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Periodicity of the Clifford Algebras

B. Askari[1], F. Fattahi

Department of Mathematics, Ghorveh Branch, Islamic Azad University, Ghorveh, Iran.

Abstract

In this paper we study the structure of Clifford Algebras $Cl_{p,q}$ associated with a non degenerate symmetric bilinear form of signature (p, q), where p, q are positive integer. Also we present a description of these algebras as matrix algebras, and then we will discuss the periodicity of these algebras completely. As a consequence, We create the related algebra matrix tables for these algebras, when $0 \leq p \leq 8$ and $8 \leq q \leq 13$. We also present an isomorphism between $Cl^0_{q,p}$ and $Cl^0_{p,q}$.

Keywords: Tensor algebra, Exterior algebra, Clifford algebra, Quadratic form, Bilinear form.

1 Introduction

Given any vector space, V,over a field, K, there is a special K-algebra, $T(V)$, together with a linear map, $i : V \to T(V)$, following the universal mapping property [1]. The algebra, $T(V)$, is the *tensor algebra* of V. It may be constructed as the direct sum $T(V) = \bigoplus_{i\geq 0} V^{\otimes i}$, Where $V^0 = K$, and $V^{\oplus i}$ is the i-fold tensor product of V with itself. For every $i \geq 0$, there is a natural injection $\iota_n : V^{\otimes n} \to T(V)$ and in particular, an injection $\iota_0 : K \to T(V)$. The multiplicative unit, $\mathbf{1}$, of $T(V)$ is the image, $\iota_0(1)$, in $T(V)$ of the unit, 1, of the field K. Since every $v \in T(V)$ can be expressed as a finite

[1]Corresponding author. E-mail: Bahman_Askari2003@yahoo.com

sum $v = v_1 + v_2 + \ldots + v_k$, where $v_i \in V^{\otimes n_i}$ and n_i the are natural numbers with $n_i \neq n_j$ if $i \neq j$, to define multiplication in $T(V)$, using bilinearity [1], it is enough to define the multiplication $V^{\otimes m} \times V^{\otimes n} \to V^{\otimes(m+n)}$. Of course, this is defined by:

$$(v_1 \otimes \ldots \otimes v_m).(w_1 \otimes \ldots \otimes w_n) = v_1 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots \otimes w_n.$$

It is important to note that multiplication in $T(V)$ is not commutative. Also, the unit, **1**, of $T(V)$ is not equal to 1, the unit of the field K. However, in view of the injection $\iota_0 : K \to T(V)$, for the sake of notational simplicity, we will denote **1** by 1. More generally, in view of the injections $\iota_n : V^{\otimes n} \to T(V)$, we identify elements of $V^{\otimes n}$ with their images in $T(V)$.

Most algebras of interest arise as well-chosen quotients of the tensor algebra $T(V)$. This is true for the *exterior algebra*, $\Lambda^* V$ (also called Grassmann algebra), where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v$, where $v \in V$, and for the symmetric algebra, Sym V, where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes w - w \otimes v$, where $v, w \in V$. A *Clifford algebra* may be viewed as a refinement of the exterior algebra, in which we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v - \Phi(v).1$, where Φ is the *quadratic form* associated with a symmetric bilinear form, $\varphi : V \times V \to K$, and $\cdot : K \times T(V) \to T(V)$ denotes the scalar product of the algebra $T(V)$. For simplicity, let us assume that we are now dealing with real algebras.

2 Preliminaries

Definition 2.1 *Let V be a real finite-dimensional vector space. A quadratic form on V is a mapping $\Phi : V \to \Re$ such that*

1. $\Phi(\lambda v) = \lambda^2 \Phi(v)$ *for all* $\lambda \in \Re$, $v \in V$.

