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Exact solutions for some of the fractional differential equations by using modification of He's variational iteration method

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Abstract

In this paper, the modification of He's variational iteration method (MVIM) is developed to solve fractional ordinary differential equations and fractional partial differential equations. It is used the free choice of initial approximation to propose the reliable modification of He's variational iteration method. Some of the fractional differential equations are examined to illustrate the effectiveness and convenience of the method. The results show that the proposed method has accelerated convergence.

Keywords: Fractional ordinary differential equations, fractional partial differential equations, modification of He's variational iteration method .

1 Introduction

The variational iteration method was first proposed by [9, 8, 10]. This method has been shown to effectively, easily and accurately solve a large class of nonlinear problems. Generally, one or two iterations lead to highly accurate solutions.This method is, in fact, a modification of the general Lagrange multiplier method into an iteration

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method. Applications of the method have been enlarged due to its flexibility, convenience and efficiency. The VIM was first applied to autonomous ordinary differential systems, delay differential equations, and fractional differential equations by He et al [9, 10, 18]. The convergence of the method is systematically discussed by Tatari and Dehghan [16], Odibat [15]. In this paper we propose the reliable modification of He's VIM [6] for the fractional ordinary differential equations and fractional partial differential equations by constructing an initial trial-function without unknown parameters so that one iteration leads to exact solution.

2 Basic definitions

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1: A real function $f(x),\ x > 0$, is said to be in the space $C_\mu,\ \mu \in R$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, 1)$. Clearly $C_\mu \subset C_\beta$ if $\beta \le \mu$.

Definition 2.2: A function $f(x),\ x > 0$, is said to be in the space $C_\mu^m,\ m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.

Definition 2.3: The left sided Riemann –Liouville fractional integral operator of order $\alpha \ge 0$ of a function $f \in C_\mu,\ \ \mu \ge -1$, is defined as [7]

$$\begin{aligned} &J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, \qquad x > 0, \\ &J^0 f(x) = f(x). \end{aligned} \tag{1}$$

Definition 2.4: Let $f \in C_{-1}^m,\ \ m \in N \cup \{0\}$, then the Caputo fractional derivative of $f(x)$ is defined as [13, 3]

$$D^\alpha f(x) = \begin{cases} J^{m-\alpha} f^{(m)}(x), & m-1 < \alpha < m,\ m \in N, \\ \frac{D^m f(x)}{Dx^m}, & \alpha = m. \end{cases} \tag{2}$$

Hence, we have the following properties [7, 1, 13, 3]:

$$\begin{aligned}
&1.\ J^{\alpha}J^{\nu}f = J^{\alpha+\nu}f, \quad \alpha,\nu > 0,\ f \in C\mu,\ \mu > 0. \\
&2.\ J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}, \quad \alpha > 0, \gamma > -1,\ x > 0. \\
&3.\ J^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^{+})\frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \le m.
\end{aligned} \tag{3}$$

Lemma 2.5: If $0 < \alpha < 1$ then

$$J^{\alpha}D^{\alpha}f(x) = f(x) - f(0). \tag{4}$$

Proof: Let $0 < \alpha < 1$. From (1)–(2), we have $m = 1$ and

$$J^{\alpha}D^{\alpha}f(x) = J^{1}f'(x) = \frac{1}{\Gamma(1)}\int_0^x f'(t)dt = f(x) - f(0). \tag{5}$$

The Caputo fractional derivative [7] is considered here, because it allows traditional initial and boundary conditions to be included in the formulation of the problem. For more information on the mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

3 Analysis of the modified variational method

In this section, we first present a brief review of He's variational iteration method. Then we will propose the reliable modification of the VIM [9] for solving fractional differential equations by constructing an initial trial-function without unknown parameters. Here, we consider the following fractional functional equation

$$Lu + Ru + Nu = g(x), \tag{6}$$

where L is the fractional order derivative, R is a linear differential operator, N represents the nonlinear terms, and g is the source term. By using (3) and applying the inverse operator J^{α} to both sides of (6), and using the given conditions, we obtain

$$u = f - J^{\alpha}[Ru] - J^{\alpha}[Nu] + \sum_{k=0}^{m-1} u^{(k)}(0^{+})\frac{x^k}{k!}, \tag{7}$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions, all are assumed to be prescribed.

