E(\hat{x}) \cup F(\hat{x}) = \bigcup_{i \in I^{\hat{x}}} \partial_c g_i(\hat{x}),$$

We now extend the *Arrow-Hurwicz-Uzawa constraint qualification* (AHUCQ, in brief) for SIP.

Definition 3.2 *Let $\hat{x}$ is a feasible solution of SIP. We say that the AHUCQ is satisfied at $\hat{x}$ if*

(i): *$G(.)$ is Lipschitz around $\hat{x}$.*

(ii):

$$\partial_c G(\hat{x}) \subseteq conv(E(\hat{x})). \tag{2}$$

(iii):

$$E^-(\hat{x}) \cap F^0(\hat{x}) \neq \emptyset. \tag{3}$$

Remarks 3.3

1. The definition 3.2 reduce to the classical AHUCQ -which is considered in [1]- for finite differentiable problems.

2. Owing to the [7, pp. 267], the estimate of (2) is equivalent to $\partial_c G(\hat{x}) = conv(E(\hat{x}))$, for convex SIPs.

3. It is known that if for all $i \in I$, g_i is convex function; I is a compact set in some metric space; and for each fixed $\tilde{x} \in P$ the function $i \to g_i(\tilde{x})$ is upper semicontinuous on I, then (2) verifies at every $\hat{x} \in P$ (see [7, pp. 267]).

There is no relation of implication between the virtues of (2) and (3). Indeed, for any finite I, inclusion of (2) is trivially true, but it may not satisfy in inequality of (3); while the following example the system actually satisfies in (3) at $\hat{x} = 0$, but the estimation (2) does not hold at this point.

Example: Let $I = \{0,\ 1,\ 2\ \ldots\}, \hat{x} = 0$, and

$$\begin{aligned} &g_0(x) = 2x, \\ &g_{2k+1}(x) = x - \frac{1}{k+1}, \qquad k = 0, 1, 2, \ldots \\ &g_{2k}(x) = 3x - \frac{1}{k}, \qquad k = 1, 2, \ldots. \end{aligned}$$

Since g_is are linear, we obtain that $V = \emptyset$ and $W = I$.

We observe that:

$$P = (-\infty, 0], \qquad I^{\hat{x}} = \{0\}, \qquad E(\hat{x}) = K(\hat{x}) = \{2\},$$

and

$$G(x) = \sup_{i \in \{1,\ 2,\ \ldots\}} \{g_0(x), g_i(x)\} = \begin{cases} x & if \quad x < 0 \\ 3x & if \quad x \geq 0. \end{cases}$$

Since

$$\begin{aligned} &E^-(\hat{x}) = (-\infty, 0), \\ &F^0(\hat{x}) = \Re, \\ &\partial_c G(\hat{x}) = [1, 3] \\ &conv(E(\hat{x})) = \{2\}, \end{aligned}$$

the system does not satisfy in (3) but the virtue of (2) is true.

Now, the optimality condition of KKT-type for SIP is stated as follows.

Theorem 3.4 *(KKT condition) Suppose that $\hat{x}$ is an optimal solution of SIP, and assume that the AHUCQ satisfies at $\hat{x}$.*

(a): *One has*

$$0 \in \partial_c f(\hat{x}) + cl(cone(K(\hat{x}))). \tag{4}$$

(b): *If, in addition, $cone(K(\hat{x}))$ is closed cone, then there exist scalers λ_i, $i \in I^{\hat{x}}$, which finite numbers of them are nonzero, such that*

$$0 \in \partial_c f(\hat{x}) + \sum_{i \in I^{\hat{x}}} \lambda_i \partial_c g_i(\hat{x}). \tag{5}$$

Proof:

(a): Since $E^-(\hat{x}) \cap F^0(\hat{x}) \neq \emptyset$, we can choice a vector $d \in E^-(\hat{x}) \cap F^0(\hat{x})$. Thus

$$\langle \xi, d \rangle < 0, \qquad \forall \xi \in E(\hat{x}), \tag{6}$$

$$\langle \eta, d \rangle \leq 0, \qquad \forall \eta \in F(\hat{x}). \tag{7}$$

Let $\widehat{\xi} \in conv(E(\hat{x}))$. Then, there exist scalers $\gamma_1, \ldots, \gamma_s \geq 0$, and vectors $\xi_1, \ldots, \xi_s \in E(\hat{x})$, such that

$$\sum_{v=1}^{s} \gamma_v = 1, \qquad \widehat{\xi} = \sum_{v=1}^{s} \gamma_v \xi_v.$$

