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On the asymptotic behaviour of graded generalized local cohomology modules

F. Dehghani-Zadeh

Department of Mathematics, Yazd Branch, Islamic Azad University, Yazd, Iran.

Abstract

Let $R = \bigoplus_{n\geq 0} R_n$ be a graded Noetherian ring with local base ring R_0 and let $R_+ = \bigoplus_{n\geq 1} R_n$. Let M and N be finitely generated graded R-modules. In this paper, we prove some results on the asymptotic behaviour of the n-th graded components $H^i_{R_+}(M,N)_n$ of $H^i_{R_+}(M,N)$ for $n \to -\infty$. We also study the tameness and asymptotical stability of the homogeneous components of $H^i_{R_+}(M,N)$ for some i's with a specified property.

Keywords: Associated primes, Asymptotic behaviours, Graded generalized local cohomology, Tameness.

1 Introduction

There is plenty of current interest in the theory of graded local cohomology modules and in recent years many papers have appeared in this area of research. The main purpose of this paper is to study the asymptotic behaviour of associated prime ideals of graded components of generalized local cohomology modules.

For an ideal I of a commutative Noetherian ring R and R-modules M and N, the i-th generalized local cohomology module

$$H^i_I(M,N) = \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(\frac{M}{I^n M}, N)$$

E-mail Address: f.dehghanizadeh@yahoo.com

has been introduced by Herzog in [11] and studied further by Suzuki in [18]. It is well known that for $M = R$, $H^i_I(M,N)$ is converted to $H^i_I(N)$, the i-th ordinary local cohomology module with respect to I.

Throughout this paper, we assume that $R = \bigoplus_{n\geq 0} R_n$ is a positive graded commutative Noetherian ring with local base ring (R_0, m_0) and that $R_+ = \bigoplus_{n>0} R_n$ is the irrelevant graded ideal of R. Also we use $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ to denote non-zero finitely generated graded R-modules(here $\mathbb{Z}$ denotes the set of all integers). It is well known that $H^i_{R_+}(M,N)$ carry a natural grading and its n-th components $H^i_{R_+}(M,N)_n$ is finitely generated R_0-module for all n and is zero for all $n \gg 0$. This raises the following question. What can be said about $H^i_{R_+}(M,N)_n$ and $\mathrm{Ass}_{R_0}(H^i_{R_+}(M,N)_n)$ for $n \ll 0$ and their relations with $\mathrm{Ass}_R\, H^i_{R_+}(M,N)$? In this connection, there are three open problems:

Problems [cf 2] Let $i \in \mathbb{N}_0$ and let M be a finitely generated graded R-module.

(i) $H^i_{R_+}(M,N)$ has the property of asymptotic stability of associated primes ,

(ii) $H^i_{R_+}(M,N)$ has the property of asymptotic stability of supports ,

(iii) $H^i_{R_+}(M,N)$ is tame .

Note that (i) implies (ii) and (ii) implies (iii) .

In this paper we investigate the above questions for the module $H^i_{R_+}(M,N)$ for some indices i.

2 preliminaries

Throughout this paper M and N are finitely generated graded R-modules over a Noetherian local ring (R,m). Let $pd_R(M)$ denote the projective dimension of M. For any ideal I of R we denote by $I_M = ann(\frac{M}{IM})$the annihilator of the module $\frac{M}{IM}$ and Γ_I the I-torsion functor.

Definition 2.1. *Let $s \geq 0$ be an integer, and $x_1, x_2, ..., x_n \in R$ be a sequence. We say that $x_1, ..., x_n$ is an N-sequence in dimension$>s$ if and only if $x_i \notin p$ for all $p \in Ass(\frac{M}{(x_1,...,x_{i-1})M})$ satisfying $\dim \frac{R}{p} > s$, for all $i = 1, ..., n$. Assume that R is local. Then $x_1, ..., x_n$ is an N-sequence in dimension>0 if and only if it is a filter regular sequence with respect to N in the sense of [13]. Moreover, $x_1, x_2, ..., x_n$ is an N- sequence in dimension>1 if and only if it is a generalized regular sequence with respect to N in the sense of [17].*

Reminder 2.2. *Let I be a proper ideal of R. The f-depth (resp. gdepth) of I on N is defined as the length of any maximal N-filter (resp. generalized) regular sequence in I, denoted by f-depth(I, N)(resp. gdepth (I, N)). By [17], it follows that gdepth (I, N) is the least integer i such that $SuppH^i_I(N)$ is an infinite set. Also using [8] yields that f-depth$(I + Ann(M), N)$ is the least integer such that $H^i_I(M, N)$ is not Artinian.*

