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Numerical solution of nonlinear Volterra-Fredholm integral equation by using Chebyshev polynomials

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Abstract

In this paper, we have used Chebyshev polynomials to solve linear and nonlinear Volterra-Fredholm integral equations, numerically. First we introduce these polynomials, then we use them to change the Volterra-Fredholm integral equation to a linear or nonlinear system. Finally, the numerical examples illustrate the efficiency of this method.

Keywords: Linear and nonlinear Volterra-Fredholm integral equation; Chebyshev polynomials; Operational matrix

1 Introduction

Nonlinear Volterra-Fredholm integral equations are defined as follows:

$$y(x) - \int_{-1}^{x} k_1(x,t)[y(t)]^p\,dt - \int_{-1}^{1} k_2(x,t)[y(t)]^q\,dt = f(x), \quad x \in [-1,1]. \tag{1}$$

where, the functions $f(x), k_1(x,t)$ and $k_2(x,t)$ are known and $y(x)$ is the unknown function to be determined and $p, q \geq 1$ are two positive integers.

Previously, some kinds of Volterra-Fredholm integral equations had been solved numerically, by different methods that are indicated below. First of all Kauthen used

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Taylor polynomials to solve this kind of equations [1], then Yalsinbas and Sezer used them in obtaining approximate solution of high-order linear Vloterra-Fredholm integro-differential equations [2]. In [3], Yalsinbas developed numerical solution of nonlinear Volterra-Fredholm integral equations by using Taylor polynomials. Also, Maleknejad and Mahmodi [4] introduced a method for numerical solution of high-order nonlinear Volterra-Fredholm integro-differential equations based on Taylor polynomials. In [5], Yousefi and Razzaghi solved nonlinear Volterra-Fredholm integral equation by using Legendre wavelets. Yusufoglu and Erbas presented the method based on interpolation in solving linear Volterra-Fredholm integral equations [6]. Before that, Chebyshev polynomials also were applied for solving nonlinear Fredholm-Volterra integro-differential equations by Cerdik-Yaslan and Akyuz-Dascioglu [7].

We know that Chebyshev polynomials of the first kind of degree n are defined by [3]:

$$T_n(x) = \cos(n \arccos(x)), \quad n \geq 0.$$

Also these polynomials are derived from the following recursive formula [3]:

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, 3, ...$$

These polynomials are orthogonal on [-1,1] with respect to weight function $\omega(x) = (1 - x^2)^{-1/2}$ [3]:

$$\int_{-1}^{1} T_i(x)T_j(x)\omega(x)\,dx = \begin{cases} \pi, & i = j = 0, \\ \\ \frac{\pi}{2}\delta_{ij}, & i, j > 0. \end{cases}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ \\ 0, & i \neq j. \end{cases}$$

We apply these polynomials, as basis in [-1,1], to solve the equation (1) and reduce it to a system of equations. The generated system, which according the type of equation (1) would be either linear or nonlinear, can be solved through various methods and accordingly the unknown coefficients can be found.

2 Approximation of function by using Chebyshev polynomials

We suppose the function $y(x)$ defined in [-1,1]. This function may be represented by first kind Chebyshev polynomials series as [8]:

$$y(x) = \sum_{i=0}^{\infty} y_i T_i(x), \tag{2}$$

if we truncated the series (2), then we can write (2) as follows:

$$y(x) \simeq \sum_{i=0}^{N} y_i T_i(x) = Y^T T(x), \tag{3}$$

where

$$Y = [y_0, y_1, y_2, ...y_N]^T, \tag{4}$$

$$T(x) = [T_0(x), T_1(x), T_2(x), ..., T_N(x)]^T. \tag{5}$$

Clearly Y and T are $(N+1) \times 1$ vectors and coefficients y_i are given by [8]:

$$y_i = (y(x), T_i(x)) = \begin{cases} \frac{1}{\pi} \int_{-1}^{1} \omega(x) y(x)\, dx, & i = 0, \\ \\ \frac{2}{\pi} \int_{-1}^{1} \omega(x) T_i(x) y(x)\, dx, & i > 0. \end{cases} \tag{6}$$

where $\omega(x)$ is the weight function as $(1-x^2)^{-1/2}$.

