



Homotopy analysis and Homotopy Padé methods for the modified Burgers-Korteweg-de Vries and the Newell-Whitehead equations

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Abstract

In this paper, analytic solutions of the modified Burgers-Korteweg-de Vries equation (mBKdVE) and the Newell-Whitehead equation are obtained by the Homotopy analysis method (HAM) and the Homotopy Padé method (HPadéM). The obtained approximation by using HAM contains an auxiliary parameter which is a way to control and adjust the convergence region and rate of the solution series. The approximation solution by $[m, m]$ HPadéM is often independent of auxiliary parameter \bar{h} and this technique accelerates the convergence of the related series.

Keywords: Homotopy analysis method, Homotopy Padé method, The modified Burgers-Korteweg-de Vries equation, The Newell-Whitehead equation.

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1 Introduction

Several methods have been suggested to solve nonlinear equations. These methods include the Homotopy perturbation method [1], Luapanov's artificial small parameter method, δ -expansion method, the tanh-coth method [2], $(\frac{G'}{G})$ -expansion method [3], Adomian decomposition method and variational iterative method and so on [4, 5].

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The HAM, first proposed by Liao in his Ph.D Dissertation [6], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [7, 8]. Liao in his book [9] proved that HAM is a generalization of some previously used techniques such as the δ -expansion method, artificial small parameter method [10] and Adomian decomposition method. Moreover, unlike previous analytic techniques, the HAM provides a convenient way to adjust and control the region and rate of convergence [11]. There exist some techniques to accelerate the convergence of a given series. Among them, the so-called Padé method is widely applied [7, 9].

The mBKdVE is presented as

$$u_t + p u^2 u_x + 6r u_{xx} - r u_{xxx} = 0, \quad (1)$$

where p and r are constants. Many physical problems can be described by Burgers-kdv equation (BKdVE) and mBKdVE. Typical examples are provided by the behavior of long waves in shallow water and waves in plasmas. In [12] traveling wave solution is obtained for (1).

The general nonlinear parabolic equation is of the form

$$u_t = u_{xx} + au + bu^n, \quad (2)$$

where a, b are real constants. Eq. (2) gives rise to three well-known models. For $a = -4, b = 4$, and $n = 3$ Eq. (2) becomes the Allen-Cahn equation [13]. If for $n = 3$ the coefficient b is replaced by $-b$, then Eq. (2) becomes the Newell-Whitehead equation. The Newell-Whitehead equation describes the dynamical behavior near the bifurcation point for the Rayleigh-Benard convection of binary fluid mixtures [14]. However, for $n = 2$ and $b = -a$, Eq. (2) reduces to the well-known Fishers equation [8]. Because Eq. (2) represents at least three of the well-known parabolic equation, it will be named as the general parabolic equation. Eq. (2) arise in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. It is well-known that wave phenomena of plasma media and fluid dynamics are modelled by kink shaped

tanh solution or by bell shaped sech solutions. Several different approaches, such as Backlund transformation, a bilinear form [15], inverse scattering method [16], the tanh-coth method [2], Jacobi elliptic functions, and a Lax pair [17] have been used to solve (2)

In this paper we apply HAM [18]-[22] and HPadéM [23]-[25] for solving of the Newell-Whitehead equation and the mBKdVE. These methods have good results for solving many types of linear and nonlinear equations.

2 Homotopy analysis method

In this section we describe the main points of HAM method. For more details of the HAM, the reader is referred to Liao's book [9]. Consider the following equation

$$N[u(x, t)] = 0, \quad (3)$$

where N is a nonlinear operator and x and t are spatial and temporal independent variables and $u(x, t)$ is unknown function. By means of generalizing the traditional Homotopy method, the zero order deformation equation is constructed as

$$(1 - q)L[\phi(x, t, q) - u_0(x, t)] = q\bar{h}H(x, t)N[\phi(x, t, q)], \quad (4)$$

where L is a linear operator, $q \in [0, 1]$ is the embedding parameter, \bar{h} is a nonzero auxiliary parameter, $H(x, t) \neq 0$ is an auxiliary function and $u_0(x, t)$ is an initial guess of $u(x, t)$. When $q = 0$, the zero order deformation equation (4) becomes