Lemma 2.3. *Let I denote an ideal of a Noetherian local ring R. The following statements are true:*

(i) *(cf.[9]) Let$E^{\bullet}$ be an injective resolution of N. Then, for any $j \geq 0$, we have*

$$H^i_I(M, N) \cong H^i(\Gamma_I(Hom(M, E^{\bullet}))) \cong H^i(Hom(M, \Gamma_I(E^{\bullet}))) \cong H^i(Hom(M, \Gamma_{I_M}(E^{\bullet}))).$$

(ii) *(cf.[1]) Let $t =$gdepth(I, M). Then $H^i_I(M)$ is I-cofinite for all $i < t$.*

(iii) *(cf.[15]) For every I-cofinite module $T, \frac{T}{IT}$ is a finite R- module.*

Lemma 2.4. *Assume that I is an ideal of R generated by elements of positive degrees, and r is equal to the length of maximal N-sequences in dimension$>s$ in I. Then there exists an homogeneous N-sequence in dimension$>s$ in I of length r.*

Proof It follows immediately from the definition (2.1) and [7, 1.5.10].

Remark 2.5. *Let $f : R_0 \longrightarrow R'_0$ be a faithful flat ring homomorphism and let $R' = R'_0 \otimes_{R_0} R$, $M' = R'_0 \otimes_{R_0} M$ and $N' = R'_0 \otimes_{R_0} N$. Then $H^k_{R'_+}(M', N')_n \cong H^k_R(M, N)_n \otimes_{R_0} R'_0$. Therefore when this is the case $H^k_{R'_+}(M', N')_n = 0$ if and only if $H^k_R(M, N)_n = 0$.*

Also, if R'_0 is local with unique maximal ideal m'_0, then $\dim \frac{N}{m_0 N} = \dim \frac{N'}{m'_0 N'}$. *Note that $H^k_{R_+}(M,N)$ is Artinian (resp. Noetherian) if and only if $H^k_{R'_+}(M',N')$ is Artinian (resp. Noetherian). Moreover*

$$\operatorname{Ass}_{R_0} H^i_{R_+}(M,N)_n = \{\mathfrak{p}'_0 \cap R_0 | \mathfrak{p}'_0 \in \operatorname{Ass}_{R'_0} H^i_{R'_+}(M',N')_n\}$$

for all $n \in \mathbb{Z}$ [14,23.2]. It follows that $\operatorname{Ass}_{R_0} H^i_{R_+}(M,N)_n)_{n\in\mathbb{Z}}$ is asymptotically stable if and only if $(\operatorname{Ass}_{R'_0} H^i_{R'_+}(M',N')_n)_{n\in\mathbb{Z}}$ is asymptotically stable.
For any unexplained terminology, the reader can refer to[6]and[19].

3 Results

In this section, we study the concept of asymptotic behaviour on generalized local cohomology modules. We keep the previous notations and hypotheses, and consider the following statements.

(i) $Ass_{R_0} H^i_{R_+}(M,N)_n$ is asymptotically stable for $n \to -\infty$ (e.g. there is some $n_0 \in \mathbb{Z}$ such that $Ass_{R_0} H^i_{R_+}(M,N)_n = Ass_{R_0} H^i_{R_+}(M,N)_{n_0}$ for all $n \leq n_0$),

(ii) $H^i_{R_+}(M,N)$ is tame or asymptotically gap free (e.g there is some $n_0 \in \mathbb{Z}$ such that $H^i_{R_+}(M,N)_n = 0$ for all $n \leq n_0$ or else $H^i_{R_+}(M,N)_n \neq 0$ for all $n \leq n_0$).

It is easy to see that asymptotic stability implies that $H^i_{R_+}(M,N)$ is asymptotically gap free and $Ass_R H^i_{R_+}(M,N)$ is finite. However, according to Brodmann et al. [5] the converse is not true.

Lemma 3.1. *If $H^i_{R_+}(N)$ is Artinian for each $i < r$. Then $H^i_{R_+}(M,N)$ is Artinian for all $i < r$.*

Proof It follows immediately by using induction on r.

It is easy to see that the analogue of [3,3.1] holds true whenever ordinary local cohomology is replaced by generalized local cohomology. Thus we have the following.

