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Fibonacci Length of an Efficiently Presented Metabelian p-Group

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Abstract

For every integer $n \geqslant 1$ and every odd prime p, an efficient presentation is given for the wreath product $\mathbb{Z}_p \wr \mathbb{Z}_{p^n}$ which, is of nilpotency class $(p-1)n+1$. Moreover, the Fibonacci length is proved to be 8×3^n when $p = 3$, and in general, $k(p) \times p^n$, where $k(p)$ is the period of Fibonacci length modulo p (the Wall number of p).

Keywords: Presentation of groups, Efficient presentation, Schur multiplier, Fibonacci length.

1 Introduction

A finite presentation $\langle X \mid R\rangle$ of a group G is said to be an efficient presentation for G, if $|R| - |X| = rank(M(G))$ where, $M(G)$ is the Schur multiplier of G (see [2, 18, 19]).

Through this paper n is a positive integer and p is an odd prime. We consider the wreath product $G(p,n) = \mathbb{Z}_p \wr \mathbb{Z}_{p^n}$ -studied by H. Neumann(see [24])- where, the standard wreath product $G \wr H$ of the finite groups G and H is defined to be a semidirect product of G by the direct product B of $|G|$ copies of H.

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Presenting the groups efficiently depending on the Schur multiplier of the group, is of interest and many efforts have been done during the years (one may see [5, 7, 8, 9, 10, 11, 12, 20, 23, 25, 26, 27, 29, 30, 31, 32, 33]). The article [21] also studies a 2-group of nilpotency class $n+1$, and our group $G(p,n)$ for odd primes p, is indeed the generalization of [13].

Our notations are standard, we use $H:G$ for the semidirect product of H by G.

2 An efficient presentation for $G(p,n)$

First we recall the celebrated theorems of Blackburn and Schur of [19]:

Theorem 2.1. *Let m be the number of involutions in a group G. Then the Schur multiplier $M(G \wr H)$ is a direct product of $M(G)$, $M(H)$, $\frac{1}{2}(|G|-m-1)$ copies of $H \otimes H$ and m copies of $H\#H$. Where, $A \otimes B$ is used for the tensor product and $A\#A$ is the factor group of $A \otimes A$ by the subgroup generated by all of the elements of the form $a \otimes b + b \otimes a$, when A is abelian. If S is an arbitrary group then we put $S\#S = \frac{S}{S'}\#\frac{S}{S'}$.*

Theorem 2.2. *For the groups A and B, $A \otimes B \cong Hom(A,B)$. Moreover, for every positive integers m and n, $Hom(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_k$ where, $k = g.c.d\ (m,n)$.*

An elementary presentation for the group

$$G = G(p,n) = \mathbb{Z}_p : (\underbrace{\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times \cdots \times \mathbb{Z}_{p^n}}_{p-copies})$$

may be given by:

$$G = \langle y, x_1, \ldots, x_p | y^p = {x_i}^{p^n} = 1, \quad x_i^y = x_{i+1}, (1 \leqslant i \leqslant p), \\ [x_j, x_k] = 1, (1 \leqslant j < k \leqslant p) \rangle,$$

where indices are reduced modulo p.

Proposition 2.3. *The group G has an efficient presentation isomorphic to*

$$G = \langle x, y | y^p = x^{p^n} = 1, [x, x^{y^i}] = 1, (i = 1, 2, \ldots, \frac{p-1}{2}) \rangle.$$

Proof. Eliminate the generators $x_2, \dots, x_p$. Indeed, we have

$$x_2 = x_1^y, x_3 = x_1^{y^2}, x_4 = x_1^{y^3}, \dots, x_p = x_1^{y^{p-1}},$$

then $[x_i, x_j] = 1$ yields $[x_1^{y^{i-1}}, x_1^{y^{j-1}}] = 1$. Particularly, $[x_1^{y^0}, x_1^{y^k}] = [x_1, x_1^{y^k}] = 1$ for all $1 \leqslant k \leqslant p-1$. Now let $x = x_1$.We have $y^p = 1$,so $[x, x^{y^i}] = 1$ implies (by conjugation)$[x^{y^{p-i}}, x] = 1$ for, $1 \leq i \leq \frac{p-1}{2}$. So, we have the claimed presentation for G.