2. *the mapping* $(x, y) \to (\Phi(x+y) - \Phi(x) - \Phi(y)) = \varphi(x, y)$ *of* $V \times V$ *into* $\Re$ *is bilinear.*

Then φ *is called the bilinear form associated to* Φ*.*

It is obvious from the definition that φ is symmetric:

$$\varphi(x, y) = \varphi(y, x)$$

and $\varphi(x, x) = \Phi(x)$.
Two elements x, y of V such that $\varphi(x, y) = 0$ are said to be orthogonal to each other.

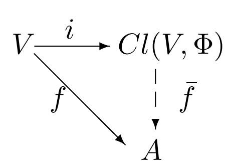
Definition 2.2 *Let* V *be a real finite-dimensional vector space together with a symmetric bilinear form* $\varphi : V \times V \to \Re$*, and associated quadratic form,* $\Phi(x) = \varphi(x, x)$*. A Clifford algebra associated with* V *and* Φ *is a real algebra,* $Cl(V, \Phi)$*, together with a linear map,* $i : V \to Cl(V, \Phi)$ *satisfying the condition* $(i(v))^2 = \Phi(v).1$ *for all* $v \in V$ *and so that for every real algebra,* A*, and every linear map,* $f : V \to A$*, with*

$$(f(v))^2 = \Phi(v).1 \quad for\ all \quad v \in V,$$

there is a unique algebra homomorphism, $\bar{f} : Cl(V, \Phi) \to A$ *so that*

$$f = \bar{f} o i,$$

as in the diagram below:



We use the notation, λu*, for the product of a scalar,* $\lambda \in \Re$ *and of an element,* u*, in the algebra* $Cl(V, \Phi)$ *and juxtaposition,* uv*, for the multiplication of two elements,* $u, v \in Cl(V, \Phi)$*.*

By a familiar argument, any two Clifford algebras associated with V and Φ are isomorphic.

To show the existence of $Cl(V,\Phi)$, observe that $T(V)/U$ does the job, where U is the ideal of $T(V)$ generated by all elements of the form $v\otimes v-\Phi(v).1$, where $v\in V$ The map $i:V\rightarrow Cl(V,\Phi)$ is the composition

$$V\xrightarrow{\iota_1}T(V)\xrightarrow{\pi}\frac{T(V)}{U}$$

where π is the natural quotient map. We often denote the Clifford algebra $Cl(V,\Phi)$ simply by $Cl(\Phi)$.
Observe that when $\Phi\equiv 0$ is the quadratic form identically zero everywhere, then the Clifford algebra $Cl(V,0)$ is just the exterior algebra, Λ^*V.

Remark: As in the case of the tensor algebra, the unit of the algebra $Cl(\Phi)$ and the unit of the field $\Re$ are not equal.
Since

$$\Phi(u+v)-\Phi(u)-\Phi(v)=2\varphi(u,v)$$

and

$$(i(u+v))^2=i(u)^2+i(v)^2+i(u)i(v)+i(v)i(u),$$

using the fact that

$$(i(u))^2=\Phi(u).1,$$

We get:

$$i(u)i(v)+i(v)i(u)=2\varphi(u,v).1.$$

As a consequence, if $(u_1,\ldots,u_n)$ is an orthogonal basis w.r.t. φ(which means that $\varphi(u_j,u_k)=0$ for all $j\neq k$), we have:

$$i(u_j)i(u_k)+i(u_k)i(u_j)=0 \quad for\ all \quad j\neq k.$$

Proposition 2.3 *For every vector space, V, of finite dimension n, the map $i : V \to Cl(\Phi)$ is injective. Given a basis $(e_1, e_2, \ldots, e_n)$ of V the $2^n - 1$ products*

$$i(e_1)i(e_2)\cdots i(e_k), \qquad 1 \le i_1 < i_2 < \ldots < i_k \le n,$$

and 1 form a basis of $Cl(\Phi)$. Thus, $Cl(\Phi)$ has dimension 2^n.

Proof. See[4].