Using the virtue of (6) we have

$$\langle \widehat{\xi}, d \rangle = \sum_{v=1}^{s} \gamma_v \langle \xi_v, d \rangle < 0,$$

and hence –in view of (2)– we conclude

$$d \in \Big(conv(E(\hat{x}))\Big)^- \subseteq (\partial_c G(\hat{x}))^-.$$

Thus

$$G^0(\hat{x}; d) < 0,$$

and consequently, there exists a scaler $\delta_1 > 0$, such that

$$g_i(\hat{x} + \beta d) \leq G(\hat{x} + \beta d) < G(\hat{x}) \leq 0, \quad \forall\, 0 \leq \beta \leq \delta, \quad \forall\, i \in W. \tag{8}$$

On the other hand, in regard to (7), we have

$$g_j^0(\hat{x}; d) \leq 0, \qquad \forall\, j \in V.$$

Thus, for all $\hat{\beta} \in (0,1]$ we obtain

$$g_j^0(\hat{x}; \frac{1}{\hat{\beta}}[(\hat{x}+\hat{\beta}d)-\hat{x}]) = g_j^0(\hat{x}; d) \leq 0, \quad \forall\, j \in V.$$

Using the pseudoconcavity of g_j $(j \in V)$, we get

$$g_j(\hat{x}+\hat{\beta}d) \leq g_j(\hat{x}) \leq 0, \quad \forall\, \hat{\beta} \in (0,1], \quad \forall\, j \in V. \tag{9}$$

Therefore, in view of (8)-(9), we have

$$\hat{x}+td \in P, \qquad \forall\, 0 \leq t \leq \min\{1, \delta_1\},$$

and by minimality of $\hat{x}$, we conclude that

$$\frac{1}{\hat{\beta}}(f(\hat{x}+td) - f(\hat{x})) \geq 0, \qquad \forall\, 0 \leq t \leq \min\{1, \delta_1\}.$$

Summarizing, –since d is an arbitrary element of $E^-(\hat{x}) \cap F^0(\hat{x})$– we have

$$f^0(\hat{x}; d) \geq 0, \qquad \forall\, d \in E^-(\hat{x}) \cap F^0(\hat{x}).$$

Since

$$\Big(cl(cone(K(\hat{x})))\Big)^0 = K^0(\hat{x}) = E^0(\hat{x}) \cap F^0(\hat{x}) = cl(E^-(\hat{x}) \cap F^0(\hat{x})),$$

and since each $g_i(\hat{x}; .)$ is continuous, we obtain that

$$f^0(\hat{x}; d) \geq 0, \qquad \forall\, d \in \Big(cl(cone(K(\hat{x})))\Big)^0 := X.$$

Thus, the following convex function attains its minimum at $\hat{d} = 0$:

$$\Psi(.) := \Phi_X(.) + f^0(\hat{x}; .),$$

where $\Phi_X(.)$ denotes the indicator function of X, it is defined as

$$\Phi_X(y) := \begin{cases} 0 & if \quad y \in X, \\ +\infty & if \quad y \notin X. \end{cases}$$

Hence –in view of Theorem 2.4– we get

$$0 \in \partial\Psi(0) = \partial\Phi_X(0) + \partial f^0(\hat{x}; .)(0) = cl(cone(K(\hat{x}))) + \partial_c f(\hat{x}).$$

(b): It is follows from virtue of (4) and Theorem 2.2.

Remark 3.5 If I is a finite set and for each $i \in I$, g_i is differentiable, then $cone(K(\hat{x}))$ is closed.

Recall, the following definition from [9, Definition 3.6].

Definition 3.6 We say that the SIP satisfies the *Slater constraint qualification* (SCQ, in brief) if g_is are convex function (for all $i \in I$); $I \subseteq \Re^m$ is a compact set; $g_i(x)$ is a continuous function of (i, x) in $I \times \Re^n$; and there is a point $x_0 \in \Re^n$ such that $g_i(x_0) < 0$, for all $i \in I$.