Now, we show that the above presentation is efficient. By Theorem 2.2, we have

$$\mathbb{Z}_{p^n} \otimes \mathbb{Z}_{p^n} \cong Hom(\mathbb{Z}_{p^n}, \mathbb{Z}_{p^n}) \cong \mathbb{Z}_{p^n}.$$

So, by Theorem 2.1, we have

$$\begin{aligned} M(\mathbb{Z}_p \wr \mathbb{Z}_{p^n}) = M(\mathbb{Z}_p) \quad & \times M(\mathbb{Z}_{p^n}) \\ & \times \underbrace{(\mathbb{Z}_{p^n} \times \cdots \times \mathbb{Z}_{p^n})}_{\frac{1}{2}(p-m-1)-copies} \quad \times \underbrace{(\mathbb{Z}_{p^n} \times \cdots \times \mathbb{Z}_{p^n})}_{m-copies} \end{aligned}$$

where, m is the number of involutions in $\mathbb{Z}_p$. But p is odd and so $m = inv(\mathbb{Z}_p) = 0$. This gives us $M(\mathbb{Z}_p) = M(\mathbb{Z}_{p^n}) = 1$ (for every cyclic group has the trivial multiplicator). So we have

$$M(\mathbb{Z}_p \wr \mathbb{Z}_{p^n}) = \underbrace{(\mathbb{Z}_{p^n} \times \cdots \times \mathbb{Z}_{p^n})}_{\frac{1}{2}(p-1)-copies}.$$

(By Lemma 6.3.4 of [19], if H is a finite group, $\frac{H}{H'} = \bigoplus_{i=1}^{t} \mathbb{Z}_{n_i}$, and s is the number of even n_i's then,

$$H \# H \cong \bigoplus_{1 \leqslant i,j \leqslant t}^{t} \mathbb{Z}_{(n_i, n_j)} \oplus \mathbb{Z}_2^{(s)},$$

where $\mathbb{Z}_2^{(s)}$ is a direct sum of s copies of $\mathbb{Z}_2$. Now, if $H = \mathbb{Z}_{p^n}$, then $\frac{H}{H'} \cong \mathbb{Z}_{p^n}$ and so, $t = 1$, $(p^n, p^n) = p^n$ and $s = 0$. Thus, $H \# H = \mathbb{Z}_{p^n} \# \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^n}$).

So, $rank(M(\mathbb{Z}_p \wr \mathbb{Z}_{p^n})) = \frac{p-1}{2}$. On the other hand, by the above presentation for G, we get $|R| - |X| = \frac{p-1}{2} + 2 - 2 = \frac{p-1}{2}$. Therefore, $rank(M(G)) = |R| - |X|$ and the given presentation for G, is efficient. This completes the proof. □

Now, we prove some statements to obtain the nilpotency class of G. The following lemma is the key statement in this way.

Lemma 2.4. *In the group G,*

$$[y^i, x][y^j, x] = [y^j, x][y^i, x], \quad 1 \leqslant i, j \leqslant p-1.$$

Proof. Let $1 \leqslant i, j \leqslant p-1$. Since x and x^{y^i} commute,we have:
$[y^i, x][y^j, x] = x^{-y^i} x x^{-y^j} x = x^{-y^j} x x^{-y^i} x = [y^j, x][y^i, x]$. □

Also, we will use the following property on commutators which holds in every group.

Lemma 2.5. *For every z and u in a group T we have:*

$$[z, u^i] = \prod_{k=0}^{i-1} [z, u]^{u^k}, \quad (i \geqslant 2).$$

Proof. We use an induction method on i. If $i = 2$, then

$$[z, u^2] = z^{-1}u^{-2}zu^2 = z^{-1}u^{-1}zuu^{-1}z^{-1}u^{-1}zu^2 = [z, u][z, u]^u,$$

and if $i = 3$, then

$$[z, u^2][z, u]^{u^2} = z^{-1}u^{-2}zu^2u^{-2}(z^{-1}u^{-1}zu)u^2 = z^{-1}u^{-3}zu^3 = [z, u^3].$$

Now, let the assertion holds for $i-1$, namely,

$$[z, u^{i-1}] = \prod_{k=0}^{i-2} [z, u]^{u^k},$$

and then,

$$\begin{aligned}
\textstyle\prod_{k=0}^{i-1}[z,u]^{u^k} &= (\textstyle\prod_{k=0}^{i-2}[z,u]^{u^k}) \cdot [z,u]^{u^{i-1}} \\
&= [z, u^{i-1}] \cdot [z,u]^{u^{i-1}} \qquad \text{(by induction hypothesis)} \\
&= z^{-1}u^{-(i-1)}zu^{i-1}.u^{-(i-1)}(z^{-1}u^{-1}zu)u^{i-1} \\
&= z^{-1}u^{-(i-1)}u^{-1}zuu^{i-1} \\
&= z^{-1}u^{-i}zu^i \qquad = [z, u^i].
\end{aligned}$$