Remark: Since i is injective, for simplicity of notation, from now on, we write u for $i(u)$ Proposition 2.3 implies that if $(e_1, e_2, \ldots, e_n)$ is an orthogonal basis of V, then $Cl(\Phi)$ is the algebra presented by the generators $(e_1, e_2, \ldots, e_n)$ and the relations

$$e_j^2 = \Phi(e_j).1, \quad 1 \le j \le n, \quad and \quad e_j e_k = -e_k e_j, \quad 1 \le j,\ k \le n, \quad j \ne k.$$

In other words, Clifford algebra $Cl(\Phi)$ consists of certain kinds of "polynomials," linear combinations of monomials of the form $\sum_J \lambda_J e_J$, where $J = \{i_1, i_2, \ldots, i_k\}$ is any subset (possibly empty) of $\{1, \ldots, n\}$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$, and the monomial e_j is the "product"$e_{i_1} e_{i_2} \ldots e_{i_k}$.

Definition 2.4 *The even-graded elements (the elements of $Cl^0(\Phi)$) are those generated by 1 and the basis elements consisting of an even number of factors, $e_{i_1} e_{i_2} \ldots e_{i_2 k}$, and the odd-graded elements (the elements of $Cl^1(\Phi)$) are those generated by the basis elements consisting of an odd number of factors, $e_{i_1} e_{i_2} \ldots e_{i_{2k+1}}$.*

Remark: we assume that Φ is the quadratic form on $\Re^n$defined by

$$\Phi(x_1, \ldots, x_n) = -(x_1^2 + \ldots + x_n^2)$$

Let Cl_n denote the Clifford algebra $Cl(\Phi)$.

Example 2.5 *Cl_1 is spanned by the basis $(1, e_1)$. We have*

$$e_1^2 = -1.$$

Under the bijection

$$e_1 \mapsto i$$

Cl_1 *is isomorphic to the algebra of complex numbers,* $\mathbb{C}$.

Example 2.6 *Let* (e_1, e_2) *be the canonical basis of* $\Re^2$*, then* Cl_2 *is spanned by the basis by* $(1, e_1, e_2, e_1e_2)$*. Furthermore, we have:*

$$e_2e_1 = -e_1e_2, \quad e_1^2 = -1, \quad e_2^2 = -1, \quad (e_1e_2)^2 = -1.$$

Under the bijection

$$e_1 \mapsto i, \qquad e_2 \mapsto j, \qquad e_1e_2 \mapsto k,$$

it is easily checked that the quaternion identities

$$i^2 = j^2 = k^2 = -1 \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

hold, and thus, Cl_2*, is isomorphic to the algebra of quaternions,* $\mathbb{H}$.

Definition 2.7 *For every non degenerate quadratic form* Φ *over* $\Re$ *there is an orthogonal basis with respect to which* Φ *is given by*

$$\Phi(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)$$

where p *and* q *only depend on* Φ *. The quadratic form corresponding to* (p, q) *is denoted* $\Phi_{p,q}$ *and we call* (p, q) *the signature of* $\Phi_{p,q}$*. Let* $n = p+q$ *We denote the Clifford algebra associated with* $\Re^n$ *and* $\Phi_{p,q}$ *where has* $\Phi_{p,q}$ *signature* (p, q) *by* $Cl_{p,q}$*. Note that with this new notation,* $Cl_n = Cl_{0,n}$.

Example 2.8 *Let* $Cl_{p,q} = Cl(\Re^{p+q}, \Phi_{p,q})$*, where* Φ *has signature* (p, q)*, and orthonormal basis is written as* $\{e_1, \ldots, e_p, \varepsilon_1, \ldots, \varepsilon_q\}$ *where* $e_1^2 = \cdots = e_p^2 = 1$*,* $\varepsilon_1^2 =$

$\cdots = \varepsilon_q^2 = -1$. *Thus, we have:*

$$Cl_{1,0} = \Re \oplus \Re \quad \textit{with} \quad e_1 = \pm 1;$$

$$Cl_{0,1} = \not{C}, \quad \textit{with} \quad \varepsilon_1 = i;$$

$$Cl_{2,0} = M_2(\Re), \quad \textit{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$Cl_{0,2} = \not{H}, \quad \textit{with} \quad \varepsilon_1 = i \quad \varepsilon_2 = j, \quad \varepsilon_1\varepsilon_2 = k;$$

$$Cl_{1,1} = M_2(\Re), \quad \textit{with} \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1\varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3 Main Results

