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Obtaining D'Alembert's wave formula from variational iteration and homotopy perturbation methods

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Abstract

In this paper, the variational iteration and the homotopy perturbation methods are adopted to obtain the solution of the one-dimensional wave equation. The solutions procedure reveal that the applied methods are very efficient. In addition, the result obtained is the same as the D'Alembert's formula for the wave equation.

Keywords: Wave equation, D'Alembert's formula, Variational iteration method, Homotopy perturbation method.

1 Introduction

Variational iteration method (VIM) is an analytical method which was established by He [1-10]. This method has been successfully used by many authors [11-26] as a powerful mathematical tool for solving many problems. The basic idea of VIM is to construct a correction functional with a general Lagrange multiplier which can be identified optimally via variational theory. The method can solve the problems without any need to discrete the variables, therefore there is no need to compute the round off errors and one does not require a large computer memory and a lot of time.

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Another type of analytical methods namely the homotopy perturbation method (HPM) also exists which is a powerful and efficient technique to find the solutions of the linear and non-linear equations. The method was first introduced by He [27]. In this method, the non-linear problem is transferred to an infinite number of sub-problems and then the solution is approximated by the sum of solutions of the first few sub-problems. HPM is a combination of the perturbation and homotopy methods. This method can take the advantages of the conventional perturbation method while eliminating its restrictions. In general, this method has been successfully applied to solve many types of linear and non-linear equations in science and engineering by many authors [28-35].

In the present article, we aim to achieve D'Alembert's formula for the wave equation using the variational iteration and the homotopy perturbation methods. To accomplish this goal, consider the following one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2},$$

where $u(x,t)$ is a unknown function and c is a constant. The variables x and t are the independent variables as well. This equation is also known as the equation of vibration of a string. Mathematically, the above equation is a linear second order partial differential equation of hyperbolic type. The wave equation is often encountered in different fields such as elasticity, aerodynamics, acoustics, and electrodynamics.

The mathematical model of a physical process includes not only the governing differential equation but also the side conditions to be imposed. Therefore, we solve the wave equation for the special case when we clamp the string ends at $x = 0$ and $x = L$. Factually, we find the solution of the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad t > 0, \qquad 0 < x < L, \tag{1}$$

with the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \qquad 0 \le x \le L, \tag{2}$$

and the boundary conditions

$$u(0,t)=0, \qquad u(L,t)=0, \qquad t\geq 0, \tag{3}$$

also it is necessary to express that $f(x)$ and $g(x)$ are the initial displacement and velocity, respectively.

D'Alembert in 1747, employed the method of change of variable for solving Eq.(1) with the initial conditions and obtained the following formula

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2}[G(x+ct)-G(x-ct)],$$

where $G(x)=\frac{1}{c}\int g(x)dx$. This is also known as D'Alembert's formula and gives the solution of the wave equation (1) subject to the initial conditions [36].

The rest of this article has been organized as follows: in section two, the basic ideas of VIM are explained. In section three, the basic concepts of HPM are illustrated. In section four, both methods are used for obtaining the solution of the wave equation and finally, a comparison between the employed methods and D'Alembert's method is provided in section five.

2 Variational iteration method (VIM)

Consider the following functional equation, $L(u)+N(u)=g(x,t)$, where L is a linear operator, N is a non-linear operator and $g(x,t)$ is a specific analytical function. As usual in this method, we consider the following correction functional

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\tau)[Lu_n(x,\tau)+N\tilde{u}_n(x,\tau)-g(x,\tau)]d\tau,$$

where λ is a general Lagrange multiplier and can be identified optimally via variational theory. Also $\tilde{u}_n$ is considered as a restricted variation [5,6], i.e. $\delta\tilde{u}_n=0$. In this method, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1},\ n\geq 0$, of the solution u will be readily obtained upon using the determined Lagrange multiplier and by using the initial approximation u_0. Consequently, the solution is given by $u(x,t)=\lim_{n\to\infty}u_n(x,t)$.