Lemma 3.2. *Let $S \subsetneqq \mathbb{N}_0$. Assume that R-module $\frac{R_0}{m_0} \bigotimes_{R_0} H^i_{R_+}(M,N) = T_i$ is Artinian for each $i \in S$. Then, there is some $n_0 \in \mathbb{Z} \bigcup \{\infty\}$ such that for each $x \in R_1 - \bigcup_{P \in AttT_i - V(R_+)} P$ the multiplication maps $H^i_{R_+}(M,N)_n \stackrel{x}{\longrightarrow} H^i_{R_+}(M,N)_{n+1}$ are surjective for all $i \in S$ and for all $n < n_0$.*

Theorem 3.3. *Let $R = R_0[R_1]$ and $H^i_{R_+}(M,N)$ be Artinian for all $i < r$, then $\mathrm{Ass}_{R_0} H^r_{R_+}(M,N)_n$ is asymptotically stable and $\mathrm{Ass}_R H^r_{R_+}(M,N)$ is finite set.*

Proof Let x be an indeterminate and apply remark (2.5) with $R'_0 = R_0[x]_{m_0 R_0[x]}$, the localization of the polynomial ring $R_0[x]$ at the prime ideal $m_0 R_0[x]$. We note that R'_0 is Noetherian local faithfully flat R_0-algebra, with maximal ideal $m_0 R'_0$ and with the residue field isomorphic to $(\frac{R_0}{m_0})(x)$ which is infinite. Hence, if we replace R, M and N by $R', M'$and N', respectively, then we are able to assume that $\frac{R_0}{m_0}$ is infinite. By using reminder (2.2), $r =$f-depth(I,N) where $I = R_+ + Ann(M)$. Also, in view of lemma (2.3), $H^i_{R_+}(M,N) \cong H^i_I(M,N)$. Therefore, we may assume that $\mathrm{Ann}(M) = 0$. We now argue by induction on r $(r \geq 0)$. It is straightforward to see that the result is true when $r = 0$. Now, suppose inductively, that $r \geq 1$ and that the result has been proved for $r-1$. Since $r \geq 1$, it follows that $R_+ \nsubseteq \bigcup_{P \in AssN - V(m)} P$. Therefore, $A = (AssN - V(m)) \bigcup (\bigcup_{i<r} Att \frac{H^i_{R_+}(M,N)}{m_0 H^i_{R_+}(M,N)} - V(R_+))$ is a finite set of graded primes in R, none of which contains R_1. As $\frac{R_0}{m_0}$ is infinite, the set $B = R_1 - \bigcup_{P \in A} P$ is not empty [7,1.5.12]. Let $x \in B$, it follows that x is a filter regular sequence with respect to N and there is some $n_0 \in \mathbb{Z} \bigcup \{\infty\}$ such that the multiplication maps $H^i_{R_+}(M,N)_n \stackrel{x}{\longrightarrow} H^i_{R_+}(M,N)_{n+1}$ are epimorphism for all $i < r$ and for all $n < n_0$. As $(0 :_N x)$ has finite length, we get that $H^i_{R_+}(M,(0 :_N x))_n = 0$ for all $n \ll 0$. Therefore, we can use the exact sequences $0 \longrightarrow (0 :_N x) \longrightarrow N \longrightarrow xN \longrightarrow 0$, $0 \longrightarrow xN \longrightarrow N \longrightarrow \frac{N}{xN} \longrightarrow 0$, to obtain the exact sequence

$$0 \longrightarrow H^{r-1}_{R_+}(M, \frac{N}{xN})_{n+1} \longrightarrow H^{r}_{R_+}(M,N)_n \longrightarrow H^{r}_{R_+}(M,N)_{n+1}$$

for all $n \ll 0$. So, the proof is complete by induction.

Proposition 3.4. *Let $R = R_0[R_1]$. If $H^i_{R_+}(N)$ is Artinian for all $i < t$, then $\mathrm{Ass}_{R_0} H^t_{R_+}(M,N)_n$ is asymptotically stable and $AssH^t_{R_+}(M,N)$ is finite set.*

Proof In view of lemma (3.1), $H^i_{R_+}(M.N)$ is Artinian for all $i < t$. By using the argument as in the method of theorem (3.3) is proved.
lemma (3.4) and lemma(3.3) are similar to the results of [12, 2.5 and 2.6] where the finitely generatedness is replaced by the Artinianness.