It turns out that the real algebras $Cl_{p,q}$ can be build up as tensor products of the basic algebras $\Re$, $\not{C}$ and $\not{H}$. According to [6], the description of the real algebras $Cl_{p,q}$ as matrix algebras and the 8-periodicity was first discovered by Elie Cartan in 1908. Of course, Cartan used a very different notation. These facts were rediscovered independently by [2] in the 1960's (see Raoul Bott's comments in Volume 2 of his Collected papers.).

As mentioned in Example 2.3, we have:

$$Cl_{0,1} = \not{C}, \quad Cl_{0,2} = \not{H}, \quad Cl_{1,0} = \Re \oplus \Re, \quad Cl_{2,0} = M_2(\Re),$$

And

$$Cl_{1,1} = M_2(\Re).$$

The key to the classification is the following lemma:

Lemma 3.1 *We have the isomorphisms*

$$Cl_{0,n+2} \approx Cl_{n,0} \otimes Cl_{0,2}$$

$$Cl_{n+2,0} \approx Cl_{0,n} \otimes Cl_{2,0}$$

$$Cl_{p+1,q+1} \approx Cl_{p,q} \otimes Cl_{1,1}$$

for all $n, p, q \geq 0$.

Proof. Let $\Phi_{0,n+2}(x) = - \parallel x \parallel^2$, where $\parallel x \parallel$ is the standard Euclidean norm on $\Re^{n+2}$, and let $(e_1, \ldots, e_{n+2})$ be an orthonormal basis for $\Re^{n+2}$ under the standard Euclidean inner product. We also let $(e'_1, \ldots, e'_n)$ be a set of generators for $Cl_{n,0}$ and (e''_1, e''_2) be a set of generators for $Cl_{0,2}$. We can define a linear map $f : \Re^{n+2} \to Cl_{n,0} \otimes Cl_{0,2}$ by its action on the basis $(e_1, \ldots, e_{n+2})$ as follows:

$$f(e_i) = \begin{cases} e'_i \oplus e''_1 e''_2 & 1 \leq i \leq n \\ 1 \oplus e''_{i-n} & n+1 \leq i \leq n+2 \end{cases}$$

Observe that for $1 \leq i, j \leq n$ we have

$$f(e_i)\ f(e_j) + f(e_j)\ f(e_i) = (e'_i e'_j + e'_j e'_i) \otimes (e''_1 e''_2)^2 = -2\delta_{ij} 1 \otimes 1,$$

Since $(e''_2)^2 = (e''_1)^2 = -1,\ e''_1 e''_2 = -e''_2 e''_1$ and $e'_i e'_j = -e'_j e'_i$, for all $i \neq j$, and $(e'_i)^2 = 1$, for all i with $1 \leq i \leq n$. Also for $n+1 \leq i,\ j \leq n+2$ we have

$$f(e_i)\ f(e_j) + f(e_j)\ f(e_i) = 1 \otimes (e''_{i-n} e''_{j-n} + e''_{j-n} e''_{i-n}) = -2\delta_{ij} 1 \otimes 1,$$

and

$$f(e_i)\ f(e_k) + f(e_k)\ f(e_i) = 2e'_i \otimes (e''_1 e''_2 e''_{k-n} + e''_{k-n} e''_1 e''_2) = 0,$$

for all $1 \leq i,\ j \leq n$ and $n+1 \leq k \leq n+2$ (since $e''_{k-n} = e''_1$ or $e''_{k-n} = e''_2$). Thus, we have:

$$f(x)^2 = - \parallel x \parallel^2 .1 \otimes 1 \quad for\ all \quad x \in \Re^{n+2},$$

and by the universal mapping property of $Cl_{0,n+2}$, we get an algebra map:

$$\tilde{f} : Cl_{0,n+2} \to Cl_{n,0} \otimes Cl_{0,2}.$$

Since $\tilde{f}$ maps onto a set of generators, it is surjective. However,

$$dim(Cl_{0,n+2}) = 2^{n+2} = 2^n.2 = dim(Cl_{n,0}) dim(Cl_{0,2}) = dim(Cl_{n,0} \otimes Cl_{0,2})$$

and $\tilde{f}$ is an isomorphism.