□

Using the Lemma 2.5, we get:

Corollary 2.6. *For every x and y in a group T, we have:*

$$[x^{-1}x^{y^i}, y^j] = [x^{-1}x^{y^i}, y^{j-1}][x^{-1}x^{y^i}, y]^{y^{j-1}} = \prod_{k=0}^{j-1}[x^{-1}x^{y^i}, y]^{y^k}.$$

□

We also need the following theorem about the coefficients of the binomial theorem which proved firstly by A. Fleck in 1913:

Theorem 2.7. *For every prime p and every positive integer r,*

$$\sum_{m \equiv r (mod\ p)} (-1)^m \cdot \binom{k}{m} = 0 \ \ (mod\ p^{\lfloor \frac{k-1}{p-1} \rfloor}).$$

In particular,

$$p^n | \binom{k}{0} - \binom{k}{p} + \binom{k}{2p} - \cdots + - \binom{k}{tp},$$

and

$$p^n | -\binom{k}{1} + \binom{k}{p+1} - \binom{k}{2p+1} + \cdots - + \binom{k}{tp+1},$$

where $k = n(p-1)+1$ and $k = tp + r$, $(0 \leqslant r < t)$.

Proof. See for example [17]. □

Remark 2.8. *In the following proposition, we prove that G is a metabelian group of nilpotency class $n(p-1)+1$ (this is a special case of general formula proved by Liebeck (see [22])).*

Proposition 2.9. *The group G is a metabelian group of nilpotency class $n(p-1)+1$.*

Proof. Obviously,

$$\frac{G}{G'} \cong \langle x, y | x^{p^n} = y^p = [x, y] = 1 \rangle \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_p.$$

So,

$$|G'| = \frac{(p^n)^p \cdot p}{p^n \cdot p} = (p^n)^{p-1} = p^{n(p-1)}.$$

Now, let $H = \langle [y,x],[y^2,x],[y^3,x],\ldots,[y^{p-1},x]\rangle$. Since $[x,x^{y^i}]=1$ then, $x^{y^i}x = xx^{y^i}$, and hence,

$$(x^{y^i}x)^{p^n} = (x^{y^i})^{p^n}\cdot x^{p^n} = y^{-i}x^{p^n}y^i x^{p^n} = 1.$$

Also, by $(x^{y^i})^{-1}x = x(x^{y^i})^{-1}$ we get $|(x^{y^i})^{-1}x| = |x^{y^i}x| = p^n$. On the other hand, the equations $[y^i,x] = y^{-i}x^{-1}y^i x = (x^{y^i})^{-1}x$ give us,

$$|[y^i,x]| = |x^{y^i}x| = p^n, \quad \forall i = 1,\ldots,p-1.$$

However by Lemma 2.4, we calculate that

$$[y^i,x][y^j,x] = [y^j,x][y^i,x], \quad 1 \leqslant i,j \leqslant p-1.$$

Thus,

$$H \cong \underbrace{(\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^n}\times\cdots\times\mathbb{Z}_{p^n})}_{(p-1)-copies}.$$

Hence, $|H| = |G'| = p^{n(p-1)}$. But, $H \subseteq G'$ and therefore, $G' = H$, and G' is abelian. Consequently, G is metabelian and G' is of the form

$$G' \cong \underbrace{(\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^n}\times\cdots\times\mathbb{Z}_{p^n})}_{(p-1)-copies}.$$

Now, let

$$\gamma_0 = G \supseteq \gamma_1 = G' = [G,G] \supseteq \gamma_2 = [G,G'] \supseteq \cdots \supseteq \gamma_j = [G,\gamma_{j-1}] \supseteq \cdots$$

be the lower central series for G. Since $[x,x^{y^i}]=1$ holds for every i so, γ_j is generated by the commutators as follows:

$$\gamma_j = \langle [z,y^i] | z \in \gamma_{j-1},\ i = 1,2,\ldots,p-1\rangle.$$

Now, the Lemma 2.5 shows that:

$$[z,y^i] = \prod_{k=0}^{i-1}[z,y]^{y^k}.$$

Thus, it suffices to prove that for all $z \in \gamma_{j-1}$, $[z, y] = 1$, and then $\gamma_j = \{1\}$, i.e.; G will be a nilpotent group of nilpotency class j. We have:

$$[x, y] = x^{-1}x^y \in \gamma_1,$$

$$[[x, y], y] = [x^{-1}x^y, y] = x^{y^2}(x^{-2})^y x \in \gamma_2,$$

$$[[[x, y], y], y] = x^{y^3}(x^{-3})^{y^2}(x^3)^y x^{-1} \in \gamma_3,$$

$$[[[[x, y], y], y], y] = x^{y^4}(x^{-4})^{y^3}(x^6)^{y^2}(x^{-4})^y x \in \gamma_4,$$

