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Numerical solution of fifth-order boundary-value problems in off step points

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Abstract

Non-polynomial sextic spline in off step points is used to solve special fifth order linear boundary value problems. Associated boundary formulas are developed. We compare our results with the results produced by non-polynomial sextic spline method [10]. However, it is observed that our approach produce better numerical solutions in the sense that max$|e_i|$ is a minimum.

Keywords: Fifth-order boundary-value problem, boundary formulae, non-polynomial spline.

1 Introduction

The solution of fifth order boundary value problems are not very much found in the analysis literature. Rashidinia et al. in [1,2] developed the spline approximate solutions of fifth-order and eighth-order boundary- value problems.The conditions for existence and uniqueness of solution of such boundary value problems are explained by theorems presented in Agarwal [3]. Caglar et al. [4] solved third order linear and nonlinear boundary value problems using fourth degree B-spline. Siddiqi and Twizell[5-8] developed the solutions of 6th,8th,10th and 12th order boundary value problems using the

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6th,8th,10th and 12th degree spline, respectively. Siddiqi and Ghazla Akram[9] Applied the quintic spline for solution of fourth order B.V.Ps.Siddiqi et al. [10] presented non-polynomial sextic spline method for the solution of special case linear fifth-order two-point boundary value problems. Siddiqi and Ghazla Akram[11] presented the solutions of fifth-order linear boundary value problems using non-polynomial spline, This method is second-order convergent, in [12] siraj-ul islam et al. given a method based on sextic spline solution for the solution of fifth order boundary value problem in grid points and also in [13] Khan et al. derived a numerical method based on non-polynomial spline. Scott and Watts [15] described the numerical solution of linear BVP using a combination of superposition and orthonormalization and in [16] described several computer codes that were developed using the superposition and orthonormalization technique and invariant imbedding. The basic motivation of this paper is development of boundary formulas. In this paper we used non-polynomial spline approximation to develop a family of new numerical methods to obtain the solution of fifth-order differential equation.The sextic Non-polynomial spline function proposed in this paper, have the form $T_9 = \text{span}\{1, x, x^2, x^3, x^4, \cos(kx), \sin(kx)\}$ where k is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and which will be used to raise the accuracy of the method. Thus in each subinterval $x_i \leq x \leq x_{i+1}$, we have

$$\text{span}\{1, x, x^2, x^3, x^4, \cos(|k|x), \sin(|k|x)\},$$

or

$$\text{span}\{1, x, x^2, x^3, x^4, x^5, x^6\}, \ \ (\text{when} k \to 0).$$

In this manuscript the following fifth-order boundary value problem is consider:

$$y^{(5)}(x) + g(x)y(x) = q(x), \quad x \in [a, b], \tag{1}$$

with boundary conditions

$$y(a) = \alpha_0 \ , \ y^{(1)}(a) = \alpha_1 \ , \ y^{(3)}(a) = \alpha_2 \quad \text{and} \quad y(b) = \beta_0 \ , \ y^{(1)}(b) = \beta_1, \tag{2}$$

where $\alpha_i, i = 0, 1, 2$ and $\beta_i, i = 0, 1$ are finite real constants and also the functions $g(x)$ and $q(x)$ are continuous on [a,b]. In this paper, in Section 2, the new non-polynomial spline methods in off step points are developed for solving equation (1) along with boundary condition (2). The boundary formulas have been developed in Section 3 and the Section 4 dealing with numerical experiment, discussion and comparison with method in [10].

2 Numerical methods

To develop the spline approximation to the solution of fifth-order boundary-value problem (1)-(2), the given interval [a,b] is divided into n equal subintervals using the grid $x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h, i = 1, ..., n$, where $h = \frac{b-a}{n}$. Consider the following non-polynomial sextic spline $S_i(x)$ is each subinterval $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], i = 1, ..., n-1, x_0 = a, x_n = b$,

$$S_i(x) = a_i \text{cos} k(x - x_i) + b_i \text{sin} k(x - x_i) + c_(x - x_i)^4 + d_i(x - x_i)^3 +$$

$$e_i(x - x_i)^2 + f_i(x - x_i) + g_i, \tag{3}$$

where $a_i, b_i, c_i, d_i, e_i, f_i$ and g_i are real finite and k is free parameter. The spline S is defined in terms of its 1th and 2th derivatives and we denote these values at knots as:

