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Approximate cyclic amenability of Banach algebras

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Abstract

In the current paper, we deal with generalized notion of amenability which is called approximate cyclic amenability. We introduce this concept and we show by means of an example, its distinction with its classic analogs. Moreover we show the relationship between approximate trace extension property and approximate cyclic amenability. This answers partially, question 9.1 of [3] for approximate cyclic amenability.

Keywords: Banach algebra, Cyclic amenability, Approximate cyclic amenability.

1 Introduction

The concept of amenability for Banach algebras was initiated by B. E. Johnson in [6]. Later, Gronbaek in [5] investigated properties of cyclic amenable Banach algebras. He also identified the relationship between hereditary properties of cyclic amenability and the trace extension property.

The concept of approximately amenable Banach algebras was introduced and studied for the first time by Ghahramani and Loy in [3]. In the mentioned paper, they characterized the structure of approximately amenable Banach algebras through several ways. At the beginning, authors asked which of the standard results on amenability

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work for the approximate concepts (page 233 of [3]), a question which identified the main direction of [3, 4] and the present paper. Motivated by this question, we introduce and study new concept of the approximate cyclic amenability. In question 9.1 of [3] the authors ask, What are the hereditary properties of the approximate concepts of amenability? We answer this question for approximate cyclic amenability by defining approximate trace extension property and constructing approximate analogs of certain results of Gronbaek in [5]. In the section 2 we discuss hereditary properties of approximate cyclic amenability and the relationship between approximate cyclic amenability of $\mathcal{A}$ and its unitization. Some of our arguments were inspired by their classic analogs mostly in [1, 2, 5].

Before proceeding further we recall some terminology. Throughout $\mathcal{A}$ is a Banach algebra and $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. Then $\mathcal{X}^*$ is a Banach $\mathcal{A}$-bimodule with module action given by

$$\langle a.\lambda, x\rangle = \langle \lambda, x.a\rangle \qquad \langle \lambda.a, x\rangle = \langle \lambda, a.x\rangle \qquad (a \in \mathcal{A}, \quad x \in \mathcal{X} \quad \lambda \in \mathcal{X}^*).$$

A linear mapping $D : \mathcal{A} \longrightarrow \mathcal{X}$ is a derivation if $D(ab) = a.Db + Da.b$ for $a, b \in \mathcal{A}$. For any $x \in \mathcal{X}$, the mapping $\delta_x : a \longmapsto ax - xa$, $a \in \mathcal{A}$, is a continuous derivation which is called an inner derivation. We will denote the set of all bounded [resp. inner] derivations from $\mathcal{A}$ into $\mathcal{X}$ by $Z^1(\mathcal{A}, \mathcal{X})$ [resp. $B^1(\mathcal{A}, \mathcal{X})$]. Also set $H^1(\mathcal{A}, \mathcal{X}) = Z^1(\mathcal{A}, \mathcal{X})/B^1(\mathcal{A}, \mathcal{X})$. A bounded derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is called *cyclic* if $\langle Da, b\rangle + \langle Db, a\rangle = 0$ for all $a, b \in \mathcal{A}$. The set of all cyclic derivations is denoted by $Z^1_{\lambda_0}(\mathcal{A}, \mathcal{A}^*)$ and $Z^1_{\lambda_0}(\mathcal{A}, \mathcal{A}^*)/B^1(\mathcal{A}, \mathcal{A}^*)$ by $H^1_{\lambda_0}(\mathcal{A}, \mathcal{A}^*)$. $\mathcal{A}$ is called *cyclic amenable* if $H^1_{\lambda_0}(\mathcal{A}, \mathcal{A}^*) = 0$.

A derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ is said to be approximately inner if there exists a net $(x_\alpha) \subseteq \mathcal{X}$ such that $D(a) = \lim_\alpha(a.x_\alpha - x_\alpha.a)$ for all $a \in \mathcal{A}$.

2 Main Results

Definition 2.1 *Let $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. We say that $\mathcal{I}$ has the approximate trace extension property if for each $\lambda \in \mathcal{I}^*$ with $a.\lambda = \lambda.a (a \in \mathcal{A})$ there is a net $(\Lambda_\alpha) \subseteq \mathcal{A}^*$ such that for each α, $\Lambda_\alpha|_I = \lambda$ and*

$$a.\Lambda_\alpha - \Lambda_\alpha.a \longrightarrow 0 \quad (a \in \mathcal{A}).$$

Definition 2.2 *Let $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. We say that a bounded approximate identity $\{e_\alpha\}$ of $\mathcal{I}$ is quasi central of $\mathcal{A}$ if $\lim_\alpha \|ae_\alpha - e_\alpha a\| = 0$ for all $a \in \mathcal{A}$.*

Definition 2.3 *$\mathcal{A}$ is called approximately cyclic amenable if every cyclic derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is approximately inner.*

The following example shows the distinction between cyclic amenability and approximate cyclic amenability of Banach algebras.