$$[[[[[x, y], y], y], y], y] = x^{y^5}(x^{-5})^{y^4}(x^{10})^{y^3}(x^{-10})^{y^2}(x^5)^y x^{-1} \in \gamma_5,$$

$$[[[[[[x, y], y], y], y], y], y] = x^{y^6}(x^{-6})^{y^5}(x^{15})^{y^4}(x^{-20})^{y^3}(x^{15})^{y^2}(x^{-6})^y x \in \gamma_6,$$

$$[[[[[[[x, y], y], y], y], y], y], y]$$

$$= x^{y^7}(x^{-7})^{y^6}(x^{21})^{y^5}(x^{-35})^{y^4}(x^{35})^{y^3}(x^{-21})^{y^2}(x^7)^y x^{-1} \in \gamma_7.$$

Finally,

$$[[\cdots[x, \underbrace{y], y], \ldots, y]}_{k-times} = (x^{\binom{k}{0}})^{y^k} \quad (x^{-\binom{k}{1}})^{y^{k-1}} (x^{\binom{k}{2}})^{y^{k-2}}$$
$$\cdots (x^{(-1)^{k-1}\binom{k}{k-1}})^{y^1} (x^{(-1)^k\binom{k}{k}})^{y^0} \in \gamma_k.$$

Now, when $k = n(p-1)+1$, since $y^k = y^{n(p-1)+1} = y^{-n+1}$, after simplifying it by the Fleck's congruence, Theorem 2.7, we have

$$[[\ldots[x, \underbrace{y], y], \ldots, y]}_{k-times} = 1,$$

and so $\gamma_k = \{1\}$, where $k = n(p-1)+1$, as desired. This completes the proof. $\square$

3 The Fibonacci length of $G(p,n)$

Let G be a finitely generated group, $G=\langle A\rangle$, where $A=(a_1,\ldots,a_m)$ an ordered m-tuple. Then we define the Fibonacci orbit of G with respect to the generating m-tuple A, written $F_A(G)$, is the sequence $x_1=a_1,\cdots,x_m=a_m, x_{i+m}=\prod_{j=1}^{m} x_{i+j-1},\ i\geqslant 1$. If $F_A(G)$ is periodic then the length of the period of the sequence is called the Fibonacci length of G with respect to the generating m-tuple A, written $LEN_A(G)$. If $F_A(G)$ is not periodic then we say that the group G has infinite Fibonacci length on the generating m-tuple A, written $LEN_A(G)=\infty$. The minimal length of the period of the Fibonacci series modulo a positive integer m is denoted by $k(m)$ and is called the Wall number of m. For more information on the Fibonacci length of groups and Wall numbers one may consult [1, 3, 4, 6, 14, 15, 16, 28].

In this section, we want to calculate the Fibonacci length of the p-groups considered in the previous section, i.e. the $G(p,n)$. First we prove a lemma:

Lemma 3.1. *Let G be a group and $x,y\in G$. Then for every positive integer j and every integers t and s we have*

$$\left(x^{y^{t-1}}\right)^s\cdot y^j = y^j\cdot\left(x^{y^{t+j-1}}\right)^s.$$

Proof. The proof is by induction on j. If $j=1$ we have

$$\begin{aligned}\left(x^{y^{t-1}}\right)^s\cdot y &= \underbrace{(y^{-t+1}xy^{t-1})(y^{-t+1}xy^{t-1})\cdots(y^{-t+1}xy^{t-1})}_{s-times}\cdot y\\ &= y\cdot\underbrace{(y^{-t}xy^{t})(y^{-t}xy^{t})\cdots(y^{-t}xy^{t})}_{s-times}\\ &= y\cdot\left(x^{y^t}\right)^s.\end{aligned}$$

So,

$$\left(x^{y^{t-1}}\right)^s\cdot y = y\cdot\left(x^{y^t}\right)^s. \qquad (*)$$

Now, suppose that the assertion holds for $j-1$ and we prove that the assertion is true for j:

$$\begin{aligned}
\left(x^{y^{t-1}}\right)^s \cdot y^j &= \left(x^{y^{t-1}}\right)^s \cdot y^{j-1} \cdot y \\
&= y^{j-1} \cdot \left(x^{y^{t+j-2}}\right)^s \cdot y && \text{(by the induction hypothesis)} \\
&= y^{j-1} \cdot y \cdot \left(x^{y^{t+j-1}}\right)^s && \text{(by } (*)\text{)} \\
&= y^j \cdot \left(x^{y^{t+j-1}}\right)^s .
\end{aligned}$$