$$S_i(x_{i-\frac{1}{2}}) = y_{i-\frac{1}{2}} \ , \ S_i^{(1)}(x_{i-\frac{1}{2}}) = m_{i-\frac{1}{2}} \ , \ S_i^{(2)}(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}}, \ , \ S_i^{(5)}(x_{i-\frac{1}{2}}) = L_{i-\frac{1}{2}},$$

$$S_i(x_{i+\frac{1}{2}}) = y_{i+\frac{1}{2}} \ , \ S_i^{(1)}(x_{i+\frac{1}{2}}) = m_{i+\frac{1}{2}}, \ , \ S_i^{(2)}(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}} \ , \ S_i^{(5)}(x_{i+\frac{1}{2}}) = L_{i+\frac{1}{2}}$$

$$\text{for } \ i = 1, 2, ..., n-1. \tag{4}$$

Assuming $y(x)$ to be the exact solution of the boundary value problem (1) and y_i be an approximation to $y(x_i)$, obtained by the spline $S(x_i)$, we can obtained the coefficients in (3) in the following form

$$
\begin{aligned}
&a_i = -\frac{Csc(\frac{\theta}{2})(-L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})}{2k^5},\\
&b_i = -\frac{Sec(\frac{\theta}{2})(-L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})}{2k^5},\\
&c_i = -\tfrac{1}{6h^4k^5}[(-6hk+2(-3+h^2k^2)Cot(\theta)+(6+h^2k^2)Csc(\theta))L_{i-\frac{1}{2}}-(-6+2h^2k^2+\\
&(6+h^2k^2)Cos(\theta))Csc(\theta)L_{i+\frac{1}{2}}+k^5(h(6m_{i-\frac{1}{2}}+h(2M_{i-\frac{1}{2}}+M_{i+\frac{1}{2}}))+6(y_{i-\frac{1}{2}}+y_{i+\frac{1}{2}}))],\\
&d_i = -\tfrac{1}{6hk^3}[-k^3(M_{i-\frac{1}{2}}-M_{i+\frac{1}{2}})+(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})Tan(\theta)],\\
&e_i = -\tfrac{1}{4h^2k^5}[(6hk+(6-h^2k^2)Cot(\theta)-6Csc(\theta))L_{i-\frac{1}{2}}-(-6+2h^2k^2+\\
&(6Cos(\theta))Csc(\theta)L_{i+\frac{1}{2}}+k^5(hm_{i-\frac{1}{2}}+h^2M_{i-\frac{1}{2}}+6y_{i-\frac{1}{2}}-6y_{i+\frac{1}{2}}))],\\
&f_i = -\tfrac{1}{24hk^5}[k^5(h^2(M_{i-\frac{1}{2}}-M_{i+\frac{1}{2}})+24(y_{i-\frac{1}{2}}+y_{i+\frac{1}{2}}))-(24+h^2k^2)(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})Tan(\theta)],\\
&g_i = -\tfrac{1}{96k^5}[(-30hk+(-78+4h^2k^2)Cot(\theta)-(18+h^2k^2)Csc(\theta))L_{i-\frac{1}{2}}+(78-4h^2k^2+\\
&(18+h^2k^2)Cos(\theta))Csc(\theta)L_{i+\frac{1}{2}}+k^5(h(30m_{i-\frac{1}{2}}+4hM_{i-\frac{1}{2}}-hM_{i+\frac{1}{2}})+78y_{i-\frac{1}{2}}+18y_{i+\frac{1}{2}})].
\end{aligned}
$$

where $\theta = kh$ and $i = 1, 2, ..., n-1$. Applying the Continuity condition of the second, third and fourth derivatives at $(x_{i-\frac{1}{2}}, y_{i-\frac{1}{2}})$, that is $S_{i-1}^{(\lambda)}(x_{i-\frac{1}{2}}) = S_i^{(\lambda)}(x_{i-\frac{1}{2}})$, where $\lambda = 1, 3$ and 4, yields the following equations:

$$
\begin{aligned}
&\frac{1}{hk}(k^5(6hm_{i-\frac{3}{2}}+6hm_{i-\frac{1}{2}}+h^2M_{i-\frac{3}{2}}-h^2M_{i-\frac{1}{2}}+12y_{i-\frac{3}{2}}-12y_{i-\frac{1}{2}}))-\\
&L_{i-\frac{3}{2}}(6hk+(-12+h^2k^2)Tan(\frac{\theta}{2}))-L_{i-\frac{1}{2}}(6hk+(-12+h^2k^2)Tan(\frac{\theta}{2}))=0,
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&\frac{1}{hk}(k^5(12m_{i-\frac{3}{2}}+12m_{i-\frac{1}{2}}+h(3M_{i-\frac{3}{2}}+8M_{i-\frac{1}{2}}+M_{i+\frac{1}{2}})+12(y_{i-\frac{3}{2}}-y_{i-\frac{1}{2}}))+\\
&L_{i-\frac{3}{2}}(hk(-4+hkCot(\frac{\theta}{2}))+4Tan(\frac{\theta}{2}))-2L_{i-\frac{1}{2}}(6hk+(-12+h^2k^2)Tan(\frac{\theta}{2}))+\\
&L_{i+\frac{1}{2}}(-3h^2k^2Cot(\frac{\theta}{2})-2(-6+h^2k^2)Tan(\frac{\theta}{2})))=0,
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&\frac{1}{hk}((-24hk+8(-3+h^2k^2)Cot(\theta)+(24+4h^2k^2+h^4k^4)Csc(\theta))L_{i-\frac{3}{2}}-\\
&2hk(-12+hk((6+h^2k^2)Cot(\theta)+6Csc(\theta)))L_{i-\frac{1}{2}}+(4(6+h^2k^2)Cot(\theta)+
\end{aligned}
$$

$$(-24+8h^2k^2+h^4k^4)Csc(\theta)L_{i+\frac{1}{2}}+4k^5(h(6m_{i-\frac{3}{2}}-6m_{i-\frac{1}{2}}-h(-2M_{i-\frac{3}{2}}+M_{i-\frac{1}{2}}+M_{i+\frac{1}{2}}))-6(M_{i-\frac{3}{2}}-2M_{i-\frac{1}{2}}+M_{i+\frac{1}{2}}))) = 0. \quad (7)$$

In order to eliminate m's,M's and T's in the above equation (5)-(7) get nine additional equation i is replaced by $i-1, i+1, i+2$, in each of the Eqs.(5), (6) and (7). After lengthy calculations, the following recurrence relation is obtained:

$$-y_{i-\frac{5}{2}}+5y_{i-\frac{3}{2}}-10y_{i-\frac{1}{2}}+10y_{i+\frac{1}{2}}-5y_{i+\frac{3}{2}}+y_{i+\frac{5}{2}} = h^5[\alpha L_{i-\frac{5}{2}}+\beta L_{i-\frac{3}{2}}+\gamma L_{i-\frac{1}{2}}+\gamma L_{i+\frac{1}{2}}+\beta L_{i+\frac{3}{2}}+\alpha L_{i+\frac{5}{2}}],$$
$$i = 3, 5, ..., n-3, \quad (8)$$

Where

$$\alpha = \frac{1}{\theta^5}(1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4-Cos(\theta))Csc(\theta),$$
$$\beta = (\frac{11}{24}\theta-\frac{1}{2\theta^3}-\frac{3}{\theta^5})Csc(\theta)+(-\frac{1}{12\theta}+\frac{1}{\theta^3}+\frac{3}{\theta^5})Cot(\theta),$$
$$\gamma = -(\frac{1}{2\theta}+\frac{2}{\theta^5})Csc(\theta)+(\frac{1}{2\theta}+\frac{1}{\theta^3}+\frac{2}{\theta^5})Cot(\theta).$$

3 Development of the boundary formulas

To obtain unique solution of the system(8) we need five more equations to be associated with it, so that we need to consider the following boundary conditions. In order to obtain the fifth-order boundary formula we define the following identities:

$$d'_0y_0+\sum_{k=0}^{4}a'_ky_{k+\frac{1}{2}}+c'hy_0^{(1)} = h^5\sum_{k=0}^{6}b'_ky^{(5)}_{k+\frac{1}{2}}+t_1, \quad (9)$$

$$d''_0y_0+\sum_{k=0}^{5}a''_ky_{k+\frac{1}{2}}+c''hy_0^{(1)} = h^5\sum_{k=0}^{7}b''_ky^{(5)}_{k+\frac{1}{2}}+t_2, \quad (10)$$