Similarly, regarding a function with two variable of $k(x,t)$, which is defined on [-1,1], we'll have [9]:

$$k(x,t) \simeq \sum_{i=0}^{N} \sum_{j=0}^{N} T_i(x) k_{ij} T_j(t) = T^T(x) K T(t), \tag{7}$$

where K is a $(N+1)\times(N+1)$ matrix, with

$$K_{ij} = (T_i(x), (k(x,t), T_j(t))). \tag{8}$$

Also the positive integer powers of a function may be approximated as:

$$[y(x)]^p \simeq [Y^T T(x)]^p = {Y_p^*}^T T(x), \tag{9}$$

where Y_p^* is called the operational vector of the pth power of the function $y(x)$. The elements of Y_p^* are nonlinear combinations of the elements of the vector Y.
For the Chebyshev polynomials with N=3 the second and third product vector operation vector of y(x) is computed as follows [10]:

$$Y_2^* = \frac{1}{2}\begin{pmatrix} 2y_0^2 + y_1^2 + y_2^2 + y_3^2 \\ 4y_0y_1 + 2y_1y_2 + 2y_2y_3 \\ y_1^2 + 4y_0y_2 + y_1y_3 \\ 2y_1y_2 + 4y_0y_3 \end{pmatrix},$$

also

$$Y_3^* = \frac{1}{4}\begin{pmatrix} 4y_0^3 + 6y_0y_1^2 + 3y_1^2y_2 + 6y_0y_2^2 + 6y_1y_2y_3 + 6y_0y_3^2 \\ 12y_0^2y_1 + 3y_1^3 + 12y_0y_1y_2 + 6y_1y_2^2 + 3y_1^2y_3 + 12y_0y_2y_3 + 3y_2^2y_3 + 6y_1y_3^2 \\ 6y_0y_1^2 + 12y_0^2y_2 + 6y_1^2y_2 + 3y_2^3 + 12y_0y_1y_3 + 6y_1y_2y_3 + 6y_2y_3^2 \\ y_1^3 + 12y_0y_1y_2 + 3y_1y_2^2 + 12y_0^2y_3 + 6y_1^2y_3 + 6y_2^2y_3 + 3y_3^3 \end{pmatrix}.$$

3 The operational matrices

In this section we introduce the operational matrix as P [11] for computing the integral of vector T(x) which defined in (5).
For $T_0(x)$ and $T_1(x)$ we have:

$$\int_{-1}^{x} T_0(t)\,dt = 1 + x = T_0(x) + T_1(x),$$

$$\int_{-1}^{x} T_1(t)\,dt = \frac{x^2}{2} - \frac{1}{2} = \frac{-1}{4}T_0(x) + \frac{1}{4}T_2(x), \tag{10}$$

and similarly for $T_{N-1}(x)$ which $N \geq 3$ we have:

$$\int_{-1}^{x} T_{N-1}(t)\,dt = \frac{1}{2N}T_N(x) - \frac{1}{2(N-2)}T_{N-2}(x) + \frac{(-1)^{N-1}}{1-(N-1)^2}T_0(x). \tag{11}$$

Equations (10) and (11) allow us to write:

$$\int_{-1}^{x} T(t)\,dt = PT(x), \tag{12}$$

where P is the $(N+1)\times(N+1)$ operational matrix as follows:

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{4} & 0 & \cdots & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \frac{(-1)^{N-1}}{1-(N-1)^2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2N} \\ \frac{(-1)^N}{1-N^2} & 0 & 0 & 0 & \cdots & -\frac{1}{2(N-1)} & 0 \end{pmatrix}. \tag{13}$$