Theorem 3.7 *Suppose that SIP satisfies the SCQ. Then*

(i): *The SIP satisfies the AHUCQ at each $\hat{x} \in P$.*

(ii): *$cone(K(\hat{x}))$ is a closed cone for all $\hat{x} \in P$.*

Proof:

(i): By definition of SCQ we have

$$V = \emptyset = F(\hat{x}), \qquad I = W, \qquad K(\hat{x}) = E(\hat{x}).$$

Let x_0 be a point which satisfies in definition of SCQ. For all $i_0 \in I^{\hat{x}}$ and $\xi \in \partial g_{i_0}(\hat{x})$, we have

$$\langle \xi, x_0 - \hat{x} \rangle \leq g_{i_0}(x_0) - g_{i_0}(\hat{x}) = g_{i_0}(x_0) < 0.$$

Thus, $(x_0 - \hat{x}) \in K^-(\hat{x})$, which implies that (3) is verify. Owning to the Remark 3.3, the proof is complete.

(ii): Since $K^-(\hat{x}) \neq \emptyset$ –by (i) in above– it is easy to see $0 \notin conv(K(\hat{x}))$. On the other hand, according to [7, Theorem 4.4.1] $K(\hat{x})$ is a compact set. Owing to the Theorem 2.3, the proof is complete.

The following result –which was proved in [9] by another approach– is immediate from Theorems 3.4 and 3.7.

Theorem 3.8 *Suppose that $\hat{x}$ is an optimal solution of a convex SIP, and SCQ holds. Then, there exist scalers λ_i, $i \in I^{\hat{x}}$, which finite numbers of them are nonzero, such that*

$$0 \in \partial f(\hat{x}) + \sum_{i \in I^{\hat{x}}} \lambda_i \partial g_i(\hat{x}).$$

4 Conclusions

We have established a Karush-Kuhn-Tucker type necessary optimality condition for nonsmooth semi-infinite programming problems under analogous of the Arrow-Hurwitcz-Uzawa constraint qualification which is based on the Clarke subdifferential. Furthermore, we have extended our results to the case where the problem satisfies in Slater constraint qualification which is based on the convex subdifferential.

References

[1] Arrow K.J., Hurwicz L., Uzawa H. (1961) "Constraint Qualifications in Maximization Problems," Naval Research Logistics Quaterly, 8, 175-191.

[2] Clarke F.H., Optimization and Nonsmooth Analysis, Wiley-Interscience, 1983.

[3] Giorgi J., Gwirraggio A., Thierselder J., Mathematics of Optimization; Smooth and Nonsmooth cases, Elsivier, 2004.

[4] Gorberna M.A., López M.A., Pastor J. (1981) "Farkas-Minkowski system in semi-infinite programming," Appl. Math. Optim., 7, 295-308.

[5] Gorberna M.A., López M.A., Linear Semi-infinite Optimazation, Wiley, Chichester, 1998.

[6] Hettich R., Kortanek K.O. (1993) "Semi-infinite programming: Theory, methods and applications," SIAM Review, 35, 380-429.

[7] Hiriart-Urruty J.B., Lemarechal C., Convex Analysis and Minimization Algorithms, I, Springer, Berlin, Heidelberg, 1991.

[8] Li W., Nahak C. Singer I. (2000) "Constraint qualifications in semi-infinite systems of convex inequalities," SIAM J. Optim., 11, 31-52.

[9] López M.A., Vercher E. (1983) "Optimality conditions for nondifferentiable convex semi-infinite Programming," Math. Programming, 27, 307-319.

[10] López M.A., Still G. (2007) "Semi-infinite programming," European J. Opera. Res., 180, 491-518.

[11] Michel P., Penot J.P. (1984) "Calcul sous- differentiel pour des fonctions lipschitziennes et non lipschitziennes," C.R. Acad. Sci. Paris sér. I Math., 12, 269-272.

[12] Mordukhovich B.S., Variational analysis and generalized differentiation. I. Basic theory, Springer-Verlag, 2006.

[13] Mordukhovich B.S., Variational analysis and generalized differentiation. II. Applications, Springer-Verlag, 2006.

[14] Puente R., Vera De Serio V.N. (1999) "Locally Farkas Minkowski linear inequality systems," Top, 7, 103-121.

[15] Kanzi N., Nobakhtian S. (2008) "Optimality conditions for nonsmooth semi-infinite programming," Optimization.

[16] Kanzi N., Nobakhtian S. (2008) "Nonsmooth semi-infinite programming problems with mixed constraints," J. Math. Anal. Appl., 351, 170-181.

[17] Rockafellar R.T., Wets J.B., Variational analysis, Springer-Verlag, 1998.