Proposition 3.5. *Let $R_0[R_1]$ and $m = m_0R$. If supp $H^i_{R_+}(N)$ is a finite set for all $i < s$, then $Ass_{R_0}H^s_{R_+}(N)_n$ is asymptotically stable and $n \to -\infty$ and $Ass_R H^s_{R_+}(N)$ is finite set.*

Proof First, we use the ideas of remark (2.5) similar theorem (3.3) and assume that the local ring R_0 has infinite residue field. In view of [17, 5.2], s=gdepth (R_+, N). We proceed this by induction on s. As $H^0_{R_+}(N)_n = 0$ for all $n \ll 0$, the statements of theorem are obvious for $s = 0$. By using [4, 5.6], there is nothing to prove for $s = 1$. Note that $\frac{H^i_{R_+}(N)}{mH^i_{R_+}(N)}$ is Artinian for all $i < s$ in view of lemma (2.3,ii and iii). Let $s \geq 2$ and $B = \{p \in spec(R) \mid \dim(\frac{R}{p}) \leq 1\}$. Hence, $A = (AssN - B)\bigcup(\bigcup_{i<s} Att\frac{H^i_{R_+}(N)}{mH_{R^i_+}(N)} - V(R_+))$ is a finite set of graded primes in R, none of which contains R_1. As $\frac{R_0}{m_0}$ is infinite, the set $C = R_1 - \bigcup_{P\in A} P$ is not empty [7, 1.5.12]. Let $x \in C$. We see that x is a generalized regular sequence with respect to N. Hence, gdepth$(R_+, \frac{N}{xN}) = s - 1$. By [17, 2.3], $\dim(0 :_N x) \leq 1$, then $H^i_{R_+}(0 :_N x) = 0$ for all $i > 1$. It follows $H^i_{R_+}(N) \cong H^i_{R_+}(\frac{N}{0:_N x})$ for all $i \geq 2$. Therefore, the exact sequences

$0 \longrightarrow (0 :_N x) \longrightarrow N \longrightarrow (\frac{N}{0:_N x}) \longrightarrow 0$ and $0 \longrightarrow \frac{N}{0:_N x} \longrightarrow N \longrightarrow (\frac{N}{xN}) \longrightarrow 0$,

can be used to obtain an exact sequence

$$H^{i-1}_{R_+}(\frac{N}{xN}) \longrightarrow H^i_{R_+}(N) \xrightarrow{x} H^i_{R_+}(N) \longrightarrow H^i_{R_+}(\frac{N}{xN}) \longrightarrow H^{i+1}_{R_+}(N)$$

for all $i \geq 2$. So, by lemma (3.2), the above exact sequence yields the exact sequence

$$0 \longrightarrow H^{s-1}_{R_+}(\frac{N}{xN})_{n+1} \longrightarrow H^{s}_{R_+}(N)_n \xrightarrow{x} H^{s}_{R_+}(N)_{n+1}$$

for $n \ll 0$. Hence

$$Ass_{R_0} H^{s-1}_{R_+}(\frac{N}{xN})_{n+1} \subseteq Ass_{R_0} H^{s}_{R_+}(N)_n \subseteq Ass_{R_0} H^{s-1}_{R_+}(\frac{N}{xN})_{n+1} \cup Ass_{R_0} H^{s}_{R_+}(N)_{n+1}.$$

Now, one can deduce that $Ass_{R_0} H^{s}_{R_+}(N)_n$ is asymptotically stable by induction on s, and $Ass_R H^{s}_{R_+}(N)$ is a finite set.

Definition 3.6. *We call N to be relative Cohen-Macalay with respect to $R_+$of dimension t if and only if $H^{i}_{R_+}(N) = 0$ for all $i \neq t$.*

Corollary 3.7. *Let N be relative Cohen-Macalay with respect to R_+ of dimension t. Then:*

(i) *$Ass_{R_0} H^{t}_{R_+}(N)_n$ is asymptotically stable.*

(ii) *$Ass_{R_0} H^{t}_{R_+}(M, N)_n$ is asymptotically stable.*

Lemma 3.8. *Let (R_0, m_0) be a local ring, T a graded finitely generated $R-$module and $\Gamma_{R_+}(T) = T$. Then $\frac{T}{m_0 T}$ is Artinian.*

Proof It is clear.