The proof of the second identity is analogous. For the third identity, we have:

$$\Phi_{p,q}(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2),$$

And let $(e_1,\dots,e_{p+1},\varepsilon_1,\dots,\varepsilon_{q+1})$ be an orthogonal basis for $\Re^{p+q+2}$ so that $\Phi_{p+1,q+1}(e_i)=+1$ and $\Phi_{p+1,q+1}(\varepsilon_j)=-1$ for $i=1,\dots,p+1$ and $j=1,\dots,q+1$. Also, let $(e'_1,\dots,e'_p,\varepsilon'_1,\dots,\varepsilon'_q)$ be a set of generators for $Cl_{p,q}$ and (e''_1,ε''_1) be a set of generators for $Cl_{1,1}$.We define a linear map $f:\Re^{p+q+2}\to Cl_{p,q}\otimes Cl_{1,1}$ by its action on the basis as follows:

$$f(e_i)=\begin{cases} e'_i\otimes e''_1\varepsilon''_1 & 1\le i\le p\\ 1\otimes e''_1 & i=p+1\end{cases}\quad ,\quad f(\varepsilon_j)=\begin{cases} \varepsilon'_j\otimes e''_1\varepsilon''_1 & 1\le j\le q\\ 1\otimes \varepsilon''_1 & j=q+1\end{cases}$$

We can check that

$$f(x)^2=\Phi_{p+1,q+1}(x).1\otimes 1\quad for\ all\quad x\in\Re^{p+q+2},$$

and we finish the proof as in the first case.
To apply this lemma, we need some further isomorphisms among various matrix algebras.

Proposition 3.2 *The following isomorphisms hold:*

$$\begin{array}{ll} M_m(\Re)\otimes M_n(\Re)\approx M_{mn}(\Re) & for\ all\ m,n\ge 0\\ M_n(\Re)\otimes_R k\approx M_n(k) & for\ all\ K=\mathbb{C}\ or\ K=\mathbb{H}\ and\ all\ n\ge 0\\ \mathbb{C}\otimes_{\Re}\mathbb{C}\approx\mathbb{C}\oplus\mathbb{C} & \\ \mathbb{C}\otimes_{\Re}\mathbb{H}\approx M_4(\mathbb{C}) & \end{array}$$

Proof. See[5].

Proposition 3.3 *(Cartan/Bott) For all $n\ge 0$ we have the following isomorphisms:*

$$Cl_{0,n+8}\approx Cl_{0,n}\otimes Cl_{0,8}$$

$$Cl_{n+8,0}\approx Cl_{n,0}\otimes Cl_{8,0}$$

Furthermore,

$$Cl_{0,8}=Cl_{8,0}=M_{16}(\Re).$$

Proof. By Lemma 3.1 we have the isomorphisms:

$$Cl_{0,n+2} \approx Cl_{n,0} \otimes Cl_{0,2}, \qquad Cl_{n+2,0} \approx Cl_{0,n} \otimes Cl_{2,0},$$

and thus,

$$Cl_{0,n+8} \approx Cl_{n+6,0} \otimes Cl_{0,2} \approx Cl_{0,n+4} \otimes Cl_{2,0} \otimes Cl_{0,2} \approx \cdots \approx Cl_{0,n} \otimes Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2}.$$

Since $Cl_{0,2} = \not{H}$ and $Cl_{2,0} = M_2(\Re)$, by Proposition 3.1, we get:
$Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \approx \not{H} \otimes \not{H} \otimes M_2(\Re) \otimes M_2(\Re) \approx M_4(\Re) \otimes M_4(\Re) \approx M_{16}(\Re)$.
The second isomorphism is proved in a similar fashion.