The basic character of He's method is the construction of a correction functional for (6), which reads

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[Lu_n(s) + R\tilde{u}_n(s) + N\tilde{u}_n(s) - g(s)]ds, \tag{8}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory [9], u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

To solve (6) by He's VIM, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. Then the successive approximations $u_n(x)$, $n \geq 0$, of the solution $u(x)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions. With determined λ, then several approximations $u_n(x)$, $n \geq 0$, follow immediately. Consequently, the exact solution may be obtained by using

$$\lim_{n\to\infty} u_n(x) = u(x). \tag{9}$$

In summary, we have the following variational iteration formula for (7)

$$\begin{cases} u_0(x) \text{ is an arbitrary initial guess,} \\ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[Lu_n(s) + Ru_n(s) + Nu_n(s) - g(s)]ds, \end{cases} \tag{10}$$

or equivalently, for (7), according to [17]:

$$\begin{cases} u_0(x) \text{ is an arbitrary initial guess,} \\ u_{n+1}(x) = f(x) - L_x^{-1}[Ru_n(x)] - L_x^{-1}[Nu_n(x)], \end{cases} \tag{11}$$

where the multiplier Lagrange λ, has been identified.

It is important to note that He's VIM suggests that the u_0 usually defined by a suitable trial-function with some unknown parameters or any other function that satisfies at least the initial and boundary conditions. This assumption made by He [11, 12] and others will be slightly varied, as will be seen in the discussion.

The MVIM, that was introduced by Ghorbani et al [6], can be established based on the assumption that the function $f(x)$ of the iterative relation (11) can be divided into two parts, namely $f_0(x)$ and $f_1(x)$. Under this assumption, we set

$$f(x) = f_0(x) + f_1(x). \tag{12}$$

According to the assumption, (12), and by the relationship (11), we construct the following variational iteration formula

$$\begin{cases} u_0(x) = f_0(x) \\ u_1(x) = f(x) - J^{\alpha}[Rf_0(x)] - J^{\alpha}[Nf_0(x)] + \sum_{k=0}^{m-1} f_0^{(k)}(0^+)\frac{x^k}{k!}, \\ u_{n+1}(x) = f(x) - J^{\alpha}[Ru_n(x)] - J^{\alpha}[Nu_n(x)] + \sum_{k=0}^{m-1} u_n^{(k)}(0^+)\frac{x^k}{k!}, \end{cases} \tag{13}$$

where the multiplier Lagrange, λ, has been identified. An important observation that can be made here is that the success of the proposed method depends mainly on the proper choice of the functions f_0 and f_1. As will be seen from the examples below, the selection of u_0 will result in a reduction of the computational work and accelerate the convergence. Furthermore, this proper selection of the components u_0 and u_1 may provide the solution by using one iteration only. To give a clear overview of the content of this study, we have chosen several fractional differential equations with the various boundary conditions.

4 Some examples

In this section, to demonstrate the effectiveness of the modification, we give chosen several the fractional ordinary differential and fractional partial differential equations.

Example 4.1 Consider the following linear fractional differential equation, [5]

$$
\begin{aligned}
&D^{\frac{1}{2}}y(x) = x^2 - y(x) + \frac{2}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}},\\
&y(0) = 0. \qquad (14)
\end{aligned}
$$

By assuming $L = D^{\frac{1}{2}}$ and applying the inverse operator L_x^{-1} to both sides of (14), we have $f_0(x) = x^2$ and $f_1(x) = \frac{\Gamma(3)}{\Gamma(\frac{7}{2})}x^{\frac{5}{2}}$. By selecting $y_0(x) = f_0(x) = x^2$, we have

$$
\begin{aligned}
&y_0(x) = x^2\\
&y_1(x) = x^2 + \frac{\Gamma(3)}{\Gamma(\frac{7}{2})}x^{\frac{5}{2}} - J^{\frac{1}{2}}x^2 = x^2\\
&y_{n+1}(x) = x^2, \quad n \geq 1. \qquad (15)
\end{aligned}
$$

In view of (15), it follows that

$$
y(x) = x^2, \qquad (16)
$$

where is the exact solution of (14). It is noteworthy to conclude here that the exact solution was determined by using only one iteration.