Proposition 3.9. *Let (R_0, m_0) be a local ring, $d = cd(R_+, N)$ and let q_0 be an m_0-primary ideal. Then $\frac{H^{d}_{R_+}(M,N)}{q_0 H^{d}_{R_+}(M,N)}$ is Artinian and $H^{d}_{R_+}(M, N)$ is tame.*

Proof As q_0 is an m_0-primary ideal, there is some $t \in \mathbb{N}$ such that $m_0^t \subsetneqq q_0$. So, it suffices to show that $\frac{R_0}{m_0^t} \bigotimes_R H^{i}_{R_+}(M, N)$ is Artinian. First we prove that the module $\frac{R_0}{m_0} \bigotimes_R H^{i}_{R_+}(M, N)$ is Artinian. In view of [4, 3.4] $d = \dim \frac{N}{m_0 N}$. We argue by induction on d. If $d = 0$, N is R_+-torsion, and so we have $\frac{R_0}{m_0} \bigotimes H^{0}_{R_+}(M, N) \cong \frac{R_0}{m_0} \bigotimes \mathrm{Hom}_R(M, \Gamma_{R_+}(N))$ is Artinian by lemma (3.8). Now suppose, inductively that

$d > 0$, and the result has been proved for $d-1$. Now we can use the exact sequence $0 \longrightarrow \Gamma_{R_+}(N) \longrightarrow N \longrightarrow \frac{N}{\Gamma_{R_+}(N)} \longrightarrow 0$, to deduce that $H^i_{R_+}(M,N) \bigotimes \frac{R_0}{m_0}$ is Artinian if and only if $H^i_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)}) \bigotimes \frac{R_0}{m_0}$ is Artinian. Therefore, since $dim\frac{N}{m_0N} = dim\frac{\frac{N}{\Gamma_{R_+}(N)}}{m_0\frac{N}{\Gamma_{R_+}(N)}} = d > 0$, we may assume that N is R_+-torsion free. Using prime avoidance theorem, we can get a homogeneous N-regular element sequence x of positive degree and $dim\frac{\frac{N}{xN}}{m_0\frac{N}{xN}} = d-1$. Now we consider the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$. Therefore, application of the functor $H^i_{R_+}(M,\cdots)$ to this yields a long exact sequence

$$H^{d-1}_{R_+}(M, \frac{N}{xN}) \xrightarrow{\psi} H^d_{R_+}(M,N) \xrightarrow{x=\varphi} H^d_{R_+}(M,N).$$

By the inductive hypothesis, $H^{d-1}_{R_+}(M, \frac{N}{xN}) \bigotimes \frac{R_0}{m_0}$ is Artinian, hence so is $im\psi \bigotimes \frac{R_0}{m_0}$ Artinian. We have then an exact sequence ,

$$\frac{im\psi}{m_0 im\psi} \longrightarrow \frac{H^d_{R_+}(M,N)}{m_0H^d_{R_+}(M,N)} \xrightarrow{x} \frac{im\varphi}{m_0\varphi} \longrightarrow 0,$$

which shows that the kernel of multiplication by x on $\frac{H^d_{R_+}(M,N)}{m_0H^d_{R_+}(M,N)}$ is an Artinian $R-$module. Since $\frac{H^d_{R_+}(M,N)}{m_0H^d_{R_+}(M,N)}$ is an (x)-torsion $R-$module. Now the result follows from [16,1.3] and Nakayama's Lemma. Using induction on t, we see that $\frac{H^d_{R_+}(M,N)}{q_0H^d_{R_+}(M,N)}$ is Artinian.

Definition 3.10. Define $q(M,N) = q = \sup\{i \mid H^i_{R_+}(M,N)$ is not Artinian $\}$. If $H^i_{R_+}(M,N)$ is Artinian for all i, we write $q(M,N) = -\infty$.

Theorem 3.11. *Let $p\,d(M) = l < \infty$ and P be m-primary ideal. Then* $\frac{H^q_{R_+}(M,N)}{PH^q_{R_+}(M,N)}$ *is an Artinian R-module.*