3 Homotopy perturbation method (HPM)

To illustrate the basic ideas of this method, consider the following general non-linear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{4}$$

with the following boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad r \in \Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be decomposed into a linear part and a non-linear one, designated as L and N respectively. Hence Eq.(4) can be written as the following form

$$L(u) + N(u) - f(r) = 0.$$

Using the homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \longrightarrow R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq.(4) which satisfies the boundary conditions. Obviously, from Eq.(5) we will have

$$H(v,0) = L(v) - L(u_0) = 0,$$

$$H(v,1) = A(v) - f(r) = 0.$$

By changing the value of p from zero to unity, $v(r,p)$ changes from $u_0(r)$ to $u(r)$, in topology this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Due to the fact that $p \in [0,1]$ can be considered as a small parameter, therefore we consider the solution of Eq.(5) as a power series in p as the following

$$v = \sum_{n=0}^{\infty} v_n p^n, \tag{6}$$

setting $p=1$ results in the approximate solution for Eq.(4)

$$u=\lim_{p\to 1} v = v_0+v_1+v_2+\cdots.$$

4 Applying VIM and HPM to the wave equation

VIM: Consider Eq.(1) with (2) and (3), according to variational iteration method, we derive a correct functional as follows

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\tau)[\frac{\partial^2 u_n(x,\tau)}{\partial \tau^2}-c^2\frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2}]d\tau. \quad (7)$$

Making the above correction functional stationary, we have

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(\tau)[\frac{\partial^2 u_n(x,\tau)}{\partial \tau^2}]d\tau,$$

thus, we have

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)-\delta\lambda'(\tau)u_n(x,\tau)|_{\tau=t}+\delta\lambda(\tau)\frac{\partial u_n(x,\tau)}{\partial\tau}|_{\tau=t}+\int_0^t \delta\lambda'' u_n(x,\tau)d\tau=0.$$

This yields the following stationary conditions

$$\delta u_n : \lambda''(\tau)=0,$$

$$\delta\frac{\partial u_n}{\partial\tau} : \lambda(\tau)|_{\tau=t}=0,$$

$$\delta u_n : 1-\lambda'(\tau)|_{\tau=t}=0.$$

Therefore, the Lagrange multiplier can be identified as $\lambda=\tau-t$.

Substituting this value of the Lagrange multiplier into the functional (7) gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\tau-t)[\frac{\partial^2 u_n(x,\tau)}{\partial \tau^2}-c^2\frac{\partial^2 u_n(x,\tau)}{\partial x^2}]d\tau. \quad (8)$$

Consider $u_0(x,t)=f(x)+tg(x)$ as an initial approximation that satisfies the initial conditions. By the above iteration formula (8), one can obtain the following components

$$u_1(x,t)=f(x)+tg(x)+c^2[\frac{t^2}{2!}f^{(2)}(x)+\frac{t^3}{3!}g^{(2)}(x)],$$

$$u_2(x,t) = u_1(x,t) + c^4[\frac{t^4}{4!}f^{(4)}(x) + \frac{t^5}{5!}g^{(4)}(x)],$$
$$u_3(x,t) = u_2(x,t) + c^6[\frac{t^6}{6!}f^{(6)}(x) + \frac{t^7}{7!}g^{(6)}(x)],$$
$$\vdots$$

Therefore, the solution of Eq.(1) with the initial conditions will be as

$$u(x,t) = f(x) + tg(x) + c^2[\frac{t^2}{2!}f^{(2)}(x) + \frac{t^3}{3!}g^{(2)}(x)] + c^4[\frac{t^4}{4!}f^{(4)}(x) + \frac{t^5}{5!}g^{(4)}(x)]$$
$$+c^6[\frac{t^6}{6!}f^{(6)}(x) + \frac{t^7}{7!}g^{(6)}(x)] + \cdots,$$

or

$$u(x,t) = [f(x)+c^2\frac{t^2}{2!}f^{(2)}(x)+c^4\frac{t^4}{4!}f^{(4)}(x)+\cdots]+[tg(x)+c^2\frac{t^3}{3!}g^{(2)}(x)+c^4\frac{t^5}{5!}g^{(4)}(x)+\cdots]. \tag{9}$$

Using the Taylor series, we have

$$f(x) + c^2\frac{t^2}{2!}f^{(2)}(x) + c^4\frac{t^4}{4!}f^{(4)}(x) + \cdots = \frac{1}{2}[f(x+ct) + f(x-ct)], \tag{10}$$

on the other hand, by assuming $G(x) = \frac{1}{c}\int g(x)dx$, we will have

$$tg(x) + c^2\frac{t^3}{3!}g^{(2)}(x) + c^4\frac{t^5}{5!}g^{(4)}(x) + \cdots = \frac{1}{2}[G(x+ct) - G(x-ct)], \tag{11}$$

now substituting (10, 11) into (9), gives the solution of the wave equation with (2)

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2}[G(x+ct) - G(x-ct)],$$

which is similar to the solution obtained by D'Alembert's method.