Example 2.4 *An approximately cyclic amenable Banach algebra which is not cyclic amenable.*

For each $n \in N$ as in [3, Example 6.2] equip M_{2^n} with the ℓ^2 norm and let $\mathcal{A}_n$ be its unitization. If $\mathcal{A} = c_0(\mathcal{A}_n)$ then as it was shown in [3], $\mathcal{A}$ is approximately amenable, then $\mathcal{A}$ is approximately cyclic amenable. Let

$$P_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and inductively define

$$P_{n+1} = \begin{pmatrix} 0 & -P_n \\ P_n & 0 \end{pmatrix}.$$

Then define a derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^* = \ell^1(A_n^*)$ by $D((x_n)) = (\frac{1}{n^2}\delta_{P_n}(x_n))$. Then using the identity $\delta_P(B) = PB^T - B^T P$, one can observe that D is cyclic. However as it was pointed out in [3, Example 6.2], D is not inner.

Theorem 2.5 *Let $\mathcal{I}$ be a closed ideal in $\mathcal{A}$.*

(i) Suppose that $\mathcal{A}/\mathcal{I}$ is approximately cyclic amenable. Then $\mathcal{I}$ has the approximate trace extension property.

(ii) Suppose that $\mathcal{A}$ is cyclic amenable and $\mathcal{I}$ has the approximate trace extension property. Then $\mathcal{A}/\mathcal{I}$ is approximately cyclic amenable.

(iii) If $\mathcal{A}/\mathcal{I}$ is approximately cyclic amenable, $\overline{\mathcal{I}^2} = \mathcal{I}$, and $\mathcal{I}$ is cyclic amenable, then $\mathcal{A}$ is approximately cyclic amenable.

(iv) Suppose that $\mathcal{A}$ is approximately cyclic amenable and $\mathcal{I}$ has a quasi-central bounded approximate identity for $\mathcal{A}$. Then $\mathcal{I}$ is approximately cyclic amenable.

Proof (i) Take $\Lambda \in \mathcal{A}^*$ with $\Lambda|_{\mathcal{I}} = \lambda$. Define

$$D : a+\mathcal{I} \longmapsto a.\Lambda - \Lambda.a, \quad \mathcal{A}/\mathcal{I} \longrightarrow \mathcal{I}^{\perp} = (\mathcal{A}/\mathcal{I})^*.$$

We see immediately that $D \in Z^1_{\lambda_0}(\mathcal{A}/\mathcal{I}, (\mathcal{A}/\mathcal{I})^*)$. Since $\mathcal{A}/\mathcal{I}$ approximately cyclic amenable, there exists a net $(\lambda_\alpha) \in \mathcal{I}^{\perp}$ such that

$$D(a+\mathcal{I}) = \lim_\alpha a.\lambda_\alpha - \lambda_\alpha.a \quad (a \in \mathcal{A}).$$

Set $\tau_\alpha = \Lambda - \lambda_\alpha \in \mathcal{A}^*$. Then, for each α, $\tau_\alpha|_{\mathcal{I}} = \lambda$ and

$$\lim_\alpha a.\tau_\alpha - \tau_\alpha.a = 0 \quad (a \in \mathcal{A}).$$

(ii) Suppose $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{I}$ is the quotient map and $D \in Z^1_{\lambda_0} \in (\mathcal{A}/\mathcal{I}, (\mathcal{A}/\mathcal{I})^*)$. Set $\widetilde{D} = \pi^* \circ D \circ \pi$. Then $\widetilde{D} \in Z^1_\lambda(\mathcal{A}, \mathcal{A}^*)$, and so there exists $\lambda \in \mathcal{A}^*$ with

$$\widetilde{D}a = a.\lambda - \lambda.a \quad (a \in \mathcal{A}).$$

Thus there exists a net $(\tau_\alpha) \subseteq \mathcal{A}^*$ such that for every α we have $\tau_\alpha|_{\mathcal{I}} = \lambda|_{\mathcal{I}}$ and

$$\lim_\alpha a.\tau_\alpha - \tau_\alpha.a = 0 \quad (a \in \mathcal{A}).$$

Then $\lambda - \tau_\alpha \in \mathcal{I}^{\perp}$ and

$$D(a+\mathcal{I}) = \lim_\alpha a.(\lambda - \tau_\alpha) - (\lambda - \tau_\alpha).a \quad (a \in \mathcal{A}).$$

It follow that $\mathcal{A}/\mathcal{I}$ is approximately cyclic amenable.