$$d'''_0y_0+\sum_{k=0}^{6}a'''_ky_{k+\frac{1}{2}}+c'''hy_0^{(1)}+d^\circ h^3y_0^{(3)} = h^5\sum_{k=0}^{8}b^\circ_ky^{(5)}_{k+\frac{1}{2}}+t_3, \quad (11)$$

$$d_0^{\star} y_0 + \sum_{k=0}^{5} a_k^{\star} y_{k+n-\frac{9}{2}} + c^{\star} h y_0^{(1)} = h^5 \sum_{k=0}^{7} b_k^{\star} y_{k+n-\frac{9}{2}}^{(5)} + t_{n-2}, \tag{12}$$

$$d_0^{\star\star} y_0 + \sum_{k=0}^{4} a_k^{\star\star} y_{k+n-\frac{7}{2}} + c^{\star\star} h y_0^{(1)} = h^5 \sum_{k=0}^{6} b_k^{\star\star} y_{k+n-\frac{7}{2}}^{(5)} + t_{n-1}, \tag{13}$$

where all of the coefficients are real parameters to be determined. In order to obtain the fifth-order accuracy ,Then we have:

$$y_0 - \frac{3675}{2816} y_{\frac{1}{2}} + \frac{1225}{2816} y_{\frac{3}{2}} - \frac{441}{2816} y_{\frac{5}{2}} + \frac{75}{2816} y_{\frac{7}{2}} + \frac{105}{352} h y_0^{(1)} = h^5 \Big(\frac{213647}{69206016} y_{\frac{1}{2}}^{(5)} + \frac{318045}{23068672} y_{\frac{3}{2}}^{(5)} - \frac{23555}{11534336} y_{\frac{5}{2}}^{(5)} + \frac{74795}{34603008} y_{\frac{7}{2}}^{(5)} - \frac{18655}{23068672} y_{\frac{9}{2}}^{(5)} + \frac{2961}{23068672} y_{\frac{11}{2}}^{(5)} + t_1, \tag{14}$$

$$y_0 - \frac{15920415}{8738944} y_{\frac{1}{2}} + \frac{1865115}{1092368} y_{\frac{3}{2}} - \frac{6146847}{4369472} y_{\frac{5}{2}} + \frac{351405}{546184} y_{\frac{7}{2}} - \frac{1068235}{8738944} y_{\frac{9}{2}} + \frac{45045}{273092} h y_0^{(1)} = h^5 \Big(y_{\frac{1}{2}}^{(5)} - \frac{54045086865}{8948678656} y_{\frac{3}{2}}^{(5)} + \frac{16713930285}{1118584832} y_{\frac{5}{2}}^{(5)} - \frac{89499736057}{4474339328} y_{\frac{7}{2}}^{(5)} + \frac{8388932991}{559292416} y_{\frac{9}{2}}^{(5)} - \frac{53690933205}{8948678656} y_{\frac{11}{2}}^{(5)} + y_{\frac{13}{2}}^{(5)} \Big) + t_2, \tag{15}$$

$$y_0 - \frac{15716726026992841}{24342416896} y_{\frac{1}{2}} + \frac{378372388483332605}{24342416896} y_{\frac{3}{2}} - \frac{1572825655433447}{1058365952} y_{\frac{5}{2}} + \frac{17464229101208335}{24342416896} y_{\frac{7}{2}} - \frac{243557180886615}{1738744064} y_{\frac{9}{2}} + y_{\frac{11}{2}} - \frac{132526669163895}{760700528} h y_0^{(1)} + \frac{238888398971205}{24342416896} h^3 y_0^{(3)} = h^5 \Big(y_{\frac{1}{2}}^{(5)} - \frac{10301055489197900041}{18694976176128} y_{\frac{3}{2}}^{(5)} - \frac{8437858206963977261}{130864833232896} y_{\frac{5}{2}}^{(5)} - \frac{700649960235861641}{130864833232896} y_{\frac{7}{2}}^{(5)} - \frac{14631792463257019}{130864833232896} y_{\frac{9}{2}}^{(5)} + y_{\frac{11}{2}}^{(5)} + y_{\frac{13}{2}}^{(5)} \Big) + y_{\frac{15}{2}}^{(5)} \Big) + t_3, \tag{16}$$