Proof As P is an m-primary ideal , there is some $t' \in \mathbb{N}$ such that $m^{t'} \subseteqq P$. So, it suffices to show that $\frac{R}{m^{t'}} \bigotimes_R H^q_{R_+}(M,N)$ is Artinian. First, we prove that the module $\frac{R}{m} \bigotimes_R H^i_{R_+}(M,N)$ is Artinian. It is straightforward to see that $H^i_{R_+}(M,N) = 0$ for all $i > l+d$ whenever $d = cd(R_+,N)$. So $q \leq l+d$. Set $l+d-q = t$. We prove the result by induction on t. When $t = 0$, there is nothing to prove, since $\frac{H^{l+d}_{R_+}(M,N)}{mH^{l+d}_{R_+}(M,N)}$ is Artinian R-module by [10,2.4]. So, suppose that $t > 0$ and that the result has been proved for smaller values of t. Since Supp $\frac{N}{\Gamma_{R_+}(N)} \subseteq \mathrm{Supp}(N)$, it follows from [8,2.6] that $q(M, \frac{N}{\Gamma_{R_+}(N)}) \leq q$. We first suppose that $q(M, \frac{N}{\Gamma_{R_+}(N)}) < q$. Then $H^q_{R_+}(M, \frac{N}{\Gamma_{R_+}(N)})$ is Artinian. Now, we can use the exact sequence $0 \longrightarrow \Gamma_{R_+}(N) \longrightarrow N \longrightarrow \frac{N}{\Gamma_{R_+}(N)} \longrightarrow 0$, in conjunction with the facts that $H^i_{R_+}(M, \Gamma_{R_+}(N) \cong \mathrm{Ext}^i_R(M, \Gamma_{R_+}(N))$ is finitely generated R-module, to deduce that $\frac{H^q_{R_+}(M,N)}{mH^q_{R_+}(M,N)}$ is Artinian. Now suppose that $q(M, \frac{N}{\Gamma_{R_+}(N)}) = q$. Since $dim\frac{\frac{N}{\Gamma_{R_+}(N)}}{m_0\frac{N}{\Gamma_{R_+}(N)}} = dim\frac{N}{m_0N} = d$ (cf [4,3.2]). Hence we may assume that N is R_+-torsion free and there exists $x \in R_+$ which is nonzero-divisor on N. The short exact sequence $0 \longrightarrow N \longrightarrow N \longrightarrow \frac{N}{xN} \longrightarrow 0$, yields a long exact sequence

$$0 \longrightarrow \frac{H^q_{R_+}(M,N)}{xH^q_{R_+}(M,N)} \longrightarrow H^q_{R_+}(M, \frac{N}{xN}) \xrightarrow{\varphi} H^{q+1}_{R_+}(M,N) \longrightarrow \cdots (\star).$$

We have $q(M,N) \geqslant q(M, \frac{N}{xN})$ and $dim\frac{\frac{N}{xN}}{m_0\frac{N}{xN}} = d-1$. If $q(M, \frac{N}{xN}) < q(M,N) = q$, then $H^q_{R_+}(M, \frac{N}{xN})$ is Artinian. By using $(\star)$ $\frac{H^q_{R_+}(M,N)}{xH^q_{R_+}(M,N)}$ is Artinian, and so $\frac{H^q_{R_+}(M,N)}{mH^q_{R_+}(M,N)}$. If $q(M, \frac{N}{xN}) = q(M,N) = q$, then $l + d - 1 - q < l + d - q = t$. By using the inductive hypothesis $H^q_{R_+}(M, \frac{N}{xN}) \bigotimes_R \frac{R}{m}$ is Artinian. Again we can use the exact sequence $(\star)$ to obtain

$$tor^R_1(im\varphi, \frac{R}{m}) \longrightarrow \frac{R}{m} \bigotimes_R \frac{H^q_{R_+}(M,N)}{xH^q_{R_+}(M,N)} \longrightarrow H^q_{R_+}(M, \frac{N}{xN}) \bigotimes_R \frac{R}{m} \longrightarrow \frac{R}{m} \bigotimes_R im\varphi \longrightarrow 0.$$

Note that, since $im\varphi$ is Artinian , it follows that $tor^R_i(im\varphi, \frac{R}{m})$ is Artinian for all $i \geq 0$. Thus, the above long exact sequence shows that $\frac{H^q_{R_+}(M,N)}{mH^q_{R_+}(M,N)}$ is Artinian.The proof is complete by induction on t.

Conclusion

In this paper, without any condition on R_0, the asymptotic behaviour of the homogeneous components of $H^i_{R_+}(M,N)$ for some i's with a specified property is studied for the following cases:

(i) $i \leq r$, where r denotes the least non-negative integer i such that $H^i_{R_+}(M, N)$ is not Artinian or $SuppH^i_{R_+}(R, N)$ is not finite set.

(ii) $i \geq q$, where q denotes the most integer i such that $H^i_{R_+}(M, N)$ is not Artinian.

(iii) $i = d$ is the cohomological dimension of N with respect to R_+.

So the merit of this study is to investigate the asymptotic behaviour without imposing any constraint on R_0.

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