(iii) Assume that $\iota : I \longrightarrow \mathcal{A}$ is the natural embedding and $D \in Z^1_{\lambda_0}(\mathcal{A}, \mathcal{A}^*)$. Then $\iota^* \circ D \circ \iota \in Z^1_{\lambda_0}(\mathcal{I}, \mathcal{I}^*)$, and so, since $\mathcal{I}$ is cyclic amenable, there exists $\lambda_1 \in \mathcal{I}^*$ with

$$(\iota^* \circ D)(a) = \delta_{\lambda_1}(a) \quad (a \in \mathcal{I})$$

extend λ_1 to an element λ of $\mathcal{A}^*$. By replacing D by $D - \delta_\lambda$, we suppose that $(\iota^* \circ D)|_{\mathcal{I}} = 0$.

For $a, b \in \mathcal{I}$ and $c \in \mathcal{A}$, we have

$$\langle c, Dab\rangle = \langle ca, \iota \circ D(b)\rangle + \langle bc, \iota^*(a)\rangle = 0,$$

and so $D|_{\mathcal{I}^2} = 0$. By assumption , $\overline{\mathcal{I}^2} = \mathcal{I}$, and so $D|_I = 0$.

Set $F = \overline{\mathcal{I}\mathcal{A} + \mathcal{A}\mathcal{I}}$. Then, $F = \overline{\mathcal{I}^2} = \mathcal{I}$. For each $a \in \mathcal{A}$ and $b \in \mathcal{I}$, we have $a.Db = Dab = 0$, and thus $Da.b = 0$, take $c \in \mathcal{A}$. Then

$$\langle b.c, Da\rangle = \langle c, Da.b\rangle = 0.$$

Hence, $Da|_{\mathcal{I}\mathcal{A}} = 0$, similarly $Da|_{\mathcal{A}\mathcal{I}} = 0$, and so $Da|_{\mathcal{I}} = 0$. Thus $D(\mathcal{A}) \subseteq \mathcal{I}^\perp$ and the map

$$\widetilde{D} : a + \mathcal{I} \longmapsto Da, \quad \mathcal{A}/\mathcal{I} \longrightarrow \mathcal{I}^\perp$$

is a cyclic continuous derivation, by hypothesis, $\mathcal{A}/\mathcal{I}$ is approximately cyclic amenable, and so there exists a net $(\lambda_\alpha) \subseteq \mathcal{I}^\perp$ such that

$$Da = \lim_\alpha a.\lambda_\alpha - \lambda_\alpha.a \quad (a \in \mathcal{A}).$$

Therefore,

$$Da = \lim_\alpha a.(\lambda_\alpha + \lambda_1) - (\lambda_\alpha + \lambda_1).a \quad (a \in \mathcal{A}).$$

The above equality shows that $\mathcal{A}$ is approximately cyclic amenable.

(iv) By [5, Proposition 1.3] any bounded derivation $D : \mathcal{I} \longrightarrow \mathcal{I}^*$ can be lifted to a bounded derivation $\widetilde{D} : \mathcal{A} \longrightarrow \mathcal{A}^*$, from which, the result follows immediately.

Example 2.6 *One might ask whether the condition $\overline{\mathcal{I}^2} = \mathcal{I}$ is necessary in Theorem 2.5(iii). The following argument shows that this condition can not be removed.*

Let C be the set of complex numbers. Let $\mathcal{A} = C^2$ with zero product and $\mathcal{I} = C \oplus 0$. Then by [5, Example 2.5] $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are cyclic amenable but $\mathcal{A}$ is not cyclic amenable. Since for commutative Banach algebras, the two notions of cyclic amenability and approximate cyclic amenability coincide, then we see that $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are approximately cyclic amenable but $\mathcal{A}$ is not. Moreover $\overline{\mathcal{I}^2} \neq \mathcal{I}$.

Let $\mathcal{A}$ be a non-unital Banach algebra. Then $\mathcal{A}^{\#} = \mathcal{A} \oplus C$, the unitization of $\mathcal{A}$, is a unital Banach algebra which contains $\mathcal{A}$ as a closed ideal. We denote identity of $\mathcal{A}$ by $e = (0, 1)$.