$$L[\phi(x, t, 0) - u_0(x, t)] = 0, \quad (5)$$

so

$$\phi(x, t, 0) = u_0(x, t), \quad (6)$$

and when $q = 1$ from (4) we have

$$\bar{h}H(x, t)N[\phi(x, t, 1)] = 0. \quad (7)$$

Since $\bar{h} \neq 0$ and $H(x, t) \neq 0$, we have

$$N[\phi(x, t, 1)] = 0, \quad (8)$$

equivalently $\phi(x, t, 1)$ is the solution of (3). Thus as the embedding parameter q increases from 0 to 1, the solution $\phi(x, t, q)$ of (4) varies continuously from the initial approximation $u_0(x, t)$ to the exact solution $u(x, t)$. It is important that we have the great freedom to choose the initial guess $u_0(x, t)$, the auxiliary linear operator L and the nonzero auxiliary parameter \bar{h} . Expanding $\phi(x, t, q)$ in Taylor series with respect to q , we have

$$\phi(x, t, q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m, \quad (9)$$

where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t, q)}{\partial q^m} \Big|_{q=0}. \quad (10)$$

If the initial approximation $u_0(x, t)$, the auxiliary parameter \bar{h} , the linear operator L and the auxiliary function $H(x, t)$ are properly chosen, it is proved that the power series (9) converges at $q = 1$ and we have

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t), \quad (11)$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [9].

The governing equation of $u_m(x, t)$ can be derived from the zero order deformation equation (4). To this end, define the vector

$$\vec{u}_n = \{u_0(x, t), u_1(x, t), \dots, u_n(x, t)\}.$$

Differentiating the zero order deformation equation (4) m times with respect to q and then setting $q = 0$ and finally dividing by $m!$, we have the so-called m -th order deformation equation

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \bar{h} H(x, t) R_m(\vec{u}_{m-1}), \quad (12)$$

where

$$R_m(\vec{u}_{m-1}(x, t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x, t, q)]}{\partial q^{m-1}} \Big|_{q=0}, \quad (13)$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases}$$

By setting $\bar{h} = -1$ and $H(x, t) = 1$, then (4) becomes

$$(1-q)L[\phi(x, t, q) - u_0(x, t)] + qN[\phi(x, t, q)] = 0,$$

which is used in the homotopy perturbation method [1], where highlighting the fact that homotopy perturbation method is a special case of HAM.

3 Convergence of method

We can prove that, as long as the solution series (11) given by the homotopy analysis method is convergent, it must be the solution of the nonlinear problem (3).

Theorem 3.1 *As long as the series*

$$u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t), \quad (14)$$

is convergent, where $u_m(x, t)$ is governed by the high-order deformation (12) under definition (13), it must be a solution of equation (3).

Proof Let

$$s(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t),$$

denote the convergent series. Using (12), we have

$$\bar{h}H(x, t) \sum_{m=1}^{\infty} R_m(\vec{u}_{m-1})$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty} L [u_m(x, t) - \chi_m u_{m-1}(x, t)] \\
&= L \left[\sum_{m=1}^{\infty} u_m(x, t) - \sum_{m=2}^{\infty} u_{m-1}(x, t) \right] \\
&= L \left[\sum_{m=1}^{\infty} u_m(x, t) - \sum_{m=1}^{\infty} u_m(x, t) \right] \\
&= 0
\end{aligned}$$

which gives, since $\hbar \neq 0$, $H(x, t) \neq 0$

$$\sum_{m=1}^{\infty} R_m(\vec{u}_{m-1}) = 0 \quad (15)$$

On the other side, according to the definition (13) and from (15), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} R_m(\vec{u}_{m-1}) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m N[\phi(x, t, q)]}{\partial q^m} \Big|_{q=0} = 0.
\end{aligned} \quad (16)$$

In general, $\phi(x, t, q)$ does not satisfy the original nonlinear equation (3). Let

$$\varepsilon(x, t, q) = N[\phi(x, t, q)]$$

denote the residual of equation (3). Clearly,

$$\varepsilon(x, t, q) = 0$$

corresponds to the exact solution of the original equation (3). According to the above definition, the Maclaurin series of the residual error $\varepsilon(x, t, q)$ about the embedding parameter q is

$$\sum_{m=0}^{\infty} \frac{q^m}{m!} \frac{\partial^m \varepsilon(x, t, q)}{\partial q^m} \Big|_{q=0} = \sum_{m=0}^{\infty} \frac{q^m}{m!} \frac{\partial^m N[\phi(x, t, q)]}{\partial q^m} \Big|_{q=0}.$$

When $q = 1$, the above expression gives, using (16),

$$\varepsilon(x, t, 1) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m \varepsilon(x, t, q)}{\partial q^m} \Big|_{q=0} = 0.$$

This means, according to the definition of $\varepsilon(x, t, q)$, that we again the exact solution of the original equation (3) when $q = 1$. Thus, as long as the series (3) is convergent, it must be one solution of the original equation (3). This ends the proof.

4 Homotopy Padé method

Traditionally the $[m, n]$ Padé for $u(x, t)$ is in the form

$$\frac{\sum_{k=0}^m F_k(x)t^k}{1 + \sum_{k=1}^n F_{m+1+k}(x)t^k},$$

or

$$\frac{\sum_{k=0}^m G_k(t)x^k}{1 + \sum_{k=1}^n G_{k+m+1}(t)x^k},$$

where $F_k(x)$ and $G_k(t)$ are functions.

In Homotopy Padé approximation, we employ the traditional Padé technique to the series (9) for the embedding parameter q to gain the $[m, n]$ Padé approximation in the form of

$$\frac{\sum_{k=0}^m w_k(x, t)q^k}{1 + \sum_{k=1}^n w_{m+k+1}(x, t)q^k}, \quad (17)$$

where $w_k(x, t)$ is a function and for $i = 0, 1, \dots, m, m+2, \dots, m+n+1$, $w_i(x, t)$ is determined by product of the denominator of the above expression in the $\sum_{i=0}^{m+n} u_i(x, t)q^i$ and equating the powers of q^i , $i = 0, 1, \dots, m+n$. Thus we have $m+n+1$ equations and $m+n+1$ unknowns $w_i(x, t)$, $i = 0, 1, \dots, m, m+2, \dots, m+n+1$. By setting $q = 1$ in (17) the so-called $[m, n]$ Homotopy Padé approximation in the following form is yield

$$\frac{\sum_{k=0}^m w_k(x, t)}{1 + \sum_{k=1}^n w_{m+k+1}(x, t)}. \quad (18)$$

It is found that the $[m, n]$ homotopy Padé approximation often converges faster than the corresponding traditional $[m, n]$ Padé approximation and in many cases the $[m, m]$ Homotopy Padé approximation is independent of the auxiliary parameter \bar{h} [9].

5 Numerical Results

In this section we apply HAM and HPadéM to solve the mBKdVE and Newell-Whitehead equation by 10-th order homotopy approximation and $[5,5]$ Padé approximation. In all cases, we assume that the initial guess to be $u_0(x, t) = u(x, 0)$, i.e.

the initial condition, and use the auxiliary linear operator $L = \frac{\partial}{\partial t}$ and the auxiliary function $H(x, t) = 1$. We give approximation of error terms to show the efficiency of HAM and HPadéM.

5.1 The modified Burgers-Korteweg-de Vries equation

Let us, consider mBKdVE (1) with initial condition

$$u(x, 0) = \sqrt{\frac{6r}{p}}(1 + \tanh(x)).$$

The exact solution of this problem is

$$u(x, t) = \sqrt{\frac{6r}{p}}(1 + \tanh(x - 8rt)).$$

Employing HAM with mentioned parameters in section 2, we have the following zero-order deformation equation

$$(1-q)\{\phi_t(x, t, q) - u_{0t}\} = \hbar q\{\phi_t(x, t, q) + p\phi^2(x, t, q)\phi_x(x, t, q) + 6r\phi_{xx}(x, t, q) - r\phi_{xxx}(x, t, q)\}.$$

Subsequently solving the m -th order deformation equation with $p = 2$ and $r = 1.5$ one has

$$\begin{aligned} u_0(x, t) &= 2.1213 + 2.1213 \tanh(x), \\ u_1(x, t) &= \frac{-2 \times 10^{-38} \hbar t}{\cosh^4(x)} \{2.4559 \times 10^{19} \sinh(x) \cosh(x) \\ &\quad + 7.7202 \times 10^{18} - 1.2727 \times 10^{39} \cosh(x)\}^2, \\ &\vdots \end{aligned}$$

We use an 10-term approximation and set

$$app9 := u_0 + u_1 + \dots + u_9.$$

The influence of \hbar on the convergence of the solution series are given in Figure 1. This figure shows \hbar curve of $u(-7, 1)$. It is easy to see that in order to have a good

approximation for $u(-7,1)$, \bar{h} has to be chosen in $-1 < \bar{h} < 0.5$. In Tables 1-3 the absolute error of approximation results are given by 10-th order HAM and [5,5] HPadéM. These tables show that by growing x and t the absolute errors are increasing. The results obtained with $\bar{h} = -0.5$ are better than $\bar{h} = -1.0$. Hence, the outputs of HAM are better than the Homotopy perturbation method. Table 3 shows that the HPadéM accelerates the rate of convergence of solutions. Of course, the accuracy can be improved by computing more terms of the approximate solution. Table 4 exhibits the approximate solutions for Eq. (1) obtained for different values of n using HAM. These results numerically confirm convergence of obtained solutions whit HAM that proved in Theorem 3.1.

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Table 1: Absolute errors of app9 for mBKdVE for $\hbar = -1.0$ using HAM

x	$t = 0.1$	$t = 0.5$	$t = 1$
-15	7.7762×10^{-14}	5.7034×10^{-10}	2.1664×10^{-8}
-12	3.1371×10^{-11}	2.3009×10^{-7}	8.7402×10^{-6}
-10	1.7128×10^{-9}	1.2562×10^{-5}	4.7720×10^{-4}
-7	6.9097×10^{-7}	5.0679×10^{-3}	1.9251×10^{-1}

Table 2: Absolute errors of app9 for mBKdVE for $\hbar = -0.5$ using HAM

x	$t = 0.1$	$t = 0.5$	$t = 1$
-15	4.2662×10^{-15}	4.2678×10^{-13}	1.6747×10^{-10}
-12	1.7211×10^{-12}	1.7217×10^{-10}	6.7565×10^{-8}
-10	9.3971×10^{-11}	9.4005×10^{-9}	3.6889×10^{-6}
-7	3.7910×10^{-8}	3.7936×10^{-6}	1.4881×10^{-3}

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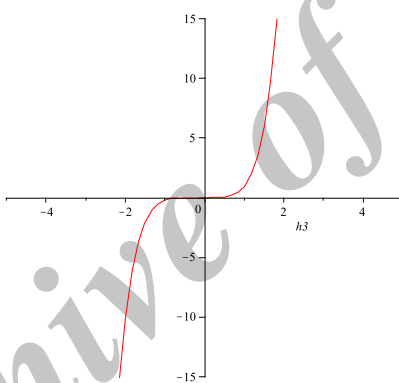


Figure 1: The \bar{h} curve for $u(-7,1)$ obtained from the 10-th order HAM approximation solution of the mBKdVE.

Table 3: Absolute errors for mBKdVE using [5,5] HPadéM

x	$t = 0.1$	$t = 0.5$	$t = 1$
-15	8.5836×10^{-20}	3.4443×10^{-15}	3.3721×10^{-14}
-12	2.5092×10^{-17}	1.3895×10^{-12}	1.3604×10^{-11}
-10	1.3726×10^{-15}	5.5373×10^{-13}	7.4277×10^{-10}
-7	5.5373×10^{-13}	3.0606×10^{-8}	2.9965×10^{-7}

Table 4: Comparing results for different values of n for mBKdVE for $\hbar = -0.5$ using HAM

(x, t)	$n = 5$	$n = 9$	$n = 13$
$(-15, 0.1)$	1.6765×10^{-14}	4.2662×10^{-15}	3.9557×10^{-16}
$(-12, 0.1)$	6.7635×10^{-12}	1.7211×10^{-12}	1.5958×10^{-13}
$(-10, 0.1)$	3.6927×10^{-10}	9.3971×10^{-11}	8.7129×10^{-12}
$(-07, 0.1)$	1.4897×10^{-7}	3.7910×10^{-8}	3.5151×10^{-9}

5.2 The Newell-Whitehead equation

In this section we apply HAM and HPadéM for Newell-Whitehead equation (2) with $a = 4$, $b = 4$, and $n = 3$, with initial condition

$$u(x, 0) = \frac{(\cosh(\frac{1}{2}\sqrt{2}x) + \sinh(\frac{1}{2}\sqrt{2}x))^2}{4 \cosh^2(\frac{1}{2}\sqrt{2}x)}.$$

The exact solution of this equation is

$$u(x, t) = \frac{(\cosh(\frac{1}{2}\sqrt{2}(x - 3\sqrt{2}t)) + \sinh(\frac{1}{2}\sqrt{2}(x - 3\sqrt{2}t)))^2}{4 \cosh^2(\frac{1}{2}\sqrt{2}(x - 3\sqrt{2}t))}.$$

Employing HAM with $L = \frac{\partial}{\partial t}$, $H(x, t) = 1$ and $u_0 = u(x, 0)$, we have the following zero-order deformation equation

$$(1 - q)\{\phi_t(x, t, q) - u_{0t}\} = \hbar q\{\phi_t(x, t, q) - \phi_{xx}(x, t, q) - 4\phi(x, t, q) + 4\phi^3(x, t, q)\}.$$

Subsequently solving the $m - th$ order deformation equation, one has

$$\begin{aligned} u_0(x, t) &= \frac{(\cosh(\frac{1}{2}\sqrt{2}x) + \sinh(\frac{1}{2}\sqrt{2}x))^2}{4 \cosh^2(\frac{1}{2}\sqrt{2}x)}, \\ u_1(x, t) &= \frac{(\cosh(\frac{1}{2}\sqrt{2}x) + \sinh(\frac{1}{2}\sqrt{2}x))^2}{16 \cosh^6(\frac{1}{2}\sqrt{2}x)} \hbar t \{-24 \cosh^4(\frac{1}{2}\sqrt{2}x) \\ &\quad + 24 \sinh(\frac{1}{2}\sqrt{2}x) \cosh^3(\frac{1}{2}\sqrt{2}x) + 4 \cosh^2(\frac{1}{2}\sqrt{2}x) \\ &\quad - 4 \cosh(\frac{1}{2}\sqrt{2}x) \sinh(\frac{1}{2}\sqrt{2}x) + 1\}, \\ &\vdots \end{aligned}$$

We use an 10-term approximation and set

$$app9 := u_0 + u_1 + \dots + u_9.$$

Absolute errors for two different value of \bar{h} and Homotopy Padé approximation are reported in Tables 5-7. These tables show that by growing x and t the absolute errors are increasing. Again the results in Table 7 highlights this fact that the HPadéM is vast superior in comparison with HAM. To investigate the suitable \bar{h} for the convergence of the solution series, we should plot the so called \bar{h} curve of $u(x, t)$ for various values of x and t . Figure 2 shows \bar{h} curve of $u(-10, 10)$. However it is observed that the series of $u(x, t)$ is convergent when $-2 < \bar{h} < 1.8$.

Table 5: Absolute errors of app9 for Newell-Whitehead equation for $\hbar = -1.0$ using HAM

x	$t = 5$	$t = 10$	$t = 20$
-30	9.8577×10^{-31}	3.0210×10^{-29}	9.4642×10^{-28}
-25	1.3785×10^{-24}	4.2248×10^{-23}	1.3235×10^{-21}
-20	1.9110×10^{-18}	5.8565×10^{-17}	1.8347×10^{-15}
-15	2.6492×10^{-12}	8.1188×10^{-11}	2.5434×10^{-9}
-10	3.6724×10^{-6}	1.1254×10^{-4}	3.5258×10^{-3}

Table 6: Absolute errors of app9 for Newell-Whitehead equation for $\hbar = -0.5$ using HAM

x	$t = 5$	$t = 10$	$t = 20$
-30	4.5151×10^{-32}	1.1522×10^{-30}	3.2746×10^{-29}
-25	6.3143×10^{-26}	1.6114×10^{-24}	4.5795×10^{-23}
-20	8.7530×10^{-20}	2.2338×10^{-18}	6.3482×10^{-17}
-15	1.2134×10^{-13}	3.0966×10^{-12}	8.8004×10^{-11}
-10	1.6821×10^{-7}	4.2927×10^{-6}	1.2199×10^{-4}

Table 7: Absolute errors for Newell-Whitehead equation using [5,5] HPadéM

x	$t = 5$	$t = 10$	$t = 20$
-30	3.7882×10^{-37}	2.3021×10^{-37}	1.7933×10^{-37}
-25	5.2977×10^{-31}	3.2194×10^{-31}	2.5079×10^{-31}
-20	7.3438×10^{-25}	4.4629×10^{-25}	3.4765×10^{-25}
-15	1.0180×10^{-18}	6.1868×10^{-18}	4.8195×10^{-19}
-10	1.4113×10^{-12}	8.5767×10^{-13}	6.6812×10^{-13}

6 Conclusion

HAM provides us with a convenient way to control the convergence of approximation series by adapting \bar{h} , which is a fundamental qualitative difference in analysis between HAM and other methods.

In this work, the HAM and HPadéM were applied to obtain the analytic solution of Newell-Whitehead equation and mBKdVE. We studied the efficiency of HAM and HPadéM in solving Newell-Whitehead equation and mBKdVE. It was illustrated that the HPadéM accurates the convergence of the related series.

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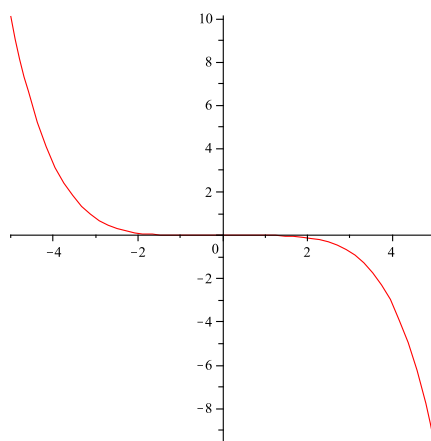


Figure 2: The \bar{h} curve for $u(-10,10)$ obtained from the 10-th order HAM approximation solution of the Newell-Whitehead equation.

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