$$-\frac{1068235}{8738944} y_{n-\frac{9}{2}} + \frac{351405}{546184} y_{n-\frac{7}{2}} - \frac{6146847}{4369472} y_{n-\frac{5}{2}} + \frac{1865115}{1092368} y_{n-\frac{3}{2}} - \frac{15920415}{8738944} y_{n-\frac{1}{2}} + y_n - \frac{45045}{273092} h y_n^{(1)} = h^5 \Big(y_{n-\frac{13}{2}}^{(5)} - \frac{53693210667}{8948678656} y_{n-\frac{11}{2}}^{(5)} + \frac{8389839489}{559292416} y_{n-\frac{9}{2}}^{(5)} - \frac{89473837063}{4474339328} y_{n-\frac{7}{2}}^{(5)} + \frac{16843614675}{1118584832} y_{n-\frac{5}{2}}^{(5)} - \frac{53339057007}{8948678656} y_{n-\frac{3}{2}}^{(5)} + y_{n-\frac{1}{2}}^{(5)} + t_{n-2}, \tag{17}$$

$$\frac{75}{2816}y_{n-\frac{7}{2}} - \frac{441}{2816}y_{n-\frac{5}{2}} + \frac{1225}{2816}y_{n-\frac{3}{2}} - \frac{3675}{2816}y_{n-\frac{1}{2}} + y_n - \frac{105}{352}hy_n^{(1)}$$

$$= h^5(-\frac{2961}{23068672}y^{(5)}_{n-\frac{11}{2}} + \frac{18655}{23068672}y^{(5)}_{n-\frac{9}{2}} - \frac{74795}{34603008}y^{(5)}_{n-\frac{7}{2}} + \frac{23555}{11534336}y^{(5)}_{n-\frac{5}{2}} -$$

$$\frac{318045}{23068672}y^{(5)}_{n-\frac{3}{2}} - \frac{213647}{69206016}y^{(5)}_{n-\frac{1}{2}}) + t_{n-1}, \qquad (18)$$

where

$$\begin{cases} t_1 = -\frac{102557}{1015021568}h^{11}y_0^{(11)}, \\ t_2 = -\frac{1068578685}{1068498944}h^{11}y_0^{(11)}, \\ t_3 = \frac{4319963216177682508l}{8291595833636290560}h^{11}y_0^{(11)}, \\ t_{n-2} = -\frac{1068419203}{1068498944}h^{11}y_n^{(11)}, \\ t_{n-1} = -\frac{102557}{1015021568}h^{11}y_n^{(11)}, \end{cases} \qquad (19)$$

The local truncation error corresponding to the method (8) is given by

$$t_i = (1-(2\alpha+2\beta+2\gamma))h^5y_i^{(5)} + (\frac{5}{24} - \frac{1}{4}(25\alpha+9\beta+\gamma))h^7y_i^{(7)}$$

$$+(\frac{23}{1152} - \frac{1}{192}(625\alpha+81\beta+\gamma))h^9y_i^{(9)} + (\frac{227}{193536} - \frac{1}{23040}(15625\alpha+729\beta+64\gamma))h^{11}y_i^{(11)}$$

$$+(\frac{631}{13271040} - \frac{1}{5160960}(390625\alpha+6561\beta+\gamma)h^{13}y_i^{(13)}$$

$$+O(h^{14}), i = 3,4,...,n-3. \qquad (20)$$

Case(i): If me choose $\alpha = \frac{1}{240}, \beta = \frac{27}{240}, \gamma = \frac{92}{240}$ the truncation errors(20) with be $O(h^7)$.

Case(ii): If me choose $\alpha = 0, \beta = \frac{1}{24}, \gamma = \frac{11}{24}$ the truncation errors(20) with be $O(h^{11})$.

4 Polynomial spline solution

The methods (8) along with boundary condition (14)-(18) when we ignore the truncation errors in (19) give a system of linear equations. Considering $Y = [y_{\frac{1}{2}}, y_{\frac{3}{2}}, ..., y_{n-\frac{1}{2}}]^T$ and $C = [c_{\frac{1}{2}}, c_{\frac{3}{2}}, ..., c_{n-\frac{1}{2}}]^T$, This system can be written the following matrix equation:

$$(A + h^5 BF)Y = C$$

where $A =$

$$\begin{pmatrix} a'_0 & a'_1 & a'_2 & a'_3 & a'_4 & 0 & 0 & 0 & \cdots & 0 \\ a''_0 & a''_1 & a''_2 & a''_3 & a''_4 & a''_5 & 0 & 0 & \cdots & 0 \\ a'''_0 & a'''_1 & a'''_2 & a'''_3 & a'''_4 & a'''_5 & a'''_6 & 0 & \cdots & 0 \\ -1 & 5 & -10 & 10 & -5 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 5 & -10 & 10 & -5 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & -1 & 5 & -10 & 10 & -5 & 1 \\ 0 & \cdots & 0 & 0 & a^\star_5 & a^\star_4 & a^\star_3 & a^\star_2 & a^\star_1 & a^\star_0 \\ 0 & \cdots & 0 & 0 & 0 & a^{\star\star}_4 & a^{\star\star}_3 & a^{\star\star}_2 & a^{\star\star}_1 & a^{\star\star}_0 \end{pmatrix} \quad (21)$$

and

$$B = \begin{pmatrix} b'_0 & b'_1 & b'_2 & b'_3 & b'_4 & b'_5 & b'_6 & & & & & & & \\ b''_0 & b''_1 & b''_2 & b''_3 & b''_4 & b''_5 & b''_6 & b''_7 & & & & & & \\ b^\circ_0 & b^\circ_1 & b^\circ_2 & b^\circ_3 & b^\circ_4 & b^\circ_5 & b^\circ_6 & b^\circ_7 & b^\circ_8 & & & & & \\ \alpha & \beta & \gamma & \gamma & \beta & \alpha & & & & & & & & \\ 0 & \alpha & \beta & \gamma & \gamma & \beta & \alpha & & & & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & & & & \alpha & \beta & \gamma & \gamma & \beta & \alpha \\ & & & & & & b^\star_7 & b^\star_6 & b^\star_5 & b^\star_4 & b^\star_3 & b^\star_2 & b^\star_1 & b^\star_0 \\ & & & & & & & b^{\star\star}_6 & b^{\star\star}_5 & b^{\star\star}_4 & b^{\star\star}_3 & b^{\star\star}_2 & b^{\star\star}_1 & b^{\star\star}_0 \end{pmatrix} \quad (22)$$

The vector C is defined by

$c_{\frac{1}{2}}=h^5(\frac{213647}{69206016}g_{\frac{1}{2}}+\frac{318045}{23068672}g_{\frac{3}{2}}-\frac{23555}{11534336}g_{\frac{5}{2}}+\frac{74795}{34603008}g_{\frac{7}{2}}-\frac{18655}{23068672}g_{\frac{9}{2}}+\frac{2961}{23068672}g_{\frac{11}{2}})$
$-\frac{105}{352}hy_0^{(1)}-y_0,$
$c_{\frac{3}{2}}=h^5(g_{\frac{1}{2}}-\frac{54045086865}{8948678656}g_{\frac{3}{2}}+\frac{16713930285}{1118584832}g_{\frac{5}{2}}-\frac{89499736057}{4474339328}g_{\frac{7}{2}}+\frac{8388932991}{559292416}g_{\frac{9}{2}}-\frac{53690933205}{8948678656}g_{\frac{11}{2}}$
$+g_{\frac{13}{2}})-\frac{45045}{273092}hy_0^{(1)}-y_0,$
$c_{\frac{5}{2}}=h^5(g_{\frac{1}{2}}-\frac{10832115164261467}{185034687774720}g_{\frac{3}{2}}+\frac{2540337475055785}{27755203166208}g_{\frac{5}{2}}-\frac{11677088273220341}{92517343887360}g_{\frac{7}{2}}+\frac{1316026985450353}{15419557314560}g_{\frac{9}{2}}$
$-\frac{15013471080229577}{555104063324160}g_{\frac{11}{2}}+g_{\frac{13}{2}}+g_{\frac{15}{2}})-\frac{350672533}{26889574}hy_0^{(1)}+\frac{26429938269}{1720932736}h^3y_0^{(3)}-y_0,$
$\vdots$
$c_{i+\frac{1}{2}}=h^5(\alpha g_{i-\frac{5}{2}}+\beta g_{i-\frac{3}{2}}+\gamma g_{i-\frac{1}{2}}+\gamma g_{i+\frac{1}{2}}+\beta g_{i+\frac{3}{2}}+\alpha g_{i+\frac{5}{2}}),$
$i=3,4,...,(n-3)$
$\vdots$
$c_{n-\frac{3}{2}}=h^5(g_{n-\frac{13}{2}}-\frac{53693210667}{8948678656}g_{n-\frac{11}{2}}+\frac{8389839489}{559292416}g_{n-\frac{9}{2}}-\frac{89473837063}{4474339328}g_{n-\frac{7}{2}}+\frac{16843614675}{1118584832}g_{n-\frac{5}{2}}$
$-\frac{53339057007}{8948678656}g_{n-\frac{3}{2}}+g_{n-\frac{1}{2}})+\frac{45045}{273092}hy_n^{(1)}-y_n,$
$c_{n-\frac{1}{2}}=h^5(-\frac{2961}{23068672}g_{n-\frac{11}{2}}+\frac{18655}{23068672}g_{n-\frac{9}{2}}-\frac{74795}{34603008}g_{n-\frac{7}{2}}+\frac{23555}{11534336}g_{n-\frac{5}{2}}-\frac{318045}{23068672}g_{n-\frac{3}{2}}$
$-\frac{213647}{69206016}g_{n-\frac{1}{2}})+\frac{105}{352}hy_n^{(1)}-y_n.$