Lemma 3.4 $Cl_{p+4,q} \approx Cl_{p,q} \otimes M_2(\not{H}) \approx Cl_{p,q+4}$.

Proof. We will prove the first isomorphism. Take $A = Cl_{p,q} \otimes M_2(\not{H})$, define
$f : \Re^{p+4,q} \to A$

$$f(e_r) = e'_r \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \quad r = 1, \ldots, p, \qquad f(\varepsilon_s) = \varepsilon'_s \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \qquad s = 1, \ldots, q,$$

and on the remaining four basic vectors, define

$$f(e_{p+1}) = 1 \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad f(e_{p+2}) = 1 \otimes \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix},$$
$$f(e_{p+3}) = 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad f(e_{p+4}) = 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From all this, we can deduce the following Theorem:

Theorem 3.5 *For $0 \le p \le 8$ and $8 \le q \le 13$ matrix representations of the Clifford* *algebras $Cl_{p,q}$ are exhibited in the following table:*

	$q \rightarrow$					
p	$M_{16}(\Re)$	$M_{16}(\mathbb{C})$	$M_{16}(\mathbb{H})$	$M_{16}(\mathbb{H}) \oplus M_{16}(\mathbb{H})$	$M_{32}(\mathbb{H})$	$M_{64}(\mathbb{C})$
$\downarrow$	$M_{16}(\Re) \oplus M_{16}(\Re)$	$M_{32}(\Re)$	$M_{32}(\mathbb{C})$	$M_{32}(\mathbb{H})$	$M_{32}(\mathbb{H}) \oplus M_{32}(\mathbb{H})$	$M_{64}(\mathbb{H})$
	$M_{32}(\Re)$	$M_{32}(\Re) \oplus M_{32}(\Re)$	$M_{64}(\Re)$	$M_{64}(\mathbb{C})$	$M_{64}(\mathbb{H})$	$M_{64}(\mathbb{H}) \oplus M_{64}(\mathbb{H})$
	$M_{32}(\mathbb{C})$	$M_{64}(\Re)$	$M_{64}(\Re) \oplus M_{64}(\Re)$	$M_{128}(\Re)$	$M_{128}(\mathbb{C})$	$M_{128}(\mathbb{H})$
	$M_{32}(\mathbb{H})$	$M_{64}(\mathbb{C})$	$M_{128}(\Re)$	$M_{128}(\Re) \oplus M_{128}(\Re)$	$M_{256}(\Re)$	$M_{512}(\mathbb{C})$
	$M_{32}(\mathbb{H}) \oplus M_{32}(\mathbb{H})$	$M_{64}(\mathbb{H})$	$M_{128}(\mathbb{C})$	$M_{256}(\Re)$	$M_{256}(\Re) \oplus M_{256}(\Re)$	$M_{512}(\Re)$
	$M_{64}(\mathbb{H})$	$M_{64}(\mathbb{H}) \oplus M_{64}(\mathbb{H})$	$M_{128}(\mathbb{H})$	$M_{256}(\mathbb{C})$	$M_{512}(\Re)$	$M_{512}(\Re) \oplus M_{512}(\Re)$
	$M_{128}(\mathbb{C})$	$M_{128}(\mathbb{H})$	$M_{128}(\mathbb{H}) \oplus M_{128}(\mathbb{H})$	$M_{256}(\mathbb{H})$	$M_{512}(\mathbb{C})$	$M_{1024}(\Re)$
	$M_{256}(\Re)$	$M_{256}(\mathbb{C})$	$M_{256}(\mathbb{H})$	$M_{256}(\mathbb{H}) \oplus M_{256}(\mathbb{H})$	$M_{512}(\mathbb{H})$	$M_{1024}(\mathbb{C})$

Remark: A table of the Clifford algebras $Cl_{p,q}$ for $0 \leq p, q \leq 7$ can be found in [7].