The following equation is generated in the same way

$$\int_{-1}^{1} T(t)\,dt = PT(1). \tag{14}$$

Moreover, for Chebyshev polynomials we have:

$$T(x)T^T(x)C \simeq \tilde{C}^T T(x), \tag{15}$$

where C is $(N+1)\times 1$ vector as

$$C = [c_0, c_1, c_2, ..., c_N], \tag{16}$$

and $\tilde{C}$ is a $(N+1)\times(N+1)$ square matrix

$$\tilde{C}=\frac{1}{2}\begin{pmatrix} 2c_0 & c_1 & \cdots & c_i & \cdots & c_{N-1} & c_N \\ 2c_1 & 2c_0+c_2 & \cdots & c_{i-1}+c_{i+1} & \cdots & c_{N-2}+c_N & c_{N-1} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots \\ 2c_i & c_{i-1}+c_{i+1} & \cdots & 2c_0+c_{2i} & \cdots & c_{N-i-1} & c_{N-i} \\ \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & & \vdots & \vdots \\ 2c_{N-1} & c_{N-2}+c_N & \cdots & c_{N-i-1} & \cdots & 2c_0 & c_1 \\ 2c_N & c_{N-1} & \cdots & c_{N-i} & \cdots & c_1 & 2c_0 \end{pmatrix}, \tag{17}$$

where $i=[\frac{N}{2}]$.

4 The method of solving

Now we begin to solve the integral equation (1) by using Chebyshev polynomials of first kind.

If we compute the approximation of $y(x), k_1(x,t), k_2(x,t), [y(x)]^p$ and $[y(x)]^q$ with method of Section 2, we'll have:

$$y(x)=T^T(x)Y,$$

$$[y(x)]^m = Y_m^{*^T} T(x), \quad for \;\; m=p,q,$$

$$k_i(x,t)=T^T(x)K_iT(t), \quad for \;\; i=1,2, \tag{18}$$

where $T(x)$ is defined in (5), Y is an unknown vector and Y_m^* are operational vectors that are described in Section 2.

Using (15) and (18), we have:

$$\int_{-1}^{x} k_1(x,t)[y(t)]^p\,dt \simeq \int_{-1}^{x} T^T(x)K_1T(t)T^T(t)Y_p^*\,dt = T^T(x)K_1\int_{-1}^{x} T(t)T^T(t)Y_p^*\,dt$$

$$= T^T(x)K_1\int_{-1}^{x} \tilde{Y}_p^{*^T}T(t)\,dt = T^T(x)K_1\tilde{Y}_p^{*^T}PT(x). \quad (19)$$

Similarly we'll have:

$$\int_{-1}^{1} k_2(x,t)[y(t)]^q\,dt = T^T(x)K_2\tilde{Y}_q^{*^T}PT(1). \quad (20)$$

Then from equations (1), (18), (19) and (20) we get

$$T^T(x)Y - T^T(x)K_1\tilde{Y}_p^{*^T}PT(x) - T^T(x)K_2\tilde{Y}_q^{*^T}PT(1) = f(x). \quad (21)$$

To find the unknown coefficients, we first collocate equation (21) in $(N+1)$ at the collocation points of $\{x_i\}_{i=0}^N$ in the interval [-1,1]. Here, we can consider the x_i points as:

$$x_i = -1 + \frac{2i}{N}, \quad i = 0, 1, 2, ..., N.$$

So we'll have:

$$T^T(x_i)Y - T^T(x_i)K_1\tilde{Y}_p^{*^T}PT(x_i) - T^T(x_i)K_2\tilde{Y}_q^{*^T}PT(1) = f(x_i). \quad (22)$$

The result of equation (22) will be either a linear or nonlinear system, by solving which through direct or iterative methods we can compute the unknown coefficients.

5 Examples

In this section, we applied presented method in this paper for solving integral equation (1) and solved some examples. In examples 5.3 and 5.4 we used Newton's iterative method for solving introduced nonlinear system. The computations associated with the examples were performed using Mathematica 5.2 software.