5 Numerical results

In this section the presented method are applied to the following test problems if choosing $(\alpha,\beta,\gamma)=(\frac{1}{240},\frac{27}{240},\frac{92}{240})and(\alpha,\beta,\gamma)=(0,\frac{1}{24},\frac{11}{24})$ we obtained the method of order $O(h^2)$ and $O(h^6)$ respectively.

Problem 1. We Consider the following boundary-value problem

$$y^{(5)}(x)+y(x)Sin(x)=Cos[x](1+Sin[x])+Sin[x](Sin[x]-1),\ \ 0\leq x\leq 1,$$

$$y(0)=1,y'(0)=1,y^{(3)}(0)=-1,$$

$$y(1)=Cos[1]+Sin[1],y'(1)=Cos[1]-Sin[1], \tag{23}$$

The exact solution for this problem is $y(x) = Cos[x] + sin[x]$. We solved this Problem by the method of $O(h^2)$ and $O(h^6)$ with different values of $h = \frac{1}{13}, \frac{1}{26}, \frac{1}{52}$. The maximum absolute errors in the solutions are tabulated in Table 1 and compared with [10].

Problem 2. We Consider the following boundary-value problem

$$y^{(5)}(x) + xy(x) = \frac{(120Cos[x])}{x^5} - \frac{(20Cos[x])}{x^3} + \frac{Cos[x]}{x} - \frac{(120Sin[x])}{x^6} + \frac{(60Sin[x])}{x^4} - \frac{(5Sin[x])}{x^2} + Sin[x], \ \ 1 \le x \le 2,$$

$$y(0) = Sin[1], y'(0) = Cos[1] - Sin[1], y^{(3)}(0) = 5Cos[1] - 3Sin[1],$$

$$y(1) = \frac{Sin[2]}{2}, y'(1) = \frac{Cos[2]}{2} - \frac{Sin[2]}{4}, \tag{24}$$

The exact solution for this problem is $y(x) = \frac{Sin[x]}{x}$. We solved this problem by different values of $h = \frac{1}{13}, \frac{1}{26}, \frac{1}{52}$ with both the methods of $O(h^2)$and $O(h^6)$ and the maximum absolute errors are summarized in Table 2 and compared with [10].The results verify that the max.Abs.error in solution in case of our $O(h^6)$ are very accurate in comparison with method in [10].

Problem 3. We Consider the following boundary-value problem

$$y^{(5)}(x) - y(x) = -(15 + 10x)e^x, \ \ 0 \le x \le 1,$$

$$y(0) = 0, y'(0) = 1, y^{(3)}(0) = -3,$$

$$y(1) = 0, y'(1) = -e, \tag{25}$$

The exact solution for this problem is $y(x) = x(1 - x)e^x$. We solved this example by different values of $h = \frac{1}{13}, \frac{1}{26}, \frac{1}{52}$ The maximum absolute errors in solutions for our methods are listed in Tables 3. Also we compare the maximum absolute errors in our methods with the maximum absolute errors in [10].The results verify that the max.Abs.error in solution in case of our $O(h^6)$ are very accurate in comparison with method in [10].