Lemma 3.6 *We have the isomorphisms*

$$Cl_{p,q} \approx Cl^0_{p,q+1}$$

$$Cl^0_{p+1,q} \approx Cl_{q,p}$$

$$Cl_{p+1,q} \approx Cl_{q+1,p}$$

for all $p, q \geq 0$.

Proof. Let $(e_1, \ldots, e_p, \varepsilon_1, \ldots, \varepsilon_p)$ be an orthonormal basis for $\Re^{p+q}$, We also let $(e'_1, \ldots, e'_p, \varepsilon'_1, \ldots, \varepsilon'_{q+1})$ be a set of generators for $Cl_{p,q+1}$. We can define a linear map $f : \Re^{p+q} \rightarrow Cl^0_{p,q+1}$ by its action on the basis $(e_1, \ldots, e_n, \varepsilon_1, \ldots, \varepsilon_q)$ as follows:

$$f(e_i) = e'_i \varepsilon'_{q+1} \quad i = 1, \ldots, p,$$
$$f(\varepsilon_j) = \varepsilon'_j \varepsilon'_{q+1} \quad j = 1, \ldots, q.$$

We have

$$f(e_i)\ f(e_j) + f(e_j)\ f(e_i) = e'_i \varepsilon'_{q+1} e'_j \varepsilon'_{q+1} + e'_j \varepsilon'_{q+1} e'_i \varepsilon'_{q+1} = e'_i e'_j + e'_j e'_i = 2\delta_{ij},$$

And

$$f(\varepsilon_i)\ f(\varepsilon_j) + f(\varepsilon_j)\ f(\varepsilon_i) = \varepsilon'_i \varepsilon'_{q+1} \varepsilon'_j \varepsilon'_{q+1} + \varepsilon'_j \varepsilon'_{q+1} \varepsilon'_i \varepsilon'_{q+1} = \varepsilon'_i \varepsilon'_j + \varepsilon'_j \varepsilon'_i = -2\delta_{ij},$$

And also

$$f(e_i)\ f(\varepsilon_j) + f(\varepsilon_j)\ f(e_i) = e'_i\varepsilon'_{q+1}\varepsilon'_j\varepsilon'_{q+1} + \varepsilon'_j\varepsilon'_{q+1}e'_i\varepsilon'_{q+1} = e'_i\varepsilon'_j + \varepsilon'_j e'_i = 0.$$

Thus, by the universal mapping property of $Cl_{p,q}$, we get an algebra map:

$$\tilde{f} : Cl_{p,q} \to Cl^0_{p,q+1}.$$

Since $\tilde{f}$ maps onto a set of generators, it is surjective. However,

$$dim(Cl^0_{p,q+1}) = \frac{2^{p+q+1}}{2} = 2^{p+q} = dim(Cl_{p,q})$$

and $\tilde{f}$ is an isomorphism.

For the second identity we define $f : \Re^{q+p} \to Cl^0_{p+1,q}$ on basic vectors by:

$$f(e_r) = e'_r e'_{p+1} \quad r = 1, \ldots, q,$$
$$f(\varepsilon_s) = \varepsilon'_s e'_{p+1} \quad s = 1, \ldots, p.$$

Then

$$f(e_r)^2 = e'_r e'_{p+1} e'_r e'_{p+1} = -{e'}^2_r {e'}^2_{p+1} = -{e'}^2_r = -1,$$

$$f(e_s)^2 = \varepsilon'_s e'_{p+1} \varepsilon'_s e'_{p+1} = -{\varepsilon'}^2_s {e'}^2_{p+1} = -{\varepsilon'}^2_s = +1,$$

The rest of the proof is like the previous part. For the third identity, according to the previous parts, we have:

$$Cl_{p+1,q} \approx Cl^0_{p+1,q+1} \approx Cl_{q+1,p}.$$

Corollary: $Cl^0_{p,q} \approx Cl^0_{q,p}$.

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