Example 5.1 *Consider the following linear Volterra-Fredholm integral equation:*

$$y(x) - \int_{-1}^{x}(2x-t)y(t)\,dt - \int_{-1}^{1}(2x+3x^2t)y(t)\,dt = f(x), \qquad (23)$$

where $f(x) = -2x^3 - \frac{9}{2}x^2 + 12x + \frac{1}{2}$.
For N=10, the method gives the exact solution $y(x) = 3x - 1$. *Table 1 shows the numerical results for N=7 and N=9.*

Table 1. The Numerical results in Example 5.1.

x_i	Exact solution	Approximation solution in N=7	Approximation solution in N=9
-1	-4	-3.99106	-3.99954
-0.75	-3.25	-3.24584	-3.24952
-0.5	-2.5	-2.50061	-2.49950
-0.25	-1.75	-1.75537	-1.74948
0	-1	-1.01012	-0.99945
0.25	-0.25	-0.26486	-0.24944
0.5	0.5	0.48040	0.50006
0.75	1.25	1.22567	1.25061
1	2	1.97094	2.00063

Example 5.2 *As the second example consider the following integral equation:*

$$y(x) - \int_{-1}^{x} e^{x+t}y(t)\,dt - \int_{-1}^{1}\frac{x}{2}e^{x+t}y(t)\,dt = f(x), \qquad (24)$$

where $f(x) = e^{-x} - e^{x}$, *with exact solution* $y(x) = e^{-x}$.
Table 2 illustrate the numerical results.

Table 2. The Numerical results in Example 5.2.

x_i	Exact solution	Approximation solution in N=9	Approximation solution in N=11
-1	2.71828	2.56680	2.71980
-0.75	2.117	1.98470	2.11800
-0.5	1.64872	1.60013	1.64945
-0.25	1.28403	1.27500	1.28440
0	1	1.00000	1.00000
0.25	0.77880	0.77501	0.77844
0.5	0.60653	0.60068	0.60586
0.75	0.47237	0.47456	0.47171
1	0.36788	0.28520	0.36840

Example 5.3 *For the following nonlinear Volterra-Fredholm integral equation:*

$$y(x) - \int_{-1}^{x} (x+t)[y(t)]^2\,dt - \int_{-1}^{1} (x-t)y(t)\,dt = f(x), \tag{25}$$

where $f(x) = \frac{-1}{3}[7x^4 - 2x - 7]$, *with exact solution* $y(x) = 2x$, *Table 3 shows the numerical results for N=8 and N=10.*

Table 3. The Numerical results in Example 5.3.

x_i	Exact solution	Approximation solution in N=8	Approximation solution in N=10
-1	-2	-1.95641	-2.00093
-0.75	-1.5	-1.46166	-1.50076
-0.5	-1	-0.96692	-1.00059
-0.25	-0.5	-0.47217	-0.50042
0	0	-0.02258	-0.00025
0.25	0.5	0.51733	0.49992
0.5	1	1.01208	1.00009
0.75	1.5	1.50683	1.50026
1	2	2.00158	2.00043

Example 5.4 *Consider the following nonlinear Volterra-Fredholm integral equation:*

$$y(x) - \int_{-1}^{x} (2x - t)y(t)\, dt - \int_{-1}^{1} (2xt + 3x^2 t)[y(t)]^2\, dt = f(x), \tag{26}$$

where $f(x) = \frac{-2}{3}x^3 + \frac{11}{2}x^2 + \frac{20}{3}x - \frac{1}{6}$.

For N=12, the method gives the exact solution $y(x) = x - 1$. *Table 4 shows the numerical results for N=10 and N=11.*

Table 4. The Numerical results in Example 5.4.

x_i	Exact solution	Approximation solution in N=10	Approximation solution in N=11
-1	-2	-2.07704	-1.99995
-0.75	-1.75	-1.78054	-1.74994
-0.5	-1.5	-1.50994	-1.50005
-0.25	-1.25	-1.25609	-1.25008
0	-1	-1.00574	-1.00008
0.25	-0.75	-0.75589	-0.75007
0.5	-0.5	-0.50348	-0.50004
0.75	-0.25	-0.23144	-0.24996
1	0	0.12992	0.00009

6 Conclusion

In this work, we solved linear and nonlinear Volterra-Fredholm integral equations by using Chebyshev polynomials. Nonlinear integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solution, for this purpose the presented method can be proposed.

This method can be extended and applied to the system of linear and nonlinear integral equations, but some modifications are required.

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