Problem 4. We Consider the following boundary-value problem

$$y^{(5)}(x) - y(x) = -e^{Sin[x]} + e^{Sin[x]}Cos[x] - 19e^{x}(Cos[x])^2 -$$

$$10e^{Sin[x]}(Cos[x])^3 + e^{x}(Cos[x])^5 + 40e^{x}Cos[x]Sin[x] +$$

$$15e^{Sin[x]}Cos[x]Sin[x] - 10e^{Sin[x]}(Cos[x])^3 Sin[x] +$$

$$19e^{x}(Sin[x])^2 + 15e^{Sin[x]}Cos[x](Sin[x])^2, \;\; 0 \leq x \leq 1,$$

$$y(0) = 1, y'(0) = 2, y^{(3)}(0) = -1,$$

$$y(1) = e^{Sin[1]} + eCos[1]Sin[1],$$

$$y'(1) = e^{Sin[1]}Cos[1] + e(Cos[1])^2 + eCos[1]Sin[1] - e(Sin[1])^2, \tag{26}$$

The exact solution for this problem is $y(x) = e^{Sin[x]} + e^{x}Cos[x]Sin[x]$. We solved this example by different values of $h = \frac{1}{13}, \frac{1}{26}, \frac{1}{52}$. The maximum absolute errors in solutions for our methods are listed in Tables 3. Also we compare the maximum absolute errors in our methods with the maximum absolute errors in [10].The table shows that our $O(h^6)$ method are more accurate with respect to the method in [10] .

Table 1: Observed maximum absolute errors for example (1)

h	$\alpha = 0, \beta = \frac{1}{24}$, $\gamma = \frac{11}{24}$	$\alpha = \frac{1}{240}, \beta = \frac{27}{240}$, $\gamma = \frac{92}{240}$	[10]
$\frac{1}{13}$	2.4299×10^{-11}	2.1953×10^{-7}	3.8657×10^{-7}
$\frac{1}{26}$	1.1691×10^{-12}	6.7417×10^{-8}	2.0206×10^{-8}
$\frac{1}{52}$	9.3838×10^{-11}	1.7739×10^{-8}	6.6482×10^{-8}

Table 2: Observed maximum absolute errors for example (2)

h	$\alpha=0,\beta=\frac{1}{24}$, $\gamma=\frac{11}{24}$	$\alpha=\frac{1}{240},\beta=\frac{27}{240}$, $\gamma=\frac{92}{240}$	[10]
$\frac{1}{13}$	2.3382×10^{-12}	5.2214×10^{-8}	4.2305×10^{-7}
$\frac{1}{26}$	6.7701×10^{-13}	1.4694×10^{-8}	3.1230×10^{-8}
$\frac{1}{52}$	2.3792×10^{-11}	3.7912×10^{-9}	3.2792×10^{-9}

Table 3: Observed maximum absolute errors for example (3)

h	$\alpha=0,\beta=\frac{1}{24}$, $\gamma=\frac{11}{24}$	$\alpha=\frac{1}{240},\beta=\frac{27}{240}$, $\gamma=\frac{92}{240}$	[10]
$\frac{1}{13}$	3.7572×10^{-9}	28453×10^{-5}	$1.3767\text{e}\times10^{-4}$
$\frac{1}{26}$	1.8015×10^{-11}	7.8018×10^{-6}	7.1273×10^{-6}
$\frac{1}{52}$	3.1166×10^{-11}	1.9911×10^{-6}	4.6950×10^{-7}

Conclusion

We approximate solution of the fifth-order linear boundary-value problems by using non-polynomial spline and obtained the new boundary conditions. The new methods of order $O(h^2)$and $O(h^6)$ enable us to approximate the solution at every point of the range of integration. The application of the methods on the different kinds of test problems shows that our methods produced better result in comparison with the methods in [10].

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Table 4: Observed maximum absolute errors for example (4)

h	$\alpha=0,\beta=\frac{1}{24}$, $\gamma=\frac{11}{24}$	$\alpha=\frac{1}{240},\beta=\frac{27}{240}$, $\gamma=\frac{92}{240}$	[10]
$\frac{1}{13}$	6.0058×10^{-6}	10709×10^{-4}	$1.71e\times10^{-4}$
$\frac{1}{26}$	5.9427×10^{-6}	348608×10^{-5}	6.62×10^{-6}
$\frac{1}{52}$	5.9402×10^{-6}	1335×10^{-5}	2.96×10^{-6}

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