Now using the first boundary condition, we will have

$$u(0,t) = \frac{1}{2}[f(ct) + f(-ct)] + \frac{1}{2}[G(ct) - G(-ct)] = 0,$$

we assume $\quad f(-ct) = -f(ct), \quad G(-ct) = G(ct),$

also applying the second boundary condition, yields

$$u(L,t) = \frac{1}{2}[f(L+ct) + f(L-ct)] + \frac{1}{2}[G(L+ct) - G(L-ct)] = 0,$$

we suppose

$$f(L+ct)+f(L-ct)=0, \qquad G(L+ct)-G(L-ct)=0, \tag{12}$$

now by substituting $ct=L+\alpha$ into (12), we will obtain

$$f(2L+\alpha)=-f(-\alpha)=f(\alpha), \qquad G(2L+\alpha)=G(-\alpha)=G(\alpha).$$

Thus, the solution of problem can be expressed as follows

$$u(x,t)=\frac{1}{2}[f^*(x+ct)+f^*(x-ct)]+\frac{1}{2}[G^*(x+ct)-G^*(x-ct)],$$

where $f^*(x)$ and $G^*(x)$ are the odd and even extensions of functions $f(x)$ and $G(x)$ with period $2L$, respectively.

HPM: Consider the wave equation (1) with the indicated initial and boundary conditions. Using HPM, one can construct a homotopy in the following form

$$(1-p)[\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 u_0}{\partial t^2}]+p[\frac{\partial^2 v}{\partial t^2}-c^2\frac{\partial^2 v}{\partial x^2}]=0. \tag{13}$$

Substituting (6) into (13) and equating the terms with the identical powers of p, gives

$$\begin{aligned}
p^0 &: \frac{\partial^2 v_0}{\partial t^2}-\frac{\partial^2 u_0}{\partial t^2}=0, \qquad v_0(x,0)=f(x), \qquad \frac{\partial v_0(x,0)}{\partial t}=g(x),\\
p^1 &: \frac{\partial^2 v_1}{\partial t^2}+\frac{\partial^2 u_0}{\partial t^2}-c^2\frac{\partial^2 v_0}{\partial x^2}=0, \qquad v_1(x,0)=0, \qquad \frac{\partial v_1(x,0)}{\partial t}=0,\\
p^2 &: \frac{\partial^2 v_2}{\partial t^2}-c^2\frac{\partial^2 v_1}{\partial x^2}=0, \qquad v_2(x,0)=0, \qquad \frac{\partial v_2(x,0)}{\partial t}=0,\\
&\vdots
\end{aligned} \tag{14}$$

Consider $u_0(x,t)=f(x)+tg(x)$ as an initial approximation for the solution that satisfies the initial conditions. Successive solving of Eqs.(14) leads to

$$v_0(x,t)=f(x)+tg(x),$$

$$v_1(x,t)=c^2[\frac{t^2}{2!}f^{(2)}(x)+\frac{t^3}{3!}g^{(2)}(x)],$$

$$v_2(x,t) = c^4[\frac{t^4}{4!}f^{(4)}(x) + \frac{t^5}{5!}g^{(4)}(x)],$$
$$v_3(x,t) = c^6[\frac{t^6}{6!}f^{(6)}(x) + \frac{t^7}{7!}g^{(6)}(x)],$$
$$\vdots$$

Therefore, the solution when $p \to 1$ will be as

$$u(x,t) = f(x) + tg(x) + c^2[\frac{t^2}{2!}f^{(2)}(x) + \frac{t^3}{3!}g^{(2)}(x)] + c^4[\frac{t^4}{4!}f^{(4)}(x) + \frac{t^5}{5!}g^{(4)}(x)]$$
$$+c^6[\frac{t^6}{6!}f^{(6)}(x) + \frac{t^7}{7!}g^{(6)}(x)] + \cdots,$$

or

$$u(x,t) = [f(x)+c^2\frac{t^2}{2!}f^{(2)}(x)+c^4\frac{t^4}{4!}f^{(4)}(x)+\cdots]+[tg(x)+c^2\frac{t^3}{3!}g^{(2)}(x)+c^4\frac{t^5}{5!}g^{(4)}(x)+\cdots].$$

Now proceeding as before, the solution of problem will be obtained as follows

$$u(x,t) = \frac{1}{2}[f^*(x+ct) + f^*(x-ct)] + \frac{1}{2}[G^*(x+ct) - G^*(x-ct)],$$

where $f^*(x)$ and $G^*(x)$ are the odd and even extensions of functions $f(x)$ and $G(x)$ with period $2L$, respectively. This solution is the same as that obtained by VIM.

5 Conclusion

In this paper, the variational iteration and the homotopy perturbation methods were successfully applied to obtain D'Alembert's formula for the wave equation. It was clearly shown that these methods are very efficient in deriving D'Alembert's formula. However, a comparison between the used methods and D'Alembert's method shows that the use of VIM and HPM is much easier and more convenient to obtain D'Alembert's formula.

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