Theorem 2.7 *Let $\mathcal{A}$ be a non-unital Banach algebra. $\mathcal{A}$ is approximately cyclic amenable if and only if $\mathcal{A}^{\#}$ is approximately cyclic amenable.*

Proof Suppose $\mathcal{A}$ is approximately cyclic amenable, $D : \mathcal{A}^{\#} \longrightarrow (\mathcal{A}^{\#})^*$ is a cyclic derivation. Then D(0,1)=0 and there exists a bounded linear map $\widetilde{D} : \mathcal{A} \longrightarrow \mathcal{A}^*$ and $\Lambda \in \mathcal{A}^*$ such that $Da = \langle \Lambda, a \rangle + \widetilde{D}a$. It is easy $\widetilde{D}$ is a bounded derivation. Since D is a cyclic, for every $a, b \in \mathcal{A}$, we have

$$0 = \langle Da, b \rangle + \langle Db, a \rangle = \langle \widetilde{D}a, b \rangle + \langle \widetilde{D}b, a \rangle. \tag{2.1}$$

So $\widetilde{D}$ is cyclic. On the other hand, for every α, β in C,

$$0 = \langle D(a + \alpha), b + \beta \rangle + \langle D(b + \beta), a + \alpha \rangle$$

$$= \langle \widetilde{D}a, b \rangle + \beta \langle \Lambda, a \rangle + \langle \widetilde{D}b, a \rangle + \alpha \langle \Lambda, b \rangle.$$

Now, if $\alpha = 0$, $\beta = 1$ and by (2.1), we have $\langle \Lambda, a \rangle = 0$, so $D = \widetilde{D}$, it follow that D approximately inner.

Conversely, let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a cyclic derivation, we define bounded linear map $\widetilde{D} : \mathcal{A}^{\#} \longrightarrow (\mathcal{A}^{\#})^*$ as follows

$$\widetilde{D}(a, \alpha) = (Da, 0), \qquad (a \in \mathcal{A}, \alpha \in C).$$

Then for every $a, b \in \mathcal{A}$ and $\alpha, \beta \in C$, we have

$$\widetilde{D}((a,\alpha)(b,\beta)) = \widetilde{D}(ab + \beta a + \alpha b, \alpha\beta)$$

$$= D(ab + \beta a + \alpha b) = aDb + Da.b + \beta Da + \alpha Db.$$

On the other hand, since D is cyclic,

$$(a,\alpha).\widetilde{D}(b,\beta) + \widetilde{D}(a,\alpha).(b,\beta) = (a,\alpha).(Db.0) + (Da,0).(b,\alpha)$$

$$= \langle a, Db\rangle + a.Db + \beta Db + \langle Da, b\rangle + \alpha Da + Da.b$$

$$= a.Db + Da.b + \beta Da + \alpha Db.$$

So $\widetilde{D}$ is a bounded derivation, and also $\mathcal{A}^{\#}$ is approximately cyclic amenable, then there exists net $(a_\gamma + \alpha_\gamma) \subseteq \mathcal{A}^* \oplus C$ such that

$$Da = \widetilde{D}(a+\alpha) = \lim_\gamma((a+\alpha)(a_\gamma + \alpha_\gamma) - (a_\gamma + \alpha_\gamma)(a+\alpha))$$

$$= \lim_\gamma aa_\gamma - a_\gamma a. \quad (a \in \mathcal{A}, \alpha \in C)$$

It follow that $\mathcal{A}$ is approximately cyclic amenable.

Remark 2.8 *In the Theorem 2.5 we observed that if we impose certain conditions to a closed ideal $\mathcal{I}$ of $\mathcal{A}$, then approximate cyclic amenability of $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ follows from that of $\mathcal{A}$. For subalgebras the situation is different, as the following theorem shows.*

Theorem 2.9 *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ closed subalgebra and $\mathcal{I}$ closed ideal are in $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{I}$. If $\mathcal{A}$ approximately cyclic amenable then $\mathcal{B}$ so is.*

Proof Suppose $P : \mathcal{A} \longrightarrow \mathcal{B}$ is a natural projection map and $D : \mathcal{B} \longrightarrow \mathcal{B}^*$ is a cyclic bounded derivation. So $P^*DP : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a cyclic derivation. Because $\mathcal{A}$ is approximately cyclic amenable, then there exists a net $(\Lambda_\alpha) \subseteq \mathcal{A}^*$ such that

$$P^*DP(a) = \lim_\alpha(a.\Lambda_\alpha - \Lambda_\alpha.a) \quad (a \in \mathcal{A}).$$

If $b \in \mathcal{B}$, we have $P^*D(b) = \lim_\alpha(b.\Lambda_\alpha - \Lambda_\alpha.b)$. Set $\lambda_\alpha = \Lambda_\alpha|_\mathcal{B} \in \mathcal{B}^*$, so $P^*D(b)|_\mathcal{B} = Db$ and $(\lim_\alpha(b.\Lambda_\alpha - \Lambda_\alpha.b))|_\mathcal{B} = \lim_\alpha b.\lambda_\alpha - \lambda_\alpha.b$, then

$$Db = \lim_\alpha(b.\lambda_\alpha - \lambda_\alpha.b) \quad (b \in B).$$

It follow that $\mathcal{B}$ approximately cyclic amenable.

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