□

Theorem 3.2. *Let* $\{x_m\}_1^\infty$ *be the Fibonacci sequence of* $G(p,n)$ *with respect to* $A = \{x, y\}$*. Then,*

$$x_m = y^{a_m} x^{b_{m1}} (x^y)^{b_{m2}} \left(x^{y^2}\right)^{b_{m3}} \cdots \left(x^{y^{p-1}}\right)^{b_{mp}}, \quad m \geqslant 1,$$

where $a_1 = 0, a_2 = 1, a_m = a_{m-1} + a_{m-2}$ $(mod\ p); m > 2$ *and* $b_{11} = 1,\ b_{21} = 0,\ b_{1i} = b_{2i} = 0, \forall i > 1,$ *and for* $m \geqslant 3,$ *if* $a_{m-1} = j$ *then*

$$\begin{cases}
b_{m1} &= b_{m-1,1} + b_{m-2,\theta^j(1)} \\
b_{m2} &= b_{m-1,2} + b_{m-2,\theta^j(2)} \\
\vdots & \quad\ \vdots \quad \vdots \\
b_{mp} &= b_{m-1,p} + b_{m-2,\theta^j(p)}
\end{cases}$$

where $0 \leqslant j \leqslant p-1,$ *and* $\theta = (1\ p\ p-1 \cdots 3\ 2),$ *the* p*-cycle in the cyclic group* C_p*.*

Proof. The proof is by induction on m. We use the Lemma 3.1 and the fact that

$$y^p = x^{p^n} = 1,\ [x^{y^\ell}, x^{y^k}] = 1,\ (0 \leqslant k, \ell \leqslant p-1).$$

We have

$$\begin{aligned}
x_m &= x_{m-2}x_{m-1}\\
&= y^{a_{m-2}}x^{b_{m-2,1}}(x^y)^{b_{m-2,2}}(x^{y^2})^{b_{m-2,3}}\cdots(x^{y^{p-1}})^{b_{m-2,p}}\\
&\cdot y^{a_{m-1}}x^{b_{m-1,1}}(x^y)^{b_{m-1,2}}(x^{y^2})^{b_{m-1,3}}\cdots(x^{y^{p-1}})^{b_{m-1,p}}\\
&= y^{a_{m-2}}\cdot y^{a_{m-1}}\cdot(x^{y^{a_{m-1}}})^{b_{m-2,1}}(x^{y^{a_{m-1}+1}})^{b_{m-2,2}}(x^{y^{a_{m-1}+2}})^{b_{m-2,3}}\\
&\cdots(x^{y^{a_{m-1}+p-1}})^{b_{m-2,p}}\cdot x^{b_{m-1,1}}(x^y)^{b_{m-1,2}}(x^{y^2})^{b_{m-1,3}}\cdots(x^{y^{p-1}})^{b_{m-1,p}}\\
&= y^{a_{m-2}+a_{m-1}}\cdot x^{b_{m-1,1}+b_{m-2,\theta^j(1)}}(x^y)^{b_{m-1,2}+b_{m-2,\theta^j(2)}}\\
&(x^{y^2})^{b_{m-1,3}+b_{m-2,\theta^j(3)}}\cdots(x^{y^{p-1}})^{b_{m-1,p}+b_{m-2,\theta^j(p)}},
\end{aligned}$$

(where θ is the p-cycle $\theta=(1\ p\ p-1\cdots 3\ 2)$ in the cyclic group C_p), since for example $\theta^j(1)=p-j+1$, where $j=a_{m-1}$ and so

$$(x^{y^{j+\theta^j(1)-1}})^{b_{m-2,\theta^j(1)}}=(x^{y^{j+p-j+1-1}})^{b_{m-2,\theta^j(1)}}=x^{b_{m-2,\theta^j(1)}},$$

$$(x^{y^{j+\theta^j(2)-1}})^{b_{m-2,\theta^j(2)}}=(x^{y^{j+p-j+2-1}})^{b_{m-2,\theta^j(2)}}=(x^y)^{b_{m-2,\theta^j(1)}}$$

and so on. Now, by comparing the powers the assertions conclude. □

Corollary 3.3. $LEN_{\{x,y\}}(G(p,n))=k(p)\times p^n$.

Proof. By the Theorem 3.2, it is clear that

$$LEN_{\{x,y\}}G(p,n)=k(p)\times p^n.$$

□

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