Example 4.2 Consider the following fractional partial differential equation, [2],

$$
\begin{aligned}
&\frac{\partial y(x,t)}{\partial t} = d(x)\frac{\partial^{1.8}y(x,t)}{\partial x^{1.8}} + g(x,t),\\
&y(x,0) = x^3, \quad 0 < x < 1,\\
&y(0,t) = 0, \quad y(1,t) = e^{-t}, \quad t > 0, \qquad (17)
\end{aligned}
$$

on a finite domain $0 < x < 1$, and

$$
d(x) = \frac{\Gamma(2.2)}{6}x^{2.8}, \quad g(x,t) = -(1+x)e^{-t}x^3. \qquad (18)
$$

We rewrite (17) in the form

$$
y_x^{(\frac{9}{5})} = \frac{6y_t}{\Gamma(\frac{11}{5})x^{\frac{14}{5}}} + \frac{6}{\Gamma(\frac{11}{5})}e^{-t}(1+x)x^{\frac{1}{5}}. \qquad (19)
$$

By using the modified technique, we obtain $f_0(x,t) = x^3e^{-t}$ and $f_1(x,t) = \frac{3\Gamma(\frac{6}{5})}{\Gamma(\frac{11}{5})}e^{-t}x^2$, and by applying boundary conditions, we select $y_0(x,t) = f_0(x,t) = x^3e^{-t}$ to find

$$
y_0(x,t) = x^3e^{-t},
$$

$$y_1(x,t) = x^3e^{-t} + \frac{3\Gamma(\frac{6}{5})}{\Gamma(\frac{11}{5})}e^{-t}x^2 + \frac{6}{\Gamma(\frac{11}{5})}J^{\frac{9}{5}}[\frac{6}{\Gamma(\frac{11}{5})x^{\frac{14}{5}}}(-x^3e^{-t})]$$
$$+y_0(0,t) + y_{0_x}(0,t)x = x^3e^{-t} + \frac{3\Gamma(\frac{6}{5})}{\Gamma(\frac{11}{5})}e^{-t}x^2 - \frac{3\Gamma(\frac{6}{5})}{\Gamma(\frac{11}{5})}e^{-t}x^2 + 0 + 0 = x^3e^{-t},$$
$$y_{n+1}(x,t) = x^3e^{-t}, \quad n \geq 1. \tag{20}$$

In view of (20), it follows that

$$y(x,t) = x^3e^{-t}. \tag{21}$$

is the exact solution of (17). The results show that high accuracy of MVIM.

Example 4.3 Consider the following linear fractional differential equation, [4],

$$D^{\alpha}y(x) = \frac{40320}{\Gamma(9-\alpha)}x^{8-\alpha} - 3\frac{\Gamma(5+\frac{\alpha}{2})}{\Gamma(5-\frac{\alpha}{2})}x^{4-\frac{\alpha}{2}} + \frac{9}{4}\Gamma(\alpha+1) + (\frac{3}{2}x^{\frac{\alpha}{2}} - x^4)^3 - y(x)^{\frac{3}{2}},$$
$$y(0) = 0, \tag{22}$$

where $\alpha \in (0,1)$. By using MVIM, we get,

$$\begin{aligned} f_0(x) &= x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha}, \\ f_1(x) &= J^{\alpha}[(\frac{3}{2}x^{\frac{\alpha}{2}} - x^4)^3] = J^{\alpha}[((\frac{3}{2}x^{\frac{\alpha}{2}} - x^4)^2)^{\frac{3}{2}}] \\ &= J^{\alpha}[(x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha})^{\frac{3}{2}}]. \end{aligned} \tag{23}$$

select $y_0(x) = f_0(x) = x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha}$, we have

$$y_0(x) = x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha},$$
$$y_1(x) = x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha} + J^{\alpha}[(x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha})^{\frac{3}{2}}] - J^{\alpha}[y_0(x)^{\frac{3}{2}}]$$
$$= x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha},$$
$$y_{n+1}(x) = x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha}. \quad n \geq 1. \tag{24}$$

In view of (24), it follows that

$$y(x) = x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha}, \tag{25}$$

where is the exact solution of (22). Again the results highlight the accelerated convergence of the proposed method.

5 Conclusion

In this work, we successfully proposed the reliable modification of He's variational iteration method. The ideas have been shown to be computationally efficient in applying the proposed technique to several fractional ordinary differential equation and fractional partial differential equation that are important in research. In all cases of the applied fields, we obtained excellent performances that may lead to a promising